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AN AXIOMATIC TREATMENT OF NON-MONOTONIC ARGUMENTS

0. Introduction

An axiomatic theory of non-monotonic consequence relations patterned upon some finitistic ideas going back to Gentzen was suggested by Gabbay [1985].[1] More recently, an infinitistic approach patterned upon Tarski's theory of consequence operation was examined by Makinson [l98.]. We compare the two approaches and examine them vis-à-vis some intuitive adequacy conditions. An enlarged version of this note will appear in Studia Logica (Wójcicki [198.]), in particular the reader is referred to it for the proofs of the results stated here.

1. Consequence operation

Throughout this note we are going to deal with an arbitrary but fixed language. Greek letters $\alpha, \beta, \ldots$ will represent formulas and capitals $X, Y, \ldots$ will represent sets of formulas of it.

By a *consequence operation* we shall mean a unary operation C on sets of formulas which is both *inclusive* and *restrictive* i.e. satisfies the following two conditions:

(T1) $X \subseteq C(X)$
(T2) If $Y \subseteq C(X)$, then $C(X \cup Y) \subseteq C(X)$.

[1]Actually Gabbay applied the term *inference relation*. The terminology we apply in this note often do not coincide with that from the papers to which we refer.

One arrives at the notion of a consequence operation in the sense of Tarski by supplementing (T1) and (T2) with

(F) $\alpha \in C(X)$ iff for a finite $X' \subseteq X, \alpha \in C(X')$,

i.e. postulating C to be *finitary*. Note that each finitary consequence operation is *monotonic*, i.e. satisfies

(T3) $C(X) \subseteq C(X \cup Y)$.

The notion of a consequence operation most commonly applied in the literature coincides with that of a monotonic consequence operation in the sense defined above, i.e. one which satisfies (T1)-(T3).

As has been observed by Makinson [198.] if the monotonicity requirement is given up then it is still natural to expect the consequence operation to be *cumulative*, i.e. to satisfy the converse of (T2):

(M) If $Y \subseteq C(X)$, then $C(X) \subseteq C(X \cup Y)$.

We are going to show that the notion of a cumulative consequence operation provides an adequate formalization of some intuitive idea of non-monotonic argument. But before we undertake this question, we shall introduce the notion of a consequence relation meant to be a finitistic (defined for finite sets of formulas only) counterpart of the infinitistic notion of a consequence operation and examine the two notions vis-à-vis each other.

2. Consequence relations

By a consequence relation we shall mean any relation $\vdash$ that holds between finite sets of formulae and formulas and satisfies the following two conditions:

(G1) $X, \alpha \vdash \alpha$,
(G2) If $X \vdash \alpha$ and $X, \alpha \vdash \beta$, then $X \vdash \beta$.

If, moreover, $\vdash$ satisfies

(G3) $X \vdash \alpha$ implies $X, \beta \vdash \alpha$,

the consequence relation $\vdash$ will be said to be *monotonic*.

The three conditions stated above correspond to three 'structural rules of the Natural Deduction' defined by Gentzen and usually referred to as *Reflexivity, Cut* and *Enlargement.* Finally, consider the following condition (Gabbay [1985]):

(G) If $X \vdash \beta$ and $X \vdash \alpha$, then $X, \beta \vdash \alpha$,

A consequence relation which satisfies it will be referred to as *cumulative.*

3. Consequence relations vs. consequence operations

Let $\vdash$ be a consequence relation. We shall say that a consequence operation C is an *expansion* of $\vdash$ iff for all finite sets X of formulas and all formulas α,

$(*)$ $X \vdash \alpha$ iff $\alpha \in C(X)$.

Now given two consequence operations C, C' define C to be *weaker* than C', in symbols $C \leq C'$ iff for all $X, C(X) \subseteq C(X')$. Curiously enough, the following is satisfied:

Observation 1. *The set of all consequence operations is a complete lattice with $\leq$ being the lattice ordering. Moreover, both the set of all monotonic and that of all cumulative consequence operations form complete sublattices of that lattice.*

The following is straightforward

Observation 2. *For each consequence relation $\vdash$ the consequence operation C determined by $(*)$ and*

(Fc) $C(X) = X$ *for all infinite* X

is the weakest of all expansions of $\vdash$.

Call a consequence operation C which satisfies (Fc) *finitistic.* One easily verifies that $(*)$ establishes a one-to-one correspondence between consequence relations and finitistic consequence operations and hence the theory of consequence relations and that of finitistic consequence operations can be viewed as inessential variants of each other.

There are many alternative ways to reduce the theory of consequence relations to a subtheory of consequence operations. One obvious reason for not to be satisfied with finitistic expansions of consequence relation is that they behave on infinite set in a trivial manner. Another one is that they do not preserve monotonicity; a finitistic expansion of a monotonic consequence relation need not be monotonic.

Inductive expansions, i.e. ones that satisfy

(I) $\alpha \in C(X)$ iff for a finite $X' \subseteq X, \alpha \in C(X' \cup Y)$ for all $Y \subseteq X$.

seem to be of considerable interest.

