Fraenkel's Axiom of Restriction: Axiom Choice, Intended Models and Categoricity

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1. Introduction. As part of an increased attention to issues in the philosophy of mathematical practice there has been a renewed interest in the spectrum of justification strategies for mathematical axioms. In a recent and ongoing debate (Feferman et al. 2000, Easwaran 2008) different methodological principles underlying the practice of axiom choice have been discussed that are supposed to explain the reasoning involved in the introduction of new axioms (such as the case of large cardinal axioms (Feferman 1999)) as well as to clarify informal justification methods found in the history of mathematical axiomatics (see Maddy 1997).

My attempt in this paper is to take up and extend this discussion by drawing to a historical episode from early axiomatic set theory centered around Abraham Fraenkel's axiom of restriction (*Beschränktheitsaxiom*) (in the following AR). The paper is structured as follows: After a brief overview of different theories of axiom justification I will specifically focus on a model presented by Maddy 1999 in which the distinction between "intrinsic" and "extrinsic" arguments for the acceptance of an axiom plays a fundamental role. Her approach will be discussed in greater detail for the specific case of the Fraenkel's axiom candidate. As will be shown, Fraenkel develops different lines of argumentation for it that only partially fit Maddy's account of extrinsic justification. His main intention behind the axiom grounds on a metatheoretical consideration, i.e. to restrict the set theoretic universe to what can be termed his intended model, thereby rendering his axiom system categorical. I will give a detailed presentation of different proposed versions of AR as well as of the evolution of Fraenkel's informal arguments for it from 1922 onward. Further, I will highlight the intellectual background in which the axiom is developed and argue that Fraenkel implicitly draws on Dedekind's approach to defining sets via closures in his conception of the intended effect of AR. Finally, a number of objections directed against AR by Baldus, von Neumann and Zermelo from the late 1920s will be discussed that eventually resulted in a fundamental shift in Fraenkel's justification of his axiom candidate.

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2. Axiom types. Feferman has recently proposed a consequential distinction between two different types of axioms used in mathematics, namely *structural axioms* of the “working mathematician“ and *foundational axioms* concerning structures that “underlie *all* mathematical concepts“ (Feferman 1999, 3). The distinction follows different functions: the axioms of the first group (e.g. the axioms of rings, groups etc.) are taken to be “*definitions of kinds of structures*“ that have a unifying role for being applicable in various mathematical fields and allowing to use similar argument patterns in different mathematical contexts. Foundational axioms (e.g. the Peano axioms for arithmetic and ZFC for set theory) in contrast are intended to capture structural properties of one specific class of entities (*the* natural numbers, *the* sets etc.) that allow to reduce all other mathematical branches to such a “secure basis“ immune to rational doubt. The main rationale behind this type of axioms is epistemic reduction (see ibid, 3).[2]

Now, according a common view the two types of axioms are generally associated with genuinely different styles of justification. Structural axioms are often considered to be introduced and justified on pragmatic grounds comparable to the experimental testing of hypotheses in the natural sciences. They are primarily assessed by their theoretical fruitfulness, i.e. with an eye on the intended consequences for the resulting theory. Foundational axioms in turn share an entirely different status. Justification of them has often been based on a reference to certain epistemic norms such as those of intuition, intuitiveness, obviousness, immediacy, naturalness etc.. Examples from the history of twentieth century mathematics that suggest such a view are numerous: Discussing the (epistemological) primacy of the Peano axioms over ZF Skolem argues that in contrast to the latter the former are “immediately clear, natural and not open to questions” (Skolem 1922, 299). In opposition to this, Gödel - in a well known passage in his (1964) describes a faculty of “intuition” as a sufficient “criterion of truth” for the set theoretic axioms.[3] Following a classification developed by Penelope Maddy, these instances of arguments can be clustered as *intrinsic*. The common guiding norm is that - in contrast to “*extrinsic* justifications” in which an axiom is evaluated “in terms of its consequences” - the motivation for accepting an axiom mainly depends on to the intuitive nature of the properties it expresses (see Maddy 1997, 36-37).

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[2] It is not questioned here whether this classification is generally valid or exclusive (compare Easwaran 2008), nor do I intend to provide a discussion of the involved concepts of mathematical unification and reduction (compare e.g. Hafner & Mancosu 2008).

[3] “But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don‘t see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception, which induces us to build up physical theories (…).” (Ibid., 271)

3. Extrinsic evidence. Nevertheless, the assumed correlation between the two different types of axioms and different types of arguments is not exclusive. As Maddy has highlighted in (1997) a number of genuinely extrinsic arguments can be identified in the history of the axiomatization of set theory. In these cases (most prominently in Zermelo's defense of the axiom of choice, see ibid, 56) the motivation for the acceptance of an axiom lies outside the literal meaning expressed in it but in its theoretical consequences.

Maddy refers to Russell 1907 as the first explicit methodological modeling of axiom choice along similar lines. In fact, in his lecture Russell proposes a "regressive method" for justifying axioms standardly labelled as foundational, i.e. the logical axioms of the *Principia Mathematica*, that does not depend on any direct intrinsic support as the appeal to intuition, but on a kind of inductive confirmation through its "obvious" consequences: „Hence we tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true." (Russell 1907, 273-274). Note the use of epistemic notions of "believing" "knowing", "truth", the "intrinsic obviousness" of consequences here. Now, Russell's theory of axiom choice is clearly extrinsic in in the sense above. In its epistemological character, however, it is comparable to Gödel's and Skolem's accounts of an epistemic foundationalism for arithmetic and set theory respectively. It differs from them only in inverting the "direction of epistemic support" (Easwaran) between the axioms and its consequences.[4]

