Computability Theory (ATC-CT)

Recursion theorem and Rice's theorem

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Recursion Theorem

Theorem. *Let $g(z, x_1, x_2, \ldots x_m)$ be a partially computable function. Then there is a number e such that*

$$\Phi_e^{(m)}(x_1, x_2, \ldots x_m) = g(e, x_1, x_2, \ldots x_m)$$

What does it mean? Given g, we need to produce a "program" e that computes g in the following weird way: all except one inputs of g are supplied. The program e must compute the missing input so that it coincides with its own "source code"!

Naive attempt to prove the theorem: The program e can have the number e built in. **Does not work** as **any number** that can be **built into a program** is **smaller than** the program's **Gödel number**!

Recursion Theorem

Proof: The program $\mathcal{P}$ with $\#(P) = e$ must contain some "partial description" of itself built-in, so that it can recover its own Gödel number, e, from that description.

Let $\mathcal{Q}$ be the following program:

$$
\begin{aligned}
Z &\leftarrow S_m^1(X_{m+1}, X_{m+1}) \\
Y &\leftarrow g(Z, X_1, X_2, \ldots X_m).
\end{aligned}
$$

Now, the program $\mathcal{P}$ will consists of $\#(\mathcal{Q})$ copies of the instruction $X_{m+1} \leftarrow X_{m+1} + 1$, followed by the program $\mathcal{Q}$.

After having executed the first $\#(\mathcal{Q})$ increments as well as the instruction $Z \leftarrow S_m^1(X_{m+1}, X_{m+1})$, Z holds the Gödel number of the program consisting of $\#(\mathcal{Q})$ copies of the instruction $X_{m+1} \leftarrow X_{m+1} + 1$ followed by the program $\mathcal{Q}$ (all this is because $X_{m+1} = \#(\mathcal{Q})$ at that stage). But this is exactly the program $\mathcal{P}$!

Recursion Theorem

Proof (the real one, without programs): Consider the partially computable function

$$g\left(S_m^1(v,v),x_1,x_2,\ldots x_m\right).$$

We have a number z_0 such that

$$\begin{aligned} g\left(S_m^1(v,v),x_1,x_2,\ldots x_m\right) &= \Phi^{(m+1)}(x_1,x_2,\ldots x_m,v,z_0) \\ &= \Phi^{(m)}\left(x_1,x_2,\ldots x_m,S_m^1(v,z_0)\right) \\ &= \Phi^{(m)}_{S_m^1(v,z_0)}(x_1,x_2,\ldots x_m). \end{aligned}$$

Now set $v = z_0$ and $e = S_m^1(z_0,z_0)$. □

Applications

Corollary. *(Quine) There is a number e such that for all x*

$$\Phi_e(x) = e.$$

Take $g(z,x)$ in the statement of the Recursion theorem to be $u_1^2(z,x) = z$.

Theorem. (Fixed Point) *Let $f(z)$ be a computable function. There is a number e such that*

$$\Phi_{f(e)}(x) = \Phi_e(x)$$

for all x.

Let $g(z,x) = \Phi_{f(z)}(x)$.

Rice's Theorem

Theorem. (Rice) *Let Γ be a collection of partially computable functions of one variable such that there exist two partially computable functions $f(x)$ and $g(x)$ with $f(x) \in \Gamma$ but $g(x) \notin \Gamma$. Then the characteristic function of Γ, R_Γ is not recursive.*

What does it mean? If Γ is a non-trivial property of functions, then Γ is undecidable.

Definition. *(*Index Set*)* $R_\Gamma = \{t \in \mathbb{N} | \Phi_t \in \Gamma\}$.

Note that Γ is a set of functions while R_Γ is a set of programs. (Informally: a function has many, in fact infinitely many, implementations).

Rice's Theorem

Proof: We will construct a *recursive function* $\varphi(i)$ such that

$$i \in K \text{ if and only if } \varphi(i) \in R_\Gamma \text{ for all } i \in \mathbb{N}$$

(here $K =^{def} \{x | \Phi_x(x) \downarrow\}$).

If R_Γ were recursive, K would be recursive, too - a contradiction (revisit the proof of Undecidability of the Halting Problem).

Denote by $h(x)$ the function that is defined nowhere, i.e. $h(x) \uparrow$ for all x. Consider two cases:

Rice's Theorem

1. $h(x) \notin \Gamma$. Consider the following program

$$\begin{array}{lcl} Z & \leftarrow & \Phi(X_2, X_2) \\ Y & \leftarrow & f(X_1) \end{array}$$

and denote its Gödel number by q.

$i \in K \Rightarrow \Phi(i,i)\downarrow \Rightarrow \Phi^{(2)}(x,i,q) = f(x)$ for all $x \Rightarrow \Phi^{(1)}\left(x, S_1^1(i,q)\right) = f(x)$ for all $x \Rightarrow \Phi_{S_1^1(i,q)}(x) = f(x)$ for all $x \Rightarrow S_1^1(i,q) \in R_\Gamma$.

$i \notin K \Rightarrow \Phi(i,i)\uparrow \Rightarrow \Phi^{(2)}(x,i,q) = h(x)$ for all $x \Rightarrow \Phi^{(1)}\left(x, S_1^1(i,q)\right) = h(x)$ for all $x \Rightarrow \Phi_{S_1^1(i,q)}(x) = h(x)$ for all $x \Rightarrow S_1^1(i,q) \notin R_\Gamma$.

2. $h(x) \in \Gamma$. Apply the same argument as in the previous case to $\overline{\Gamma}$ instead of Γ and $g(x)$ instead of $f(x)$. Conclude that $\overline{R}_\Gamma$ is not recursive. $\square$