The complex numbers and complex exponentiation Why Infinitary Logic is necessary!

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In this article we discuss some of the uses of model theory to investigate the structure of the field of complex numbers with exponentiation and associated algebraic groups. After a sketch of some background material on the use of first order model theory in algebra, we describe the inadequacy of the first order framework for studying complex exponentiation. Then, we discuss the Zilber's program for understanding complex exponentiation using infinitary logic and the essential role of understanding models in cardinality greater than $\aleph_1$. This analysis has inspired a number of algebraic results; we summarize some of them. We close by discussing some consequences on 'semiabelian varieties' of the work on the model theory of uncountable models in infinitary logic. We place in context seminal works of Shelah [She75, She83a, She83b] and Zilber [Zil05, Zil00, Zil04, Zil03] . Shelah's work was directed at understanding model theoretic phenomena–generalizing to infinitary logic the techniques and results that were proving so successful in the first order context. Zilber's later work was motivated by the attempt to understand complex exponentiation. But he rediscovered some aspects of Shelah's work and ultimately drew on some other parts of it. The earlier works in Zilber's program use model theory to formulate problems concerning complex exponentiation; this motivates work in complex analysis, algebraic geometry and number theory. But in [Zil03] the interaction between core mathematics and model theory goes both ways; the deep work of Shelah is exploited to obtain an equivalence between categoricity conditions and non-trivial arithmetic properties (in the sense of a number theorist) of certain algebraic groups.

1 Logic and Mathematics

We begin with a discussion of the relation between logic and 'core mathematics'. The logician behaves as a 'self-conscious' mathematician. That is, by being careful of the formal language in which mathematical statements are made, one is able to find and justify generalizations that would not otherwise be available. We will give several examples of this phenomena. To set the stage we need some definitions.

Definition 1

A signature L is a collection of relation and function symbols.

A structure for that signature (L-structure) is a set with an interpretation for each of those symbols.

The first order language *($L_{\omega,\omega}$) associated with L is the least set of formulas containing the atomic L-formulas and closed under* finite *Boolean operations and quantification over finitely many individuals.*

The infinitary*language ($L_{\omega_1,\omega}$) associated with L is the least set of formulas containing the atomic L-formulas and closed under* countable *Boolean operations and quantification over finitely many individuals.*

The generalized quantifier*language ($L_{\omega_1,\omega}$) associated with L is the least set of formulas containing $L_{\omega_1,\omega}$ and closed under the quantifier $(Qx)\phi(x)$ that is true if there are uncountably many solutions of $\phi(x)$.*

Note that formulas in each case are built up inductively from the basic relations named in L by quantification over individuals. As we'll explore below, the situations when each formula is equivalent to one with no quantifiers ('quantifier eliminable') or to one with only existential quantifiers ('model complete') provide important simplifications. In general we are studying infinite structures.

There is a long history of interactions between Model Theory and Number Theory. Prior to 1980 these involve the use of basic model theoretic notions such as compactness and quantifier elimination. Some important examples are Tarski's proof that the real field admits elimination of quantifiers and the proof of the analog for algebraically closed fields. Abraham Robinson developed much of the analysis. The most significant result was the Ax-Kochen-Ershov analysis of the Lang conjecture.

Many important tools of model theory were developed between 1955 and 1975 to investigate questions concerning the number of models of a complete theory in first order logic and more generally the classification of models of a first order theory. In particular, the notion of strong minimality, which is explored in detail below, was seen to be basic for the study of a theory that is categorical in power $\kappa-$ has only one model of cardinality κ (up to isomorphism).

More recently, there has been increasing use of sophisticated first order model theory including stability theory and in particular Shelah's orthogonality calculus [She91]. The most impressive result in this direction was Hrushovski's proof of the geometric Mordell-Lang conjecture (see [Bou99]. In a different direction, building on ideas of Van Den Dries (later expounded in [dD99]), and in analogy to the notion of strong minimality, Pillay and Steinhorn introduced the notion of o-minimality. A linearly ordered structure is o-minimal if every first order definable set is a finite union of intervals. In particular, no infinite discrete set can be defined. In a tour de force, Wilkie [Wil96] proved that the real field with exponentiation was o-minimal. This inaugurated an explosive study of o-minimal expansions of the reals which continues after fifteen years. In contrast to this situation, we will see below that complex exponentiation is not susceptible to study by first order methods.