3.1. Non-epistemic arguments. Maddy's survey of the history and practice of axiom choice in set theory suggests that a number of justifications for the axioms of ZF were developed that share the extrinsic structure with Russell's approach but have a closer affinity to the practical considerations underlying the choice strategies for structural axioms mentioned above. Moreover, they are explicitly non-epistemic in character. To understand these arguments is - according to Maddy – to conceive of an alternative picture of set theory as a foundational discipline that seems to be presupposed here. In contrast to a strong "foundationalist" reading of foundational axioms in terms of ontological reduction or reduction to an epistemologically secure basis, set theoretic axioms in this "modest" version of foundations share no „preferred epistemological status".[5] Instead they provide a fruitful codification of all other branches of mathematics by allowing a set theoretic "representation" of other mathematical entities and

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[4] Maddy seems to misinterpret Russell's theory as a non-epistemic model of axiom acceptance in saying that "(…) he explicitly renounces the epistemic goal of founding mathematics on something more certain than the statements of mid-level mathematics." (Ibid, 32). However, Russell in effect seems to propose a kind of modest "structural" foundationalism by introducing a new model of inferential evidence that strengthens the epistemic support of the "logical premises" of the theory. Compare Williams 2001, 82.

[5] For a comparable account of (higher-order) logic as a "nonfoundationalist foundation" for mathematics see Shapiro 1991.

structures (see Maddy 1997, 25-26). The acceptance of an axiom then primarily depends on its specific consequences for the overall success of the foundational discipline. This success can be captured by various criteria such as its theoretical fruitfulness or indispensability for mathematics, its unifying role, its explanatory power, its simplicity etc. In the specific case of set theoretic foundations especially unification (in Maddy's understanding of the concept) plays a central role:

> (…) vague structure are made more precise, old theorems are given new proofs and unified with other theorems that previously seemed quite distinct, similar hypotheses are traced at the basis of disparate mathematical fields, existence questions are given explicit meaning, unprovable conjectures can be identified, new hypotheses can settle old problems, and so on. (Ibid, 34-35)

This strong unifying role is due to the creation of a single and common universe of discourse, the universe of sets, to which all of mathematics is reducible:

> The force of set-theoretic foundations is to bring (surrogates for) all mathematical objects and (instantiations of) all mathematical structures into one arena - the universe of sets – which allows the relations and interactions between them to be clearly displayed and investigated. (Ibid, 26)

Two points of commentary are in order here. Note first that this view on foundational disciplines and the understanding of its axioms following from it stands in direct opposition to Feferman's functional distinction between structural and foundational axioms. It especially runs against his point that unification can be seen as an identification criterion for structural axioms as Maddy reasonably argues that the unification via set theoretic reduction plays a central role for the assessment of foundational axioms as well. One could follow from this that since both types of axioms seem to have certain unifying roles a principled distinction is not possible.[6]

Secondly, one central implication of this picture of set theoretic unification through a "unified arena" is that a specific conception of the domain of set theory, i.e. the universe of sets becomes a central issue. Maddy's discussion of argument types for the axioms of set theory is mainly historical for the axioms of ZFC and Gödel's axiom of construction. What is not mentioned in her survey, however, is that there has already been a strong and ongoing debate throughout in 1920s on how to conceive this universe of sets and characterize it axiomatically. In the course of different attempts to fix a domain of set theory that is capable

[6] Of course, Feferman's classification could be vindicated by highlighting the different nature of the two informal notions of unification used here. See footnote 2.

of providing such a "unified arena" for mathematics, one specific axiom candidate, namely Fraenkel's axiom of restriction stands out as the most persistent contribution. In the remaining part of the article I will focus on this specific historical episode in the early history of the axiomatic set theory in general and Fraenkel's axiom candidate in specific. It will be seen that in his elucidations concerning set theoretic restriction, one can identify a type of extrinsic argument not discussed in Maddy that is based on metatheoretic considerations, namely those concerning the categoricity of the axiomatization.

4. Fraenkel's axiom of restriction. In 1922 Abraham Fraenkel suggested two axioms to be added the axiom system presented in Zermelo (1908): the axiom of replacement, meanwhile a standard axiom of ZF, as well as the lesser known AR. The latter was basically devised to express a restriction clause, more specifically a minimality condition for any set model satisfying the Zermelo axioms.

Now, the axiom never recieved the status of an accepted axiom and is considered in retrospect as an "ad-hoc axiom" without any real, remaining significance in modern axiomatic set theory (see e.g. Kanamori 2004). Generally, the project of restricting the universe of sets to an intended model through such an axiom is commonly regarded as seriously flawed for several reasons, mainly due to its vague "metatheoretical" character. Nevertheless, Fraenkel's attempts to define such an axiom remains highly interesting from a historical point of view since the axioms takes a central and so far neglected place in a discussion about the (non-) categoricity of set theory. Beside this historical interest, its closer study is also instructive for the general methodology of axiom choice due to the specific justifications that Fraenkel provides for his axiom candidate. One can in fact identify several types of arguments in his works on the axiom, some of which can be classified as extrinsic, others as intrinsic in character. Before going into details, I will first give a brief reconstruction of the evolution of Fraenkel's thought on the notion of restriction in the 1920s.

The first mention of AR can be found in an article titled "Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre" in *Mathematische Annalen* from 1922 (1922a). Fraenkel's motivation for adding the axiom candidate is mainly pragmatic and concerns set theory as a foundational discipline. He states that "Zermelo's concept of set is more comprehensive than seems to be necessary for the needs of mathematics (…)." (Fraenkel 1922a, 223) He goes on to mention two types of possible sets in the "domain" ("*Grundbereich*") of set theory that are consistent with the existing axioms, however irrelevant for mathematical purposes. The first are „non-conceptual" sets consisting of physical elements. The second are non-wellfounded sets, i.e. sets with infinite membership chains as already specified by Mirimanoff (1917).

