Computational Logic Lecture 7

Herbrand's Theorem and Alternative Semantics

Tim Hinrichs and Mike Genesereth Autumn 2006

Propositional Interpretations

	p	q	r
	0	0	0
	0	0	1
$\longrightarrow$	0	1	0
	0	0	1
	1	0	0
	1	0	1
	1	1	0
	1	1	1

For a language with n constants, there are 2^n interpretations.

2

Relational Interpretations

$\|i\|$	a	b	r
{○,●}	○	○	{}
{○,●}	○	○	{○}
{○,●}	○	○	{●}
{○,●}	○	○	{○, ●}
{○,●}	○	●	{}
{○,●}	○	●	{○}
{○,●}	○	●	{●}
{○,●}	○	●	{○, ●}
{○,●}	●	○	{}
{○,●}	●	○	{○}
{○,●}	●	○	{●}
{○,●}	●	○	{○, ●}

. . .

Infinitely many interpretations.

3

Logical Entailment

A set of premises logically entails a conclusion if and only if *every* interpretation that satisfies the premises also satisfies the conclusion.

In the case of Propositional Logic, the number of interpretations is finite, and so it is possible to check logical entailment directly in finite time.

In the case of Relational Logic, the number of interpretations is infinite, and so a direct check of logical entailment is not feasible.

4

Good News

Given any set of sentences, there is a special subset of interpretations called *Herbrand interpretations*.

Under *certain conditions*, checking just the Herbrand interpretations suffices to determine logical entailment.

Since there are fewer Herbrand interpretations than interpretations in general, checking just the Herbrand interpretations is less work than checking all interpretations.

5

HHHHerbrand

The *Herbrand universe* for a set of sentences in Relational Logic (with at least one object constant) is the set of all ground terms that can be formed from just the constants used in those sentences. If there are no object constants, then we add an arbitrary object constant, say *a*.

The *Herbrand base* for a set of sentences is the set of all ground atomic sentences that can be formed using just the constants in the Herbrand universe.

6

Herbrand Interpretation

A Herbrand interpretation for a function-free language is an interpretation in which (1) the universe of discourse is the Herbrand universe for the language and (2) each object constant maps to itself.

$$|i| = \{a,b\}$$
$$i(a) = a$$
$$i(b) = b$$
$$i(r) = \{\langle a,a\rangle, \langle a,b\rangle\}$$

7

Herbrand Interpretations

$\lvert i\rvert$	a	b	r
$\{a,b\}$	a	b	$\{\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle b,a\rangle\}$
$\{a,b\}$	a	b	$\{\langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle a,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle b,a\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,b\rangle, \langle b,a\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,b\rangle, \langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle b,a\rangle, \langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle a,b\rangle, \langle b,a\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle a,b\rangle, \langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle b,a\rangle, \langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,b\rangle, \langle b,a\rangle, \langle b,b\rangle\}$
$\{a,b\}$	a	b	$\{\langle a,a\rangle, \langle a,b\rangle, \langle b,a\rangle, \langle b,b\rangle\}$

8

Herbrand Theorem

Herbrand Theorem: A set of quantifier-free sentences in Relational Logic is satisfiable if and only if it has a Herbrand model.

Construction of Herbrand model h given i.

The model assigns every object constant to itself.

The interpretation for relation constant ρ is the set of all tuples of object constants $\tau_1, \ldots, \tau_n$ such that i satisfies the sentence $\rho(\tau_1, \ldots, \tau_n)$.

9

Example

Interpretation

$|i| = \{\bigcirc, \bullet\}$

$i(a) = \bigcirc$

$i(b) = \bullet$

$i(r) = \{\langle\bigcirc,\bullet\rangle, \langle\bullet,\bullet\rangle\}$

Herbrand Base

$\{r(a,a), r(a,b), r(b,a), r(b,b)\}$

Herbrand Interpretation

$|i| = \{a,b\}$

$i(a) = a$

$i(b) = b$

$i(r) = \{\langle a,b\rangle, \langle b,b\rangle\}$

10

Example

Interpretation

$|i|=\{○, ●, ★\}$

$i(a) = ○$

$i(b) = ●$

$i(r) = \{\langle ○,● \rangle, \langle ●,● \rangle, \langle ★,● \rangle\}$

Herbrand Base

$\{r(a,a), r(a,b), r(b,a), r(b,b)\}$

Herbrand Interpretation

$|i|=\{a,b\}$

$i(a) = a$

$i(b) = b$

$i(r) = \{\langle a,b \rangle, \langle b,b \rangle\}$

11

Herbrand Theorem

Herbrand Theorem: A set of quantifier-free sentences is satisfiable if and only if it has a Herbrand model that satisfies it.