2 Model theory of the complex field

Since we intend to expound recent work on complex exponentiation, let us begin with the model theoretic formulation of the complex field. Algebraically closed fields are the fundamental structures for the study of Algebraic Geometry. We work with the first order theory given by the axioms for fields of fixed characteristic and

$$(\forall a_1, \ldots a_n)(\exists y)\Sigma a_i y^i = 0.$$

Axioms of this form (universal quantifiers followed by existential) are designated as $\forall\exists$. The theory T_p of algebraically closed fields of fixed characteristic has exactly one model in each uncountable cardinality (Steinitz). That is, T_p is *categorical* in each uncountable cardinality.

Here are some standard examples of structures whose first order theory is categorical in an uncountable cardinal.

Example 2 *1.* $(\mathcal{C}, =)$

2. $(\mathcal{C}, +, =)$ *vector spaces over* Q*.*

3. $(\mathcal{C}, \times, =)$

4. $(\mathcal{C}, +, \times, =)$

We will return to these examples below to illuminate the classification of combinatorial geometries. The fundamental fact about categorical theories is Morley's theorem.

Theorem 3 (Morley [Mor65]) *If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.*

The significance of this result was not only the result – establishing an analog to Steinitz' theorem for all first order theories– but various specific techniques. The citation awarding Michael Morley the 2003 Steele prize for seminal paper asserts,

'... what makes his paper seminal are its new techniques, which involve a systematic study of Stone spaces of Boolean algebras of definable sets, called type spaces. For the theories under consideration, these type spaces admit a Cantor Bendixson analysis, yielding the key notions of Morley rank and ω-stability.'

The notion of type space requires some explanation. Let $A \subset N$ and $b \in N$. The type of b/A (in the sense of N) is the collection of formulas $\phi(x, \overline{a})$ with parameters from A that are satisfied by b. Each such type corresponds to an element of the dual space of the Boolean algebra of formulas with parameters from A. The parenthetical 'in the sense of N' virtually disappears from first order model theory after Shelah introduces the notion of a universal domain or monster model. But it returns with a vengeance in the study of infinitary logics.

Here are some of the consequences of categoricity.

Corollary 4 *The set of sentences true in algebraically closed fields of a fixed characteristic is decidable.*

The next two results illustrate the value of the logician's self-conciousness concerning the form of an axiomatization. Recall that a *constructible set* is one defined closing the classes defined by equation under conjunction and disjunction.

Theorem 5 (LINDSTROMS'S LITTLE THEOREM [Lin64]) *If T is $\forall\exists$-axiomatizable and categorical in some infinite cardinality then T is model complete.*

Thus we have the model completeness of algebraically closed fields, which can be phrased in algebraic terms.

Corollary 6 (Tarski, Chevalley) *The projection of a constructible set (in an algebraically closed field) is constructible.*

With a little technical but general model theoretic work, the model completeness of algebraically closed fields (which we now have from Steinitz) can be refined to:

Corollary 7 *The theory of algebraically closed fields admits elimination of quantifiers.*

The previous examples show how the basic results on elimination of quantifiers can be derived for algebraic geometry from the fundamental fact of categoricity. But categoricity has a more specific aspect. It means the notion of the dimension of a model is very clearly visible.

Definition 8 *M is* strongly minimal *if every first order definable subset of any elementary extension M' of M is finite or cofinite.*

This notion is perhaps best understood as providing the existence of a combinatorial geometry or matroid.

Definition 9 *A* pregeometry *is a set G together with a dependence relation*

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms. **A1.** $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$

A2. $X \subseteq cl(X)$

A3. $cl(cl(X)) = cl(X)$

A4. *If* $a \in cl(Xb)$ *and* $a \notin cl(X)$*, then* $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

Geometries are classified as: trivial, locally modular, non-locally modular. Exemplars are respectively examples 1 for trivial, 2 and 3 for locally modular and 4 for non-locally modular in Example 2. Zilber had conjectured that each non-locally modular geometry of a strongly minimal set was 'essentially' the geometry of an algebraically closed field. We discuss below Hrushovski's construction which gave counterexamples to this conjecture and maybe much more.

First let us note an equivalent form of strong minimality.