From their possibility within Zermelo's axiomatization, he draws an interesting consequence for the general status of Z:

Whereas sets of the first as of the second kind are not necessary for set theory considered as a mathematical discipline, it in any case follows from the fact that they have a place within Zermelo's axiomatization that the axiom system (..) does not have a "categorical character", that is to say it does not determine the totality of sets completely. (Ibid, 234)

Categoricity is understood here as a "complete fixation" of the domain of sets. A more structured presentation of his arguments for AR can be found in the second edition of his monograph *Einleitung in die Mengenlehre* (1924). Here, the introduction of the additional axiom leads to a "simplification of the set theoretic edifice" without losing its significance for mathematics due to the fact that "all mathematically relevant sets can (…) be saved with such a restricted axiomatization." (Fraenkel 1924, 218) As a independent argument – the metatheoretical property of categoricity is mentioned: "Moreover, without such a restriction it is not within reach that our axiom system captures the totality of admissible sets *completely* as it is desirable for the construction of every axiomatization." (Ibid, 218) Two points are relevant here. First, there are at least two related, however non-identical objections against Zermelo's original axiomatization: (a) the non-eliminability of extraordinary sets not necessary for the formalization of mathematics; (b) the non-categoricity of Z considered as a general theoretical deficiency of any axiomatization.[7] Secondly note that these two issues, i.e. the applicability of set theory to mathematics and the metatheoretical property of categoricity are treated independently here. One can find no remark about the possible implications of the categoricity of the extended axiom system for its foundational role in mathematics. We come back to this point in the last section.

4.1. The (non-)categoricity of set theory. In *Einleitung* (1924) we also find Fraenkel's first explicit definition of the notion of categoricity referred to in the argument above:

According to it an axiomatic system is called complete, if it determines uniquely the mathematical objects governed by it, including the basic relations between them, in such a way that between any two

[7] This is further highlighted in a passage in his lectures from 1925: "It means more than a mere flaw of our axiom system that the totality of all possible sets is not unequivocally fixed but that instead there are always narrower and more comprehensive interpretations of the concept of set that remain compatible with our axiom system." (Fraenkel 1927, 101)

interpretations of the basic concepts and relations one can effect a transition by means of a 1-1 and isomorphic correlation. (Ibid, quoted from Awodey & Reck 2002, 30)[8]

For the specific case of set theory the following informal explication is given:

If the axiom system is complete and one has chosen in two distinct ways, each in accord with the axioms, an interpretation of the concept of set – in particular also according to its extension – and of the basic relation a ∈ b, then it has to be possible to maintain a correlation between the sets of the one interpretation and those of the other such that first, to each set of the first interpretation corresponds one and only one (…) set of the other interpretation and vice versa and that secondly, if a ∈ b is a valid relation in the first interpretation (…) then the relation a' ∈ b' also holds for the sets a' and b' that have been assigned to a and b in the other interpretation and vice versa. (Ibid, 228)

This is probably the first application of the concept of categoricity via isomorphism to axiomatic set theory. Nevertheless, his presentation remains sketchy in one regard. The central concept used in these remarks about the conditions of categoricity for set theory is the notion of an isomorphic correlation between set models. In modern terminology such a correlation is commonly taken as a 1-1- mapping between two models that is structure-preserving. However, in the 1920s, Fraenkel did not provide a closer specification of the notion of isomorphism for set theory.[9] It is in the third edition of *Einleitung* (1928) that one can find an interesting remark concerning his understanding of the concept. Following a more general discussion of the categoricity of axiom systems, he adds in a footnote:

The expression "isomorphic" has a considerably more general sense than is usually common (…). In fact the isomorphism is applicable to arbitrary relations, not only to those tertiary and n-ary relations denoted as "operations." (Fraenkel 1928, 349)

Unfortunately, he does not get more explicit how such a generalized notion of isomorphism could be conceived.[10]

Irrespective of this, Fraenkel holds that Zermelo's axiomatization is non-categorical in the sense specified above. This has been a commonly acknowledged position in the 1920s shared

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[8] In the third edition of *Einleitung* the equivalence of this type of completeness with the notions of "categorical" (Veblen) and "monomorph" (Feigl-Carnap) is stated, see Fraenkel 1928, 349.

[9] An alternative notion of isomorphism for sets had already been introduced some years before Fraenkel's version in Mirimanoff (1917). His definition is based on the simple notion of equivalence between sets and does not take into account a correlation between set models (see Mirimanoff 1917, 41). For an early discussion of this definition see Sierpinski 1922.

[10] It was due to Rudolf Carnap who seems to have followed Fraenkel's informal remarks on a generalized concept of isomorphism to develop a formal definition of a "n-stage isomophism correlator" for a type-theoretic language in his works on a general methodology of axiomatics. See Carnap 2000 and Carnap & Bachmann 1936.

by such eminent figures such as Skolem, von Neumann and Zermelo himself. What has been subject of debate was what possible reasons were responsible for this fact and whether Zermelo's original axiomatization could be rendered categorical by adding additional axioms (see Shapiro 1991). As we have seen, according to Fraenkel view, the non-categoricity of Z is mainly due to the non-eliminability of "extraordinary sets" by the existing axioms. This in turn is due to the fact that the existential axioms, i.e. the empty set axiom and the axiom of infinity, do not restrict the domain of sets whereas the restrictive axioms as the axiom of separation are not restrictive enough to yield a "unequivocal specification" of the concept of set. As a solution to this Fraenkel proposes to introduce his AR which is described in analogy to Hilbert's completeness axioms in geometry:

(…) as is the case there, the mentioned deficiencies can be remedied by setting up a (…) last axiom, the "axiom of restriction" that imposes on the concept of set or more appropriate the domain [of sets] the *smallest comprehension compatible with the remaining axioms*. (Ibid, 234)

An alternative definition of the axiom can be found in Fraenkel 1924: „Aside from the sets imposed by the axioms [of Z] there exist no further sets." (Franekel 1924, 219) Now, the motivation for introducing this axiom is clearly extrinsic in Maddy's sense. The intention behind both versions is evident: to rule out non-intended and non-well founded sets by restricting either the interpretation of the concept of set or the domain of set and, by doing so, to render the axiom system categorical.