Proof. Assume the set of sentences contains at least one object constant. If a set of quantifier-free sentences is satisfiable, then there is a model that satisfies it. Take the intersection of this model with the Herbrand base. By definition, this is a Herbrand model. Moreover, it is easy to see that it satisfies the sentences. If the sentences are ground, it must agree with the original interpretation on all of the sentences, since they are all ground and mention only the constants common to both interpretations. If the sentences contain variables, the instances must all be true, including those in which the variables are replaced only by elements in the Herbrand universe.

If there is no object constant, then create a tautology involving a new constant (say a) and add to the set. This does not change the satisfiability of the sentences but satisfies proof above. QED

12

Utility of Herbrand's Theorem

$\Delta \models \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Skolemization is a process than converts any set of sentences in relational logic into a set of quantifier-free sentences while preserving satisfiability. (See the upcoming lectures.)

Herbrand Theorem: A set of quantifier-free sentences is satisfiable if and only if it has a Herbrand model that satisfies it.

$\Delta \models \varphi$ if and only if skolem[$\Delta \cup \{\neg\varphi\}$] has no Herbrand models. (See upcoming lectures.)

13

Computational Logic for Computer Scientists

Computational Logic is concerned with algorithms for automatically processing logic. The consumers of those algorithms are usually mathematicians or computer scientists.

Relational logic is good for mathematicians who must contend with uncountable infinities.

Computer scientists are chiefly concerned with representing, manipulating, and analyzing a finite machine operating in discrete steps for an arbitrary amount of time. Countably infinite is big enough!

14

A Logic for Computer Scientists

Logic is used throughout computer science.

Examples:

- Database theory
- Logic programming
- Constraint satisfaction
- Formal verification

All of these can be defined in a single logic.

That logic is more intuitive than relational logic, and it can be used to represent more of the problems computer scientists care about.

15

Herbrand Logic

Logic = Syntax + Semantics

Herbrand logic has the same syntax as relational logic, but the semantics are different.

The only interpretations that exist are the Herbrand interpretations.

A set of premises logically entails a conclusion if and only if every *Herbrand* interpretation that satisfies the premises also satisfies the conclusion.

16

Example

Premises:

$p(a)$

Conclusion:

$\forall x.p(x)$

Relational Logic: counterexample

$|i| = \{○, ●\}$

$i(a) = ○$

$i(p) = \{○\}$

Herbrand Logic: entailed

Herbrand universe: a

Herbrand base: $p(a)$

Interpretations
$\{\}$
$\{p(a)\}$

17

Finite Herbrand Logic

Without function constants, the Herbrand universe is always finite.

Finite Herbrand Logic (FHL) is the special case of Herbrand Logic where there are no function constants.

Finite Herbrand Logic has exactly the same expressive power as propositional logic: every set of propositional sentences can be represented by a set of FHL sentences, and every set of FHL sentences can be represented by a set of propositional sentences.

18

Example

Finite Herbrand:	$q(b) \Rightarrow q(c)$
	$\exists x.\ p(x)$
	$\neg p(a)$
Grounded FHL:	$q(b) \Rightarrow q(c)$
	$p(a) \vee p(b) \vee p(c)$
	$\neg p(a)$
Propositional:	$s \Rightarrow u$
	$p \vee q \vee r$
	$\neg p$

19

Interpretation Sizes in Herbrand Logic

In FHL, all the interpretations have a universe of a fixed size: the number of object constants.

When the vocabulary includes at least one function constant and one object constant, there are infinitely many ground terms. Every interpretation has an infinite universe.

In HL, there are either infinite interpretations or finite interpretations but never both.