Definition 10 *The algebraic closure is defined on a model* M *by for every* $B \subseteq M$ *and every* a*,* $a \in \mathrm{acl(B)}$ *if for some first order formula* $\phi(x, \overline{y})$*, some* $\overline{b} \in B$*,* $\phi(a, \overline{b})$ *holds in* M *and* $\phi(x, \overline{b})$ *has only finitely many solutions.*

Lemma 11 *A complete theory* T *is strongly minimal if and only if it has infinite models and*

1. *algebraic closure induces a pregeometry on models of* T*;*
2. *any bijection between acl-bases for models of* T *extends to an isomorphism of the models*

Because of exchange each combinatorial geometry has a unique dimension (cardinality of a maximal independent set). A straightforward variant of Steinitz argument shows that every strongly minimal set (indeed every homogenous geometry) is categorical in all uncountable powers. Given the quantifier elimination result discussed above it is easy to see that the complex field is strongly minimal.

In contrast, arithmetic is a much wilder structure. It follows from Gödels work in the 30's that:

1. The collection of sentences true in $(Z, +, \cdot, 0, 1)$ is undecidable.
2. There are definable subsets of $(Z, +, \cdot, 0, 1)$ which require arbitrarily many alternations of quantifiers. (Wild)

3 Complex Exponentiation

Now we consider complex exponentiation: the structure $(C, +, \cdot, e^x, 0, 1)$. It is Godelian. The integers are defined as $\{a : e^a = 1\}$. Thus, the first order theory is undecidable and 'wild'. The resources of modern first order model theory-either stability theory or o-minimality are not available.

But Zilber had the following fundamental insight. Maybe Z is the source of all the difficulty. Fix Z by adding the *infinitary* axiom:

$$(\forall x) e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi.$$

Here is our situation. The first order theory of the complex field is categorical and admits quantifier elimination. Model theoretic approaches based on Shelah's theory of orthogonality have led to advances such as Hrushovski's proof of the geometric Mordell-Lang conjecture.

The first order theory of complex exponentiation is model theoretically intractable. But working out the insight of Zilber, we explore infinitary approaches. Will it be possible to develop the model theory of infinitary logic to have similar deep connections with core mathematics?

Now we consider how to generalize the notion of strongly minimal to the infinitary setting. Here is a trial definition. M is *'quasiminimal'* if every first order ($L_{\omega_1,\omega}$?) definable subset of M is countable or cocountable.

The analog for this situation of algebraic closure as defined in Definition 10 is: $a \in \mathrm{acl}'(\mathrm{X})$ if there is a first order formula with **countably many** solutions over X which is satisfied by a.

The following fact is an exercise for the standard definition of algebraic closure. If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from acl(X) to acl(Y). Does the result remain true if acl is replaced by acl$'$. In general the answer is no. In those classes where it is true, one can begin to analyze the construction of all models of an infinitary sentence in terms of countable components. The development of such an analysis for countable models by Shelah [She91] is one of the deepest and most significant works in first order model theory. The last stages of his proof of the 'main gap' involve the study of first order theories without the 'omitting types order property'. This is another manifestation of the notion of the excellence.

In turns out that to prove this extension of isomorphism condition in the more general setting one needs the following more precise version of quasiminimal excellence.

Definition 12 *A class* $(\boldsymbol{K}, \mathrm{cl})$ *is* quasiminimal excellent *if it admits a combinatorial geometry which satisfies on each* $M \in \boldsymbol{K}$

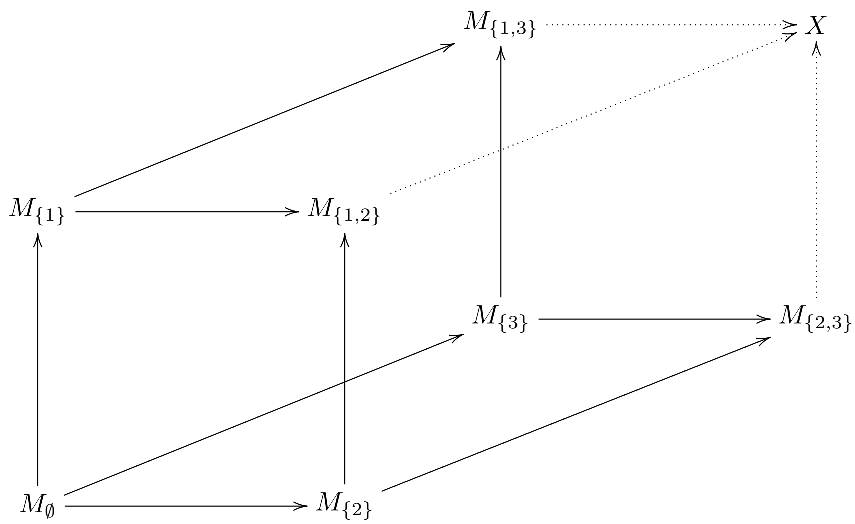
1. *there is a unique type of a basis,*
2. *a technical homogeneity condition:* $\aleph_0$*-homogeneity over* $\emptyset$ *and over models.*
3. *and the 'excellence condition' which follows.*