4.2. Versions of restriction. Fraenkel's elucidations of this intended effect of AR do not go beyond the level of informal remarks. The most detailed exposition can be found in the article "Axiomatische Begründung der transfiniten Kardinalzahlen" (1922b) in which Fraenkel develops an axiomatization for cardinal numbers. Here he formulates two versions of the axiom that also prove to be instructive for the case of standard set theory. According to the first, restriction is considered as a minimality condition on sets: There exist no sets apart from the sets implied by the given axioms. The second reading is more interesting. According to it AR can be viewed as imposing a minimal model for the axiom system: "If the domain (*Grundbereich*) B contains a smallest submodel (*Teilbereich*) B_0 satisfying the axioms (…), then B is identical with such a smallest submodel B_0." (Fraenkel 1922b, 163) This in effect rules out the existence of any possible submodel of B_0 that also satisfies the axiom system. The second definition is followed by a footnote concerning the method of constructing such a minimal model:

As is usual, a smallest submodel of the indicated character is to be understood as a model that is the intersection of all submodels of B with the property in question and that also possesses the property itself. (Ibid., 163)

Two claims are made here: first, a minimal model for Z can be conceived as the intersection of all possible models satisfying the axioms. Secondly, if such a minimal model exists, the extended axiom system Z*, i.e. the Zermelo axioms plus replacement and restriction, is categorical. Now, Fraenkel does not get more explicit about his conception of the domain or the models of set theory. How are these notions conceived? In approaching this question it will prove to be fruitful to take into consideration Fraenkel's immediate intellectual background. Specifically, a closer look at Richard Dedekind's methodological innovations concerning set formation and mapping in *Was sind and was sollen die Zahlen* from 1888 will be instructive for the understanding of how Fraenkel's ideas behind restriction evolved.

5. Dedekind's influence. Two interpretive issues concerning AR are in need of further consideration. First, how exactly did Fraenkel conceive the intended effect of his axiom on the possible set models satisfying Z? Secondly, how should it constitute the categoricity of the axiom system? My claim in this section is that on both questions Fraenkel was directly influenced by Dedekind's methodological work. More specifically I will argue that Dedekind's *theory of chains* ("*Kettentheorie*") introduced in 1888 was the driving force behind the development of AR.

Concerning the first question, we can find an insightful remark in *Einleitung* (1928) about the "special character" of the axiom compared to the "existential" and "relational axioms" of Z. Here AR is described as similar in effect to Peano's induction axiom. Fraenkel states that "in both versions [of AR], the inductive moment is essential." (p.355) What is his intuition about this "inductive character"? As we have already seen, the concept of intersection plays a central role for the intended effect of the axiom. It is supposed to impose a minimal model as the intersection of all possible models satisfying Z. From a methodological point of view, this is in effect a "pairing down" approach of defining a specific minimal structure by taking the intersection of all closed subsets of a given set. This method has first been introduced by Dedekind in (1888) and used for fixing the standard model of Peano arithmetic. One could therefore assume that Fraenkel's idea of a minimal model for set theory has been shaped in direct analogy to Dedekind's strategy of defining the standard sequence of natural numbers as a minimal set closed by induction. However, there is no immediate textual evidence that Fraenkel was guided by Dedekind's method in his thinking about restriction. My approach will be to give a short presentation of the central concepts developed in (1888) that seem of relevance for Fraenkel's axiom. In consequence I will present a number of arguments that strengthen the plausibility of my claim of this relation of influence.

5.1. The theory of chains. Dedekind's project of developing an "unambiguous foundational conception" of the natural numbers in 1888 is based on a number of well-known definitions and methodological results concerning the central concepts that allow the reduction of numbers to a logical basis.

Here also the idea of an isomorphism correlation based on a *1-1 mapping* ("*ähnliche Abbildung*") between elements of two *systems* is expressed formally for the first time. Systems that are isomorphic in this sense are terminologically fixed as "classes of similar systems" (see Dedekind 1888, 351).[11] A second newly introduced concept that will allow Dedekind to devise the sequence N of the natural numbers is that of a *chain* (relative to mapping function φ and a system S): in modern terminology, a subsystem B of S is called a *chain* if its is closed under a mapping φ (Ibid, 352). Subsequently, a system A_0 is defined as the *chain of A* ("*Kette des Systems A*") if and only if A_0 is the intersection of all chains containing A (Ibid., 353). The the way Dedekind conceives A_0 as the intersection of closures implies that it is also the smallest chain containing A, i.e. the smallest subset of S closed under φ. Again, in modern terminology, this effectively says that A_0 is the minimal closure of A under φ.[12] [13]

Dedekind's application of his theory of chains to the sequence of natural numbers cannot be discussed here. What is more important in this context is to highlight the obvious similarity between the idea of minimal chains developed here, i.e. the method of building minimal closures of a given base set and a specific operation via intersection and Fraenkel's remarks on AR. A number of additional arguments can be presented that strengthen the view that Fraenkel was immediately influenced by Dedekind's approach.

First, both positions are strikingly alike in their motivations for imposing a minimality condition on the intended model. In Fraenkel's case, as we have seen, the aim is to restrict the model to well-founded and abstract sets, thereby keeping out all types of non-standard and extraordinary sets. A comparable account can also be found in Dedekind's writings, most

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[11] Compare Sieg & Schlimm 2005 for a systematic presentation of the evolution of the concept of mapping in Dedekind's foundational work.