20

Example

Premises: $p(a)$
$p(f(a))$
$p(f(f(a)))$
…

Conclusion:
$\forall x.p(x)$

Relational Logic: counterexample

$|i|=\{○, ●\}$
$i(a) = ○$
$i(f) = \{\langle ○\rangle \rightarrow ○, \langle ●\rangle \rightarrow ●\}$
$i(p) = \{\langle ○\rangle\}$

Herbrand Logic: entailed

Herbrand universe: *a, f(a), f(f(a)), …*
Herbrand base: *p(a), p(f(a)), p(f(f(a))), …*

21

Compactness

Compactness: A set of sentences is unsatisfiable if and only if there is some finite subset that is unsatisfiable.

Theorem: Herbrand logic is not compact.
Proof: ∃x. ¬p(x)
p(a)
p(f(a))
p(f(f(a)))
…

Theorem: Relational logic is compact.

22

Example

Premises:

$p(a)$

$p(x) \Rightarrow p(f(x))$

Conclusion:

$\forall x.p(x)$

Relational Logic: counterexample

$|i|=\{○, ●\}$

$i(a) = ○$

$i(f) = \{\langle ○ \rangle \rightarrow ○, \langle ● \rangle \rightarrow ●\}$

$i(p) = \{\langle ○ \rangle\}$

Herbrand Logic: entailed

Herbrand universe: $a, f(a), f(f(a)), \ldots$

Herbrand base: $p(a), p(f(a)), p(f(f(a))), \ldots$

23

Induction

Mathematical Induction:

To prove $\forall$n.p(n),

prove the base case, e.g. p(a)

prove the inductive case, e.g. $\forall$x.(p(x)$\Rightarrow$p(f(x)))

The semantics of Herbrand logic justify using induction to prove entailment.

The semantics of Relational logic DO NOT justify using induction to prove entailment.

A lecture later in the course is devoted entirely to proof by induction.

24

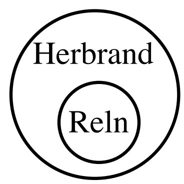
Entailment Comparison

Entailment in Herbrand logic and entailment in relational logic give different results.

Every sentence entailed in relational logic is still entailed in Herbrand logic.

There are sentences entailed in Herbrand logic that are not entailed in relational logic.

Consequences of
a set of axioms:



Computability

A problem can be categorized based on the type of algorithms that exist for solving it.

Decidable: There is an algorithm that halts and either returns a solution or states there is no solution.

Semi-decidable: There is an algorithm that halts and returns a solution when there is one.

Not semi-decidable: There is no algorithm guaranteed to find a solution.

Computability of Entailment

A set of premises logically entails a conclusion if and only if *every* interpretation that satisfies the premises also satisfies the conclusion.

Propositional Logic: decidable

Enumerate the possible models and check.

Relational Logic: semi-decidable

Proof by Goedel

Herbrand Logic: not semi-decidable

27

Reductions

Prove that entailment in Herbrand logic is not semi-decidable.

Find a problem that is not semi-decidable.

Encode the problem as an entailment query in Herbrand logic.

Answering the entailment query must then be at least as difficult as solving a problem that is not semi-decidable.

28

Diophantine Equations

$P(x_1,\ldots,x_n)$ represents a polynomial with n variables.

For example,

$P(x,y,z)$ might stand for $4x^2y^3 + 7y^4z^2$

The following problem is semi-decidable.

Is there an integral solution to $P(x_1,\ldots,x_n) = 0$?

Since solving a Diophantine is semi-decidable and not decidable, the following problem is not semi-decidable.

Is there no integral solution to $P(x_1,\ldots,x_n) = 0$?