In the following definition it is essential that $\subset$ be understood as *proper* subset.

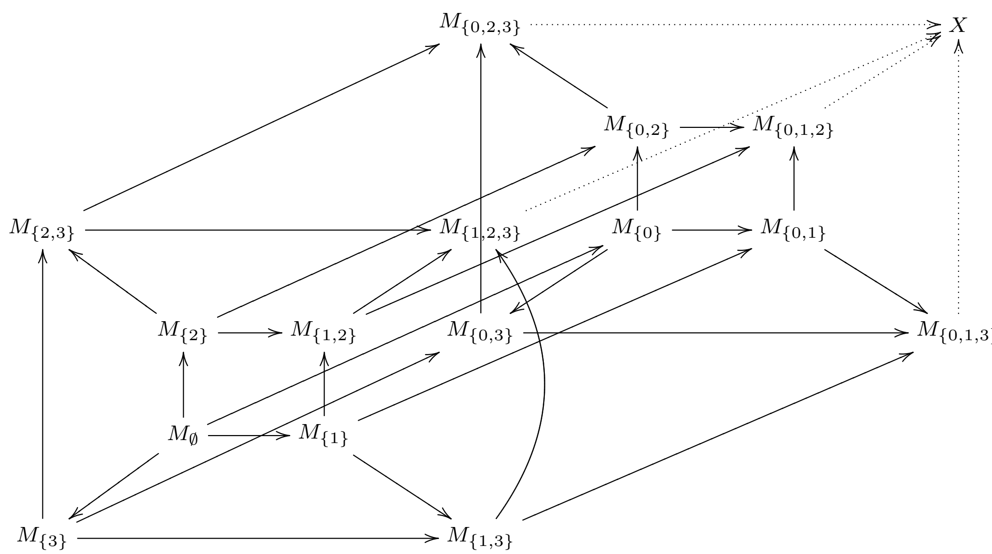
Definition 13 1. *For any* Y, $\mathrm{cl}^-(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.

2. *We call C (the union of)* an n-dimensional cl-independent system *if* $C = \mathrm{cl}^{-}(Z)$ *and Z is an independent set of cardinality n.*

We need to explore the notion of n-amalgamation. The next diagram illustrates the notion of the amalgam of a three dimensional diagram.



While, the next picture illustrates a 4-dimensional amalgam [1].



Roughly speaking, excellence asserts the existence of a prime model X over a given independent system of 2^{n-1} models. We make this precise for the special

[1]We thank Rami Grossberg for providing these diagrams.

case of quasiminimal excellence in the next few definitions. A more general notion is needed for the results of Shelah discussed in Section 7.

Definition 14 *Let $C \subseteq H \in \boldsymbol{K}$ and let X be a finite subset of H. We say* $\mathrm{tp}_{\mathrm{qf}}(X/C)$ *is* defined over *the finite C_0 contained in C if it is determined by its restriction to C_0.*

Definition 15 (Quasiminimal Excellence) *Let $G \subseteq H \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is defined over C_0.*

Excellence implies by a direct limit argument:

Lemma 16 *An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.*

This gives categoricity in all uncountable powers if the closure of each finite set is countable. And the argument is a fairly straightforward generalization of the basic Steinitz argument – given Lemma 16. More formally, we have:

Definition 17 *The structure M satisfies the* countable closure condition *if every the closure of every finite subset of M is countable.*

Theorem 18 (Zilber[Zil05]) *Suppose the quasiminimal excellent (I-IV) class $\boldsymbol{K}$ is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable. Then, for any infinite κ there is a unique structure in $\boldsymbol{K}$ of cardinality κ which satisfies the countable closure property.*

The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

4 pseudo-exponentiation

We mentioned above Hrushovski's refutation of Zilber's conjecture that all categorical first order structures were essentially known. In this section, we discuss Zilber's program to turn this construction of apparently pathological structures into a positive force for investigating complex exponentiation.