[12] Compare Sieg & Schlimm 2005 on this fact: "A_0 obviously contains A as a subset, is closed under the operation φ; and is minimal among the chains that contain A, i.e. if $A \subseteq K$ and $\varphi(K) \subseteq K$ then $A_0 \subseteq K$." (Ibid., 145)

[13] Dedekind himself is not explicit about the minimality property of *chains of A* in his (1888). There exists, however, as Sieg & Schlimm (2005) have pointed out, a note in Dedekind's earlier manuscript "Gedanken über Zahlen" from the Nachlaß in which this issue is explicitly mentioned: "(A) [i.e. the chain of A] is the "smallest" chain that contains the system A". (quoted from Sieg & Schlimm 2005, 144). I would like to thank Dirk Schlimm for drawing my attention to this passage.

explicitly in his famous letter to Kieferstein from 1890. After a short discussion of his basic concepts used for expressing N he states:

(…) however, these facts are still far from being adequate for completely characterizing the nature of the number sequence N. All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary additional elements t, to which the mapping φ could always be extended while remaining similar and satisfying φ (T) = T. (…) What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to N." (Dedekind 1890, 100)

To exclude such non-standard elements from the interpretation in question can thus be considered a common motivation behind the method of devising a minimal model. In Fraenkel's case this restriction is imposed by AR. In Dedekind's pre-axiomatic presentation of arithmetic of the natural numbers it is required by a condition equivalent to Peano's axioms.

This immediately leads to a second observation that presents additional evidence for my claim. It concerns Fraenkel's original conception of the model of ZF which seems to be directly modeled on this idea of closures. In Dedekind's account of the natural numbers 1 is the base element and the sequence N the intersection of all sets containing 1 and closed under the successor operation. Accordingly, Fraenkel's intended set model is understood as the intersection of all set models that share the properties of (a) containing the empty set and the infinite set Z and (b) being closed under the operations specified in the Zermelo axioms, i.e. pairing, union, power set etc.. This is essentially an understanding of models as "algebraic closures" (see Kanamori 2004, 515). One can find textual evidence for this conception in his work, mainly in the context of building different restricted models – e.g. as sets closed under the operations of power set or union - used for independence proofs (see e.g. Fraenkel 1922a, 233, also Fraenkel 1922b, 165-171). Here, as well as in *Einleitung* (1928) in the course of the discussion of AR he gives an informal sketch of the standard model ("*Normalbereich*") of Z as a system closed under all operations specified in the axioms. Adding the axiom to Z would impose the following effect:

This will probably result in the fact that only the empty set functioning as the primary building block for all sets is set up as the initial point. Then only those sets are admissible which emerge from the empty set and the sets imposed by [the axiom of infinity] by an arbitrary but certainly finite application of the individuals axioms. (Fraenkel 1928, 355)

Even though Dedekind's notion of chains is not mentioned in Fraenkel's remarks on model building, it seems obvious that AR can be understood here as a "restriction clause for closures" (Kanamori 2004, 515), i.e. for a universe of sets conceived in direct analogy to Dedekind's method of constructing minimal systems.

5.2. Categoricity results. Now, as I have mentioned before, there is no direct indication in Fraenkel's writings of Dedekind's influence on his conceptualization of models and AR. In first edition of *Einleitung* (1919) Dedekind is mentioned only for his existence proof of infinite systems and his definition of a finite system given in 1888. In the concluding remarks of the second edition there is a reference to his theory of chains that, as Fraenkel writes, has received a "general and fundamental significance in set theory." (Fraenkel 1924, 244)[14] However, no connection is made to his concept of restriction. There exists, however, a passage in his lectures from 1925 that allows to draw a direct link between Dedekind's minimal closures and his own approach of devising a minimal model for set theory. In a section on the "non-predicative" methods in mathematics, more specifically the debate between Poincaré and Zermelo on the indispensability of non-predicative proofs in mathematics, there is an interesting footnote mentioning Dedekind's theory:

> In a series of important and thoughtful proofs in set theory especially due to Dedekind and Zermelo (…), deductions of the following kind are taking a center stage: a set M is considered whose elements are all sets of a specific property E exclusively characteristic for it; M is thus the set of all sets sharing the property E. For the cases in question it is then shown that the sum s and the intersection d respectively of all elements of M themselves have the property E; therefore s and d respectively – which exist by virtue of the definition as *sum* and *intersection* respectively - also belong to the set M and can be characterized as the set the *most comprehensive* and the *most limited in size* respectively sharing the property E. Due to this characterization s and d respectively play a decisive role in the concerned proof. (Fraenkel 1927, 29, notation slightly changed)

The approach described here essentially follows the proof strategy introduced by Dedekind in 1888 to prove the categoricity of Paeno arithmetic. And it is precisely this idea – here formulated in Fraenkel's own words - that also most likely lies behind Fraenkel's own understanding of AR. To interpret his tacit assumptions underlying restriction in this way also sheds further light on the second issue mentioned above, namely how to understand the claim that the addition of AR to Z would render the resulting axiomatization categorical. Fraenkel's remarks alone are not conclusive on this intended effect. Here a glance at Dedekind's

[14] For the set theoretic application of the theory of chains he refers to Hessenberg 1909 and to Zermelo 1908.

categoricity proofs will be instructive to show how Fraenkel might have conceived a similar categoricity result for set theory.

Dedekind's well-known metatheoretical results (see ibid, §10) can be qualified as instances of a "categoricity based on minimal models" according to which a theory is categorical if and if only it has a minimal model and any two minimal models are isomorphic (see Grzegorczyk 1962, 63).[15] His proofs (in remarks 132 and 133) that the simple infinite system N can be captured *completely*, i.e. up to isomorphism by the conditions equivalent to the Peano axioms, strongly depends on his idea of minimal chains (see ibid, 376-377). This connection follows from Dedekind's definition of a *mapping of a number sequence through induction* used in his proofs. In remark 126 he shows that there is one and only one mapping of N into any system Ω via a function ψ that satisfies the conditions that (i) the closure of N is a subset of Ω, that (ii) $\psi(1) = \omega$, where ω is an element of Ω and that (iii) for any number n, $\psi(n') = \theta\psi(n)$, where "n'" stands for the successor of "n" and θ is a function on Ω (see ibid, 370-371). In remark 128 Dedekind then proves that there exists an equivalence between such an inductive mapping $\psi(N)$ and a minimal closure $\theta_0(\omega)$ of Ω containing ω, i.e. $\psi(N) = \theta_0(\omega)$ (Ibid, 372). It is in fact here, i.e. in the proof of this equivalence the following categoricity proofs centrally depend on that the central link between minimal chains and the mapping of a simple infinite system via induction becomes evident.