29

Encoding the Natural Numbers

Natural numbers: 0, 1, 2, 3, …

Represent the natural numbers in unary:

0, s(0), s(s(0)), s(s(s(0))), …

Define a relation that includes all the natural numbers.

num(0)
num(x) $\Rightarrow$ *num(s(x))*

30

Encoding Addition

Instead of writing $x + y = z$, we write $sum(x, y, z)$

Base case:

$x + 0 = x$

$sum(x, 0, x)$

Recursive case:

$x + y = z \Rightarrow (x+1) + y = (z+1)$

$sum(x, y, z) \Rightarrow sum(s(x), y, s(z))$

31

Encoding Multiplication

Instead of writing $x * y = z$, we write $product(x, y, z)$

Base case:

$x * 0 = 0$

$product(x, 0, 0)$

Recursive case:

$(x * y = z) \wedge (z + y = w) \Rightarrow (x+1) * y = w$

$product(x, y, z) \wedge sum(z, y, w) \Rightarrow product(s(x), y, w)$

32

Encoding a Polynomial: $2x^2 + y^2z$

$2x^2$ can be written as $(2*x)=w$ and $w*x=t_1$

$\varphi(x,t_1)$:

$$\exists w.\ (product(s(s(0)), x, w) \wedge product(w,x,t_1))$$

y^2z can be written as $(y*y)=u$ and $u*z=t_2$

$\psi(y,z,t_2)$:

$$\exists u.(product(y,y,u) \wedge product(u,z,t_2))$$

$2x^2 + y^2z = t$ uses the last two formulas

$\rho(x,y,z,t)$:

$$\exists t_1 t_2.\ (\varphi(x,t_1) \wedge \psi(y,z,t_2) \wedge sum(t_1, t_2, t))$$

33

Diophantine Equations

Is there an integral solution to $2x^2 + y^2z = 0$?

$$\exists xyz.\ \rho(x,y,z,0)$$

Is there no solution to the equation $2x^2 + y^2z = 0$?

$$\forall xyz.\ \neg\rho(x,y,z,0)$$

34

Upshot

In Herbrand logic, we have demonstrated how to encode a problem that is not semi-decidable.

$$\forall xyz.\ \neg\rho(x,y,z,0)$$

Theorem: Entailment in Herbrand logic is therefore not semi-decidable.

35

Relational Logic

Is there no integral solution to $\rho(x,y,z,0)$?

In relational logic, the following sentence DOES NOT encode this problem.

$$\forall xyz.\ \neg\rho(x,y,z,0)$$

Why? In relational logic, the $\forall$ quantifier includes the real numbers.

36

Some Good News

Theorem: $\forall^*.\Delta \models \exists^*.\varphi$ in Herbrand logic if and only if it holds in relational logic.

Corollary: $\forall^*.\Delta \models \exists^*.\varphi$ is semi-decidable in HL.

Notation:

$\forall^*.\Delta$: after pushing all the quantifiers in Δ to the front (prenex form), every quantifier is a $\forall$.

$\exists^*.\varphi$: after pushing all the quantifiers in φ to the front, every quantifier is a $\exists$.

37

More Bad News

Corollary: Whether a set of sentences is satisfiable in Herbrand logic is not semi-decidable.

More Good News

The theory of integer arithmetic, i.e. the natural numbers and 0, 1, +, *, < is finitely axiomatizable in Herbrand logic.

38

Significance

There is no algorithm for determining whether $\Delta \models \varphi$ in Herbrand logic.

There is no algorithm for determining whether Δ |# φ in Herbrand logic.

In the general case, there is no algorithm for automatically answering entailment queries either positively or negatively.

ATP in Herbrand logic relies on analyzing Δ and φ and taking advantage of special cases.

39

Herbrand and Relational Logic

Fragment	**Relational**	**Herbrand**
$\forall^* \models \exists^*$	Semi	Semi
$\forall^* \models \forall^*$	Semi	Not Semi
No functions	Semi	Decidable
$\models$	Semi	Not Semi
|#	Not Semi	Not Semi
Arithmetic	Not r.e.	Finite
Compact	Yes	No

40

Finite Relational Logic

Finite Herbrand logic is decidable, but Herbrand logic in general is not semi-decidable.

Infinite models were the source of all the trouble in Herbrand logic.

Finite Relational Logic (FRL) has the same syntax as relational logic. An interpretation has the same definition as in relational logic, except the universe is always finite.