Let $\boldsymbol{K}_0$ be a class of substuctures closed under submodel. Then we are able to define notions of dimension. The following notions arise from Hrushovski's construction of various exotic models.

A predimension is a function δ mapping finite subsets of members of $\boldsymbol{K}$ into the integers such that:

$$\delta(XY) \leq \delta(X) + \delta(Y) - \delta(X \cap Y).$$

For each $N \in \boldsymbol{K}$ and finite $X \subseteq N$, the *dimension* of X in N is

$$d_N(X) = \min\{\delta(X') : X \subseteq X' \subseteq_\omega N\}.$$

The dimension function

$$d : \{X : \ X \subseteq_{fin} G\} \to \mathbf{N}$$

satisfies the axioms:

D1. $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$

D2. $X \subseteq Y \ \Rightarrow \ d(X) \leq d(Y)$.

Definition 19 *1. For $M \in \overline{\boldsymbol{K}}_0$, $A \subseteq M$, A finitely generated (i.e. in $\boldsymbol{K}_0$), $d_M(A) = \inf\{\delta(B) : A \subset B \subseteq M, B \in \boldsymbol{K}_0\}$.*

2. For A, b contained M, $b \in \mathrm{cl}(A)$ if $d_M(bA) = d_M(A)$.

Naturally we can extend to closures of infinite sets by imposing finite character. If d satisfies:

D3. $d(X) \leq |X|$.

We get a full combinatorial (pre)-geometry with exchange.

Lemma 20 *1. The closure system defined in Definition 19 is monotone and idempotent as in Definition 9.A1.*

2. If, in addition δ is a predimension (integer range) and for any finite X, $d_M(X) \leq |X|$ then the closure system satisfies exchange, Definition 9.A4.

Let us outline the Zilber program for studying $(\mathcal{C}, +, \cdot, \exp)$.

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a 'canonical' model of an $L_{\omega_1,\omega}(Q)$-sentence.

Objective A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation. Use a Hrushovski-like dimension function to prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Objective B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Crucially, the categoricity from Objective A must be applied in Objective B to a structure of cardinality $2^{\aleph_0}$. Categoricity in $\aleph_1$ is a non-trivial algebraic result; transfer to $2^{\aleph_0}$ is a model theoretic result which requires the notion of excellence.

Here are the axioms.

Definition 21 *Let $L = \{+, \cdot, E, 0, 1\}$. Σ is the sentence of $L_{\omega_1,\omega}(Q)$ expressing the following properties.*

1. *K is an algebraically closed field of characteristic* 0.
2. *E is a homomorphism from $(K, +)$ onto $(K^x, \cdot)$ and there is $\nu \in K$ transcendental over $\mathbb{Q}$ with $\ker E = \nu Z$.*
3. $\mathrm{acl}'(\mathrm{X})$ *is countable for every finite X.*

4. E is a pseudo-exponential.

5. K is strongly exponentially algebraically closed.

We now explain the fourth axiom. In next section we introduce some of the algebra involved in the investigation and elucidate the last axiom. The Q, 'there exists uncountably many' quantifier is needed only to express property 3. Infinitary logic is needed for conditions 2 and 5.

Definition 22 *E is a* **pseudo-exponential** *if for any n linearly independent elements over $\mathbb{Q}$, $\{z_1, \ldots z_n\}$*

$$\mathrm{td}(\mathrm{z}_1, \ldots \mathrm{z}_\mathrm{n}, \mathrm{E}(\mathrm{z}_1), \ldots \mathrm{E}(\mathrm{z}_\mathrm{n})) \geq \mathrm{n}.$$

Schanuel conjectured that true exponentiation satisfies this equation. In the appropriate Hrushovski construction, we use an abstract form of the Schanuel conjecture to define a dimension function.

For a finite subset X of an algebraically closed field k with a partial exponential function. Let

$$\delta(X) = \mathrm{td}(\mathrm{X} \cup \mathrm{E}(\mathrm{X})) - \mathrm{ld}(\mathrm{X})$$

where td denotes transcendence degree and ld denotes linear dimension.