Now, unlike Dedekind, Fraenkel did not develop an actual proof of the categoricity of Z* nor does he make any remarks how such a proof based on AR might be designed. Besides this fact, the presentation of the central concepts of his theory, most importantly those of restriction, the set universe and minimal models is not comparable in technical rigor to Dedekind's foundational work in arithmetic. Nevertheless, given the textual evidence above as well as his various informal remarks on the effect of the axiom candidate as imposing a minimal model, on its "inductive character" as well as on his conception of the intended set model as a minimal closure, it seems a strongly plausible interpretation that AR was conceptualized by Fraenkel in close analogy to Dedekind's method developed in his (1888).

6. Objections to AR. A number of serious objections have been raised against Fraenkel's axiom candidate shortly after its first presentation in print that have led to general scepticism concerning the validity of AR as a set theoretical axiom and have prevented it from being added to the canonical list of ZF. Eventually the criticism has also provoked Fraenkel to substantially modify his own justifications given for the axiom. In what follows I will first

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[15] A minimal model can be defined as a model M satisfying a theory T such that for every submodel N of M that also satisfies T, N is isomorphic to M. See ibid., 63.

briefly present the main arguments adduced against Fraenkel's axiom candidate. In the final section his reaction to the objections and its impact on his own conception of restriction will be discussed.

One point of critique was first forwarded by the German mathematician Richard Baldus in the course of a discussion of Hilbert's completeness axiom in geometry (Baldus 1928). It concerns the metatheoretical character of Hilbert's and related axioms, later terminologically clustered as "extremality axioms" (Carnap & Bachmann 1936). Unlike the other axioms in Hilbert's axiomatization (e.g. that of order), the completeness axiom makes an assertion "*not only over the thought things* [of an interpretation] *but actually over all conceivable things*" (Baldus 1928, 331). This assumption of the *non-extensibility* ("*Nicht-Erweitungsfähigkeit*") of the basic elements involves a quantification over the individuals in all models. Baldus correctly indicates methodological doubts about the validity of such quantification over models:

> In order to reserve the completeness axiom its status as an axiom, one would have to allow within the axioms also assertions over other things than those thought in the respective interpretation of the axiom system, which would extend the concept of axioms in geometry in a precarious and superfluous way. (Ibid, 331)

In an attached footnote, Baldus also mentions Fraenkel's AR in this respect expressing a direct critique of it based on similar grounds:

> In a meeting in Kissingen Mr. A. Fraenkel has suggested that set theory can in no other way be rendered monomorph than by a "postulate" (…), namely by an axiom of restriction, against which similar objections can be raised as against the axiom of completeness. (Ibid, 331)

Baldus's criticism of the problematic (meta-)semantic character of the axiom has meanwhile become a standard argument against Fraenkel's axiom candidate. It basically objects that the axioms of restriction imposes no condition on sets as the individuals of set theory, but on set models (compare Ferreirós 2007).

An objection of a different nature raised specifically against Fraenkel's axiom is found in von Neumann 1925. Here an alternative axiomatization of set theory is developed based on the notions of function and argument. Von Neumann also presents in his notation a formalized version of the AR conceived to capture Fraenkel's original intention of imposing a minimal model for the theory. In von Neumann's terminology a *subsystem* of a given system is minimal if and only if it contains no subsystem that also satisfies the axioms (von Neumann

1925, 404). He then presents two "serious objections" against the axiom. According to (a) it presupposes notions of "naive set theory", most importantly that of a submodel that is not definable precisely in his own theory of sets.[16] The resulting regression to informal set theory would make the whole process of axiomatizing set theory circular (Ibid, 404). A possible remedy for this is to assume a "higher set theory" and a corresponding expanded domain P, in which the original domain Σ can be properly defined as a class of P and the subsystems Σ' of Σ as subclasses of P respectively. However, this additional "hypothesis" implies an even graver difficulty for expressing restriction clauses for his axiomatization. von Neumann argues that Fraenkel's proposed method of devising a minimal model via the intersection of all possible models need not *necessarily* lead to a model satisfying the other axioms (Ibid, 405). His argumentation cannot be discussed here since it is closely related to the distinction between sets and classes and the satisfaction conditions he devises for subsystems of a given system to satisfy his set theoretic axioms, both of which is not subject of this article.[17] What is relevant is the strong conclusion he draws from this for Fraenkel's axiom candidate:

> For these reasons we believe that we must conclude, first, that the axiom of restriction absolutely has to be rejected and, second, one cannot possibly succeed in formulating an axiom to the same effect. (Ibid, 405)

According to him this fact, together with the existence of „inaccessible sets" such as "descending sequences of sets" that lie "outside the system" in question are the main sources of the non-categoricity of set theory.

6.1. Zermelo on set models. A third and somehow the most serious objection against Fraenkel's axiom candidate has been expressed in Zermelo's paradigmatic paper "On boundary numbers and set-domains" (Zermelo 1930). Zermelo here introduces a new conception of the set universe as an open and unbounded sequence of connected set models ("*Normalbereiche*") of increasing size that all satisfy the standard axioms of ZF (including a newly introduced axiom of foundation). A comparable view of a sequence of larger and larger set models has already been presented but not further developed in von Neumann 1925. Unlike von Neumann, Zermelo, in giving a formal explication of a cumulative hierarchy of

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[16] For his distinction between *sets* and *classes* that plays a central role in his argumentation see ibid, 403.