41

Example

Dense linear order:

$$\forall xy.(x < y \Rightarrow \exists y.\ (x < z \wedge z < y))$$
$$\forall x.\ \neg(x < x)$$
$$\forall xy.(x \neq y \Rightarrow x < y \vee y < x)$$

Relational logic: satisfiable

$|i|$ = rational numbers

$i(<)$ = usual ordering

Finite relational logic: unsatisfiable

42

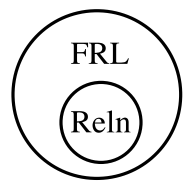
Entailment Comparison

Entailment in finite relational logic and entailment in relational logic give different results.

Every sentence entailed in relational logic is still entailed in finite relational logic.

There are sentences entailed in finite relational logic that are not entailed in relational logic.

Consequences of a set of axioms:



43

Isomorphic Finite Interpretations

Two interpretations are isomorphic if and only if they satisfy all the same sentences.

Consider the class C of all finite interpretations where the universe is a subset of the natural numbers.

Every finite interpretation of size n is isomorphic to one of the interpretations in C where the universe is {1,…,n}.

In Finite Herbrand logic, a set of premises logically entails a conclusion if and only if *every* interpretation in C that satisfies the premises also satisfies the conclusion.

44

Example

Finite interpretation

$|i|=\{○, ●\}$

$i(a) = ○$

$i(f) = \{\langle ○ \rangle \rightarrow ○, \langle ● \rangle \rightarrow ●\}$

$i(p) = \{\langle ○ \rangle\}$

Interpretation where universe is a subset of the natural numbers.

$|i|=\{1, 2\}$

$i(a) = 1$

$i(f) = \{\langle 1 \rangle \rightarrow 2, \langle 2 \rangle \rightarrow 2\}$

$i(p) = \{\langle 1 \rangle\}$

45

Satisfaction, Limited Satisfiability, Satisfiability

Satisfaction: Given an interpretation I and a sentence φ, does I satisfy φ?

Limited Satisfiability: Given a sentence φ and a positive integer n, is φ satisfied by some model of size n?

In finite relational logic, satisfaction and limited satisfiability are decidable.

Theorem: Given a finite set of sentences Δ, determining whether Δ is satisfiable is semi-decidable.

46

Trakhtenbrot's Theorem

Trakhtenbrot's Theorem: Entailment in finite relational logic is not semi-decidable.

Proof: Reduction using the halting problem shows that satisfiability in finite relational logic is undecidable.

Corollary: Satisfaction in finite relational logic is undecidable.

General Problem

The number of finite models can be very large.

n objects
m k-ary relation constants
Number of k-ary tuples: n^k
Number of k-ary relations: 2^n^k
Number of interpretations: (2^n^k)^m

10 objects
3 2-ary relation constants
Number of k-ary tuples: 100
Number of k-ary relations: 2^{100}
Number of interpretations: 2^{300}

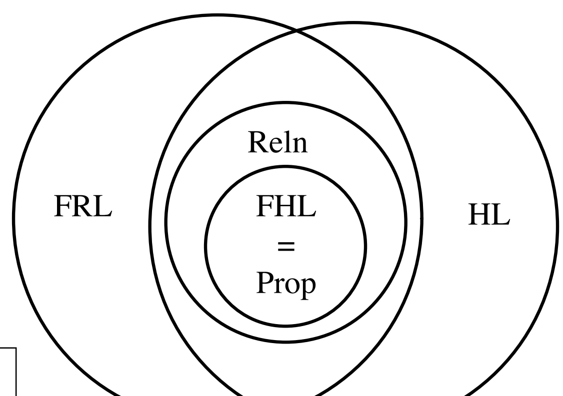
Summary

Herbrand's theorem: A set of quantifier-free sentences is satisfiable if and only if it has a Herbrand model that satisfies it.

Fragment	Relational	Herbrand	Finite
$\forall^* \models \exists^*$	Semi	Semi	
$\forall^* \models \forall^*$	Semi	Not Semi	
No functions	Semi	Decidable	
$\models$	Semi	Not Semi	Not Semi
|#	Not Semi	Not Semi	Semi
Arithmetic	Not r.e.	Finite	Not possible
Compact	Yes	No	No

Expressiveness Comparison

Consequences of a set of axioms:



FHL: Finite Herbrand
Prop: Propositional
Reln: Relational
FRL: Finite Relational
HL: Herbrand