Apply the Hrushovski construction to the collection of (k, E) with $\delta(X) \geq 0$ for all finite $X \subset k$. That is, those which satisfy the abstract Schanuel condition. The result is a quasiminimal excellent class.

5 Algebraic Results

One of the more intriguing ramifications of Zilber's program has been the results in algebra and complex analysis which have been proved in attempts to establish it. The first group concern the conjecture on intersection of tori.

I. Conjecture on Intersection of Tori.

Given a variety $W \subseteq \mathcal{C}^{n+k}$ defined over $\mathbb{Q}$, and a coset $T \subseteq (\mathcal{C}^*)^n$ of a torus.

An infinite irreducible component S of $W(\bar{b}) \cap T$ is

atypical if

$$d_f S - \dim T > d_f W(\bar{b}) - n.$$

Theorem 23 (True CIT) *There is a finite set A of nonzero elements of $\mathcal{Z}^n$,so that if S is an atypical component of $W \cap T$ then for some $\overline{m} \in A$ and some γ from $\mathcal{C}$, every element of S satisfies $\overline{x}^{\overline{m}} = \gamma$.*

Using the true CIT, the abstract Schanuel condition becomes a first order property. Replacing $\mathcal{C}$ by a semialgebraic variety gives the conjectured full CIT, which implies Manin-Mumford and more.

II. Choosing roots

Definition 24 *A* multiplicatively closed divisible subgroup *associated with* $a \in \mathcal{C}^*$, *is a* **choice** *of a multiplicative subgroup isomorphic to* $\mathbb{Q}$ *containing* a *.*

Definition 25 $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *over* F *if given subgroups of the form* $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *and* ϕ_m *such that*

$$\phi_m : F(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \to F(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \to F(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

Theorem 26 (Zilber [Zil00]) *[thumbtack lemma]*
For any $b_1, \ldots b_\ell \subset \mathcal{C}^*$, *there exists an* m *such that* $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ *over* F.

The Thumbtack Lemma implies that $\boldsymbol{K}$ satisfies the homogeneity conditions and 'excellence'.

F can be the acf of Q or a number field, or an independent system of algebraically closed fields. If $\mathcal{C}$ is replaced by a semi-abelian variety, these differences matter.

III. Towards Existential Closure

Now, we pass to the 'strongly existentially closed' axiom. Given $V \subseteq K^{2n}$ we might want to find $z_1, \ldots, z_n$ with $(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \in V$. Schanuel's conjecture prevents this for 'small' varieties. We want to say this is the only obstruction.

Definition 27 (Normal Variety) *Let* $G^n(F) = F^n \times (F^*)^n$.

If M *is a* $k \times n$ *integer matrix,*

$[M] : G^n(F) \to G^n(F)$ *is the homomorphism taking* $\langle \overline{a}, \overline{b} \rangle$ *to* $\langle M\overline{a}, \overline{b}^M \rangle$. *Act additively on first n coordinates, multiplicatively on the last n.* V^M *is image of* V *under* M. V *is* normal *if for any rank* k *matrix* M, $\dim V^M \geq k$.

Definition 28 (Free Variety) *Let* $V(\overline{x}, \overline{y})$ *be a variety in* $2n$ *variables.* $\mathrm{pr}_{\overline{\mathrm{x}}}\mathrm{V}$ *is the projection on* $\overline{x}$, $\mathrm{pr}_{\overline{\mathrm{y}}}\mathrm{V}$ *is the projection on* $\overline{y}$ V *contained in* F^{2n}, exp-*definable over* C *is* absolutely free of additive dependencies *if for a generic realization* $\overline{a} \in \mathrm{pr}_{\overline{\mathrm{x}}}\mathrm{V}$, $\overline{a}$ *is additively linearly independent over* $\mathrm{acl}(\mathrm{C})$.

V *contained in* F^{2n}, exp-*definable over* C *is* absolutely free of multiplicative dependencies *if for a generic realization* $\overline{b} \in \mathrm{pr}_{\overline{\mathrm{y}}}\mathrm{V}$, $\overline{b}$ *is multiplicatively linearly independent over* $\mathrm{acl}(\mathrm{C})$.

The variety V is exp-defined over $C \subset F$ if it is defined with parameters from $\tilde{C} = \mathbb{Q}(C \cup \exp C \cup \ker)$.