[17] Very briefly, von Neumann shows that unlike in the original system Σ the boundaries between *arguments* and *functions* are not clearly separated for the subsystem Σ'. Therefore, in order to satisfy the axiom system Σ' has to satisfy certain additional conditions that relativize the range of these concepts as well as the quantifiers ranging over them to Σ'. He argues that these satisfaction conditions are sufficient but don't necessarily imply the existence of a minimal model to satisfy the axioms (see ibid, 405-408).

sets, provides a theoretical model for this view. He also proposes a definitive clarification of the semantic notions of set models, submodels etc. that can be found both in Fraenkel and von Neumann. According to Zermelo, each set is decomposable into *layers* and cumulative *sections* that include all sets formed at earlier layers in the set theoretic hierarchy (Ibid, 32-33).[18] Set models in turn are treated "exactly like sets" that can be specified by two numbers, a *base* – i.e. the cardinality of its base set of individuals – and a *characteristic* or *boundary number*, i.e. the least ordinal greater than all ordinals contained in the model. From this it follows that each model can act as a submodel of a set model with a higher *boundary number* (see ibid, 31).

Now, it is evident that this theory of the set theoretic universe as an "unlimited sequence of well-distinguished models" differs substantially from Fraenkel's static conception of a closed and fully describable universe of sets. This divergence also results in an opposing view on the issue of the (non-)categoricity of set theory. Whereas Zermelo admits that ZF captures set models of a given "base" and "boundary number" up to isomorphism –the main results of his article in fact are a number of relative categoricity theorems and proofs (see ibid, 40-41) - categoricity in the standard meaning of capturing a unique model is not possible due to the boundlessness of the set theoretic universe, i.e. the "existence of a unlimited sequence of boundary numbers". It follows from this that talk about the one intended model captured by ZF is inadequate.

This insight also underlies his general critique of restrictive axioms. We have seen that von Neumann holds the assumption that for set theory there always exists a larger domain, a "higher set theory" in which the original model is definable as a set (in his terminology as a *class*) and in which a restriction for the lower theory yielding categoricity could at least in principle be formulated. Zermelo's theory of relative or "quasi"-categoricity essentially conforms to this view. Nevertheless, for him a domain restriction will never be a desirable from a practical point of view, because it decisively delimits the functional role of set theory as a foundational discipline:

> Our axiom system is non-categorical, which in this case is not a disadvantage but rather an advantage, for on this very fact rests the enormous importance and unlimited applicability of set theory. (Ibid, 1232)

Here lies the central objection to Fraenkel's account of restriction. Its effect is not considered a theoretical virtue of the axiomatization, but in contrary as a deficiency in a practical sense: it

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[18] For the technical details of this early version of an iterative conception of sets see e.g. Kanamori 2004 and Ferreiros 2007.

restricts set theory in its task of formalizing mathematics. Zermelo explicitly refers to Fraenkel's axiom candidate in order to underline the difference between their conceptions. He remarks that,

> Naturally one can always force categoricity artificially by the addition of further ‚axioms', but always at the cost of generality. Such postulates, like those proposed by Fraenkel (…) do not concern set theory as such, but rather only characterize a quite special model chosen by the author concerned. (…) the applicability of set theory has to be given up. (Ibid, 1232)

Note that the actual argument against restrictive axioms like that of Fraenkel mainly concerns the fruitfulness of set theory as foundational discipline. Any deliberate restriction of the set universe negatively affects the "full generality" of set theory, i.e. its "unlimited applicability" to mathematics. Recall again Maddy's account of unification as a central motive behind set theoretic axiomatization presented above. We come to see that Zermelo, in his motivation for a cumulative set universe seems to be much more attentive to this pragmatic ideal than is Fraenkel in his call for a categorical axiomatization. As has been mentioned, Fraenkel does not provide a reflection on the possible practical consequences of the adoption of restriction to ZF for its success as a foundational discipline. It was left for Zermelo to highlight the fact that an unrestricted universe of sets allows the strongest set-theoretic unification in mathematics.

7. Fraenkel's reaction. Fraenkel's reaction to the presented objections against his AR in subsequent work is in several ways instructive. For one part it better illustrates his own tacit understanding of the concepts involved in his earlier presentation of restriction. For the other part it highlights substantial shifts in his justification of the axiom as a direct result of this critique.

As far as I know Fraenkel never responded in print to Baldus' legitimate doubts about the metatheoretical character of extremality axioms and their semantic implications. Even though he acknowledged the "special character" of the axiom in comparison to the other axioms of Z he never seemed to become aware of the problem that the axiom requires quantification over set models.[19] By contrast, he immediately reacted to the objections levelled against the axiom by von Neumann. This might seem surprising at first sight because it is far from obvious that the latter's critique actually meets Fraenkel's informal presentation of the restriction on set models. First, it seems more reasonable that the technical objections against the AR rather concern von Neumann's own non-standard axiomatization of set theory (and specifically his

[19] As Shapiro points out this fact is probably due to the circumstance that a clearly delineated syntax/semantic-distinction was for from being standard by the time Fraenkel developed his theory. See Shapiro 1991, 184.

formalization of the axiom) than Fraenkel's preliminary ideas. Secondly, the general validity of his critical remarks against restriction can be challenged by drawing to a number of inconsistencies in his own treatment of set models. As Shapiro (1991) has shown, von Neumann set theory does not allow a consistent presentation of models (as classes containing its subclasses) due to the fact that proper classes cannot be conceived as elements of either sets or classes in his theory (see ibid, 186).[20]

Nevertheless, Fraenkel *in grosso modo* seems to have acknowledged von Neumann's critique. In 1927 for example he considers it "very doubtful" whether his version of restriction can be attributed "a sound meaning". He states that,

> One seriously has to take the eventuality into consideration that the possible realisations of the axiom system that differ in their size do not have a smallest common subpart that would also satisfy all the axioms. Also the previously given instruction for a "construction" of such a smallest model (…) need not lead to a definite result for the axioms IV – VI [i.e. the axioms of power set, separation and choice] themselves do have a purely constructive character. This is a serious and so far not satisfactorily solved problem from which possibly the natural necessity of a certain "boundlessness" and also a certain vagueness (so to speak at the boundaries) of the yet legitimate concept of set will follow. (Fraenkel 1927, 102)

The first remark here essentially rephrases von Neumann's critique. The second remark concerning the "boundlessness" of the set concept seems already - i.e. five years before Zermelo's (1930) - to indicate doubts about his conception of the set universe as an (algebraic) closure.[21]

Now, despite the acknowledged criticism, Fraenkel remains optimistic about the practical usefulness and general correctness of a restrictive axiom.[22] However, his subsequent presentations of the axiom candidate are marked by a number of substantial modifications. By 1958, in his *Foundations of Set Theory*, the extrinsic justification of the axiom changes substantially. Whereas the well-foundedness of the sets composing the set theoretic universe is now imposed by the axiom of foundation (as already proposed in von Neumann 1925 and

[20] Shapiro suggest a modification of this system in order to vindicate the axiom of restriction, to the effect that "sets *and proper classes* of the original theory (can be treated as) as *elements*, i.e. as sets" thereby allowing to treat models for a theory T as the subclasses of a higher-level theory T'. The effect on restriction would then be that "(…) one can state in the higher theory that a given class has no proper subclasses that are models of ordinary set theory." (Ibid, 186)

[21] Similar remarks along these lines can be found in Fraenkel (1928).

[22] In Fraenkel 1928 he concludes his discussion of the axiom by stating that: "Nevertheless I like to believe that the mentioned doubts can be resolved and that the axiom of restriction can be maintained - and then considered as a very central part of the axiomatization! – if only its formulation can be made more precise." (Ibid. 355) According to Fraenkel an adequate formal presentation of the axiom is later developed in Carnap & Bachmann 1936, see Fraenkel et al. 1958, 90.

Zermelo 1930), the main motivation for introducing the axiom of restriction is to secure the "non-existence of inaccessible numbers" (Fraenkel et al.1958, 88). The former version of AR is thus functionally divided into an independent axiom of foundation and an accessibility axiom introduced to exclude all inaccessible numbers. This change has to be interpreted as a direct reaction to the theory of set models proposed in Zermelo 1930. According to Fraenkel, the addition of this new "limitative axiom" to ZF "should enable us to prove that all models of the axiom system are isomorphic" i.e. to render the resulting axiom system categorical in the standard, absolute sense of the word (Ibid, 88).

What follows from this is that Fraenkel was still holding to his original core idea of a single and closed "intended domain of sets" in (1958) and obviously did not approve of Zermelo's conception of an ever-expanding domain composed of different set models. In a remark he critically comments on this conception of a set universe and the categoricity proofs based on it given in Zermelo (1930):

The cardinal of the basis and the ordinal α together are an invariant characteristic of the intended domain of sets. The first leads to the domain of finite sets, the second to the domain of sets up to the first inaccessible number. However, Zermelo's proof that this invariant guarantees the monomorphism (categoricalness) of the domain can hardly be considered stringent, and even the concepts used, e.g. „cardinal of the basis" are objectionable. (Ibid, 92).

Neither did he accept the latter's objections against his axiom candidate. On the contrary, restriction is thus redefined with the specific motivation to rule out the existence of boundary numbers. This version of the axiom is essentially maintained in the second and revised edition of *Foundations* (Fraenkel et al. 1973)[23]. Here, however, Zermelo's argument against AR is commented on in a more refined and almost neutral way:

The axiom of restriction points to the existence of some fixed natural universe of sets, but if the collection of all sets in this universe is again a Platonistic entity, then why should it not be admitted as a new set by allowing a wider universe than that allowed by the axiom of restriction. (Ibid, 118)

The point to be emphasised here is that Fraenkel is in fact conceding the legitimacy of an "ever-growing universe" that follows once the cumulative hierarchy of sets has been accepted.

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23 Fraenkel here defines a "first axiom of restriction" as "the conjunction" of an axiom of foundation and an accessibility axiom excluding inaccessible (cardinal) numbers. A second, "stronger" version of restriction is effectively conceived as the conjunction of an accessibility axiom and Gödel's axiom of constructability. See ibid, 115-116.

7.1. An "intuitive justification". This is related to my second and final remark. In Fraenkel et al. 1973 the given arguments for the axioms of restriction are complemented by a new type of justification, not comparable to the extrinsic justifications given by reference to non-well-founded sets or inaccessible numbers. This new justification strategy is clearly intrinsic in character in Maddy's sense: not in terms of an allusion to a set-theoretic intuition or self-evidence, but explicitly based on a prior explanation of the set universe underlying the axiomatization. The axiom in question is then justified on this ground for it captures certain structural properties of this conception (compare Maddy 1997, 37). In the second edition of *Foundations*, Fraenkel seems to have adopted Zermelo's cumulative hierarchy of sets as the underlying model for ZF. After an informal presentation of it, he makes the following remarks concerning a necessary "intuitive justification" of the axioms of restriction:

> In the case of the axiom of induction in arithmetic and the axiom of completeness in geometry, we adopt these axioms not only because they make the axiom systems categorical or because of some metamathematical properties of these axioms, but because, once these axioms are added, we obtain axiomatic systems which perfectly fit our intuitive ideas about arithmetic and geometry. In analogy, we shall have to judge the axioms of restriction in set theory on the basis of the set theory obtained after adding these axioms fits our intuitive ideas about sets. (Ibid, 117)

This passage is remarkable in the sense that it documents the substantial shift in Fraenkel's own position on the methodology of axiom choice. What is effectively said here is the justification of an axiom – here the different versions of AR – has to depend on a combination of extrinsic and intrinsic considerations. The latter are necessary to confirm (so to speak in a second loop) the choice originally made on extrinsic grounds. An acceptance solely based on extrinsic considerations – such as on "metamathematical properties" of the resulting axiomatization or simply on the "sake of economy" – do not provide sufficient evidence unless backed up by an intrinsic "absolute faith" in the sufficiency of the axioms (see ibid, 117). By 1973, Fraenkel eventually seems to have adopted the view that the cumulative hierarchy of sets can act as one such possible source of evidence.

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