OBSERVATION 3. *For each consequence relation $\vdash$ there is exactly one inductive expansion of it. Moreover, if $\vdash$ is cumulative (monotonic), then its inductive expansion is cumulative (monotonic) too.*

Note that all finitistic consequences are finitary, all finitary are inductive and neither subsumption can be reversed. On the other hand, note that C is finitary iff it is both monotonic and inductive, thus for monotonic consequences finitariness and inductiveness coincide.

4. Rules of inference

We define an *inference* to be a figure of the form $(X/\alpha)^+$ and we define a *constraint* to be a figure of the form $(X/\alpha)^-$. Now, by a *heuristic rule of inference* we shall mean a set $\mathbf{R}$ of both inferences and constraints such that the following is satisfied:

$$\text{If } (Z/\alpha)^- \in \mathbf{R}, \text{ then } (X/\alpha)^+ \in \mathbf{R} \text{ for some } X \subseteq Z.$$

From an intuitive standpoint if both $(X/\alpha)^+$ and $(X \cup Y/\alpha)^-$ are in $\mathbf{R}$ then an *instance* of the rule $\mathbf{R}$ is the instruction:

$$From\ X\ infer\ \alpha\ unless\ you\ know\ Y.$$

Now if $(X/\alpha)^+ \in \mathbf{R}$ but for no $Y, (X \cup Y/\alpha)^-$ is in $\mathbf{R}$, the instance of $\mathbf{R}$ reduces to $From\ X\ infer\ \alpha$.

A heuristic rule of inference is *unconstrained* iff it does not involve any constraints. Note that rules of inference in the ordinary sense of the word can be viewed as unconstrained heuristic rules of inference.

Finally, a heuristic rule of inference is *finitary* iff all sets of formulas it involves either in inferences or in constraints are finite.

5. Proofs

We shall write $\xi, \zeta, \eta, \ldots$ for ordinal numbers.

Let Θ be a set of heuristic rules. Given any sequence $\pi = \langle \alpha_\xi : \xi \leq \eta \rangle$ of formulas we shall say that:

(i) π is a *prima facie infinististic proof* of α from X by Θ iff $\alpha = \alpha_\eta$ and for each α_ξ, either $\alpha_\xi \in X$ or there is $Y \subseteq \{\alpha_\zeta : \zeta < \xi\}$ such that $(Y/\alpha_\xi)^+ \in \bigcup \Theta$ and for no $Z \subseteq X \cup \{\alpha_\zeta : \zeta < \xi\}, (Y \cup Z/\alpha)^- \in \bigcup \Theta$. If moreover, both η and X are finite and rules in Θ are finitary then π will be said to be a *prima facie proof* of α from X by Θ.

(ii) A prima facie (infinitistic) proof π of α from X by Θ is a (*infinitistic*) *proof* iff given any set Y of formulas which have a prima facie (infinitistic) proof from X by Θ, π is a prima facie (infinitistic) proof of α from $X \cup Y$ by Θ.

6. Characterization Theorems

There are two ways in which one may argue that inclusiveness, restrictiveness and monotonicity characterize the notion of consequence operation in an adequate manner. First, appealing to some familiar semantic ideas, one may prove that C is a monotonic consequence operation if and only if there is a set H of truth-value assignments such that for all α and all $X, \alpha \in C(X)$ iff under all $h \in H, \alpha$ is true whenever all β in X are true. This line of argument amounts to showing that the notion of a consequence operation coincides with that of *entailment*. The alternative justification of axioms for monotonic consequence operations consists in showing that for each monotonic consequence operation C there is a set Θ of rules of inference (each rule of inference being a set of instructions of the form $From\ X\ infer\ \alpha$) such that for each $X, C(X)$ is the least set of formulas which includes X and is 'closed' under all rules in Θ (see the First Characterization Theorem below). Thus each monotonic consequence operation can be viewed as 'derivability' or 'provability' operation.

While semantic analysis of non-monotonic arguments is fairly well developed no attempt to analyze them in terms of derivability has been undertaken. The two characterization theorems we state below are meant to be some preliminary contributions to the issue.

First Characterization Theorem. *C is a cumulative (monotonic) consequence operation iff there is a set Θ of (unconstrained) heuristic rules such that for all α and all $X, \alpha \in C(X)$ iff there is an infinitistic proof of α from X by Θ.*

Second Characterization Theorem. *$\vdash$ is a finitary cumulative (monotonic) consequence relation iff there is a set Θ of finitary (unconstrained) heuristic rules such that for all α and all $X, X \vdash \alpha$ iff there is a proof α from X by Θ.*

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References

[1985] Dov Gabbay, *Theoretical foundations for non-monotonic reasoning in expert systems*, [in:] K. R. Apt, ed., **Logics and Models of Concurrent Systems**, Berlin, Springler-Verlag.

[198.] David Makinson, *General Theory of Cumulative Inference*, **Studia Logica**, to appear.

[198.] Ryszard Wójcicki, *Heuristic Rules of Inference in non-monotonic arguments*, **Studia Logica**, to appear.