Assumption 29 (strong exponential algebraic closure) *Let* $V \subseteq G_n(K)$ *be free, normal and irreducible. For every finite* A, *there is* $(\overline{z}, E(\overline{z})) \in V$ *which is generic for* A.

This proposition is $L_{\omega_1,\omega}$-expressible; using uniform CIT (Holland, Zilber) it would become first order if the requirement of a generic solution were dropped. But with that requirement we have an other essential use of infinitary logic.

6 The Status of the Program

OBJECTIVE A.

Theorem 30 *The models of Σ with countable closure are categorical in all uncountable powers. This class is $L_{\omega_1,\omega}(Q)$-axiomatizable.*

This result has been proved in ZFC [Zil04].

OBJECTIVE B.

Does complex exponentiation satisfy Σ? The first two axioms are straightforward. But the rest raises a number of extremely hard questions. To verify Axiom 21.5 requires verifying the well-known Schanuel conjecture. This has been open for more than 50 years; the crucial work is that of [Ax71].

Schanuel's conjecture: If $x_1, \ldots x_n$ are $\mathbb{Q}$-linearly independent complex numbers then $x_1, \ldots x_n, e^{x_1}, \ldots e^{x_n}$ has transcendence degree at least n over $\mathbb{Q}$.

Zilber showed the following consequence to obtain Axiom 21.4.

Theorem 31 *If Schanuel holds in $\mathcal{C}$ and if the (strong) existential closure axioms hold in $\mathcal{C}$, then $(\mathcal{C}, +, \cdot, \exp)$ has the countable closure property.*

If anything, exponential completeness is even harder. We want: For any free normal V given by $p(z_1, \ldots z_n, w_1, \ldots w_n) = 0$, with $p \in \mathbb{Q}[z_1, \ldots z_n, w_1, \ldots w_n]$, and any finite A there is a solution satisfying $(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \in V$ and $z_1, \ldots z_n, E(z_1), \ldots E(z_n)$ is generic for A.

Marker [Mar] has proved the following special case.

Theorem 32 *Assume Schanuel. If $p(x, y) \in \mathbb{Q}[x, y]$ and depends on both x and y then it has infinitely many algebraically independent solutions.*

This verifies the n-variable conjecture for $n = 1$ with strong restrictions on the coefficients. The proof is a three or four page argument using Hadamard factorization.

7 Transfer of $L_{\omega_1,\omega}$-categoricity

Next we place the Zilber program in the context of the Shelah analysis of categoricity in $L_{\omega_1,\omega}$ ([She75, She83a, She83b]. Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory ($EC(T, Atomic)$-class). Shelah defined a notion of excellence; Zilber's quasiminimal excellence is the 'rank one' case.

Theorem 33 (Shelah 1983) *If $\boldsymbol{K}$ is an excellent $EC(T, Atomic)$-class then if it categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.*

Theorem 34 (Shelah 1983[She83a, She83b]) *Suppose $2^{\aleph_n} < 2^{\aleph_{n+1}}$. If an $EC(T, Atomic)$-class $\boldsymbol{K}$ has at most one model in $\aleph_n$ for all $n < \omega$, then it is excellent.*

This is an extraordinarily difficult theorem. By a simultaneous induction, Shelah proves that categoricity up to $\aleph_n$ proves the excellence condition in $\aleph_0$ for independent n diagrams and the existence of models in $\aleph_{n+1}$. Excellence of n-diagrams in $\aleph_0$ for all n, then implies of n-diagrams in all cardinalities κ. It is essential for the program that the existence of models in larger cardinals is proved as part of the induction.

An example of Hart and Shelah [HS90] shows the infinitely many instances of categoricity are necessary. The categoricity arguments were 'Morley-style'. Lessmann has given 'Baldwin-Lachlan' style proofs - showing models prime over quasiminimal sets.

For $L_{\omega_1,\omega}$ there is a straightforward analogy:

Strongly Minimal is to first order as Quasiminimal Excellent is to $L_{\omega_1,\omega}$.

But this analogy is more slippery with consideration of $L_{\omega_1,\omega}(Q)$. In particular, there is no published extension of Theorem 34, although unpublished and much more complicated arguments of Shelah may give the result.

8 Short Exact Sequences: the impact of uncountable categoricity

In the work on infinitary logic discussed so far in this article the role of model theory has been to suggest conjectures in core mathematics. But in [Zil03] certain 'arithmetic' properties of 'semiabelian varieties' are shown to be equivalent to categoricity in all uncountable powers of an $L_{\omega_1,\omega}$-description of the variety. We sketch that equivalence. By a 'semiabelian variety', Zilber means an algebraic group A whose universal cover is $\mathcal{C}^d$ for some d:

$$0 \to \Lambda \to C^d \to \mathbb{A}(\mathcal{C}) \to 0. \tag{1}$$

with $\Lambda \approx Z^{N_A}$ for some N_A with $d \leq N_A \leq 2d$.

When is the exact sequence:

$$0 \to Z^N \to V \to \mathbb{A}(\mathcal{C}) \to 0. \tag{2}$$

categorical, where V is a $\mathbb{Q}$ vector space and A is a semi-abelian variety? As we describe below, Zilber [Zil00] essentially showed 'the thumbtack lemma' is sufficient and in the special case $A(\mathcal{C}) = (\mathcal{C}, \cdot)$, the 'thumbtack lemma' holds. But he now raises the question for other choices of $\mathbb{A}$.

Fix for the following discussion a semiabelian variety $\mathbb{A}$, (that is $\mathbb{A}$ represents the formula defining a group in $\mathcal{C}$ which satisfies the short exact sequence described above). Let k_0 be the field of definition of $\mathbb{A}$.

Write T_A for the first order theory of the exact sequence and write $\Lambda = Z^N$ for the *infinitary* assertion that the kernel of the projection map is Z^N. Formally, we investigate conditions on $\mathbb{A}$ such that $T_A + \Lambda = Z^N$ is categorical in various uncountable cardinalities.

To describe the operative version of the thumbtack lemma for this application, we need some notation. The notion of an independent system in the sense of quasiminimal excellence has the following specific form in this context. Let F be an algebraically closed field containing k_0 and let B an algebraically independent over k_0 subset of F. Partition B into n subsets B_i. Then for each $s \subset n$, let

$$F_s = \mathrm{acl}(\mathrm{k_0}(\bigcup_{\mathrm{i}\in\mathrm{s}} \mathrm{B_i}).$$

Any independent system of algebraically closed fields has this form.

For $a \in \mathbb{A}(F)$, $a^{\mathbb{Q}}$ denotes

$$\{x \in \mathbb{A}(F) : x^n = a^m \text{ some } m, n \in Z, n \neq 0\}.$$

We write $\mathbb{A}_{tors}$ for the torsion points of $\mathbb{A}(F)$. Since F is algebraically closed, completeness gives that $\mathbb{A}_{tors}$ does not depend on F. But compare this with Lemma 37 below where finite torsion is found for the interpretation of $\mathbb{A}$ in other fields.

For $a = \langle a_1, \ldots a_n \rangle$, a finite sequence from $\mathbb{A}(F)$, let

$$k_a = k_0(\mathbb{A}_{tors}, a_1^{\mathbb{Q}}, \ldots a_n^{\mathbb{Q}}).$$

That is, k_a is the set of coordinates of the points in the divisible hull of the group generated by $\langle a_1, \ldots a_n \rangle$. Extend this notation to encompass extensions by an infinite set of independent points B by: Let B be a countable algebraically independent set over k_a, partitioned into n sets B_i. Then

$$k_a^B = k_0(\mathbb{A}_{tors}, \bigcup_{s \subset n} F_s, a_1^{\mathbb{Q}}, \ldots a_n^{\mathbb{Q}}).$$

Finally $\mathbb{A}_n$ denotes the points of $\mathbb{A}$ with order n.

Zilber [Zil00] had earlier proved.

Theorem 35 *If $\mathbb{A}$ denotes the multiplicative group $(\mathcal{C}, \cdot)$ then $T_A + \Lambda = Z$ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.*

The proof relies on his general analysis of quasiminimal excellence and the thumbtack lemma. Now consider the following conditions:

Notation 36 *1. Given any $b_1, \ldots b_k$ that are multiplicatively independent in $\mathbb{A}(k_0)$, there is an integer ℓ such that for any integer m:*

$$\mathrm{Gal}(k_0(\mathbb{A}_{ml}, b_1^{\frac{1}{m\ell}}, \ldots, b_k^{\frac{1}{m\ell}}) : k_0(\mathbb{A}_l, b_1^{\frac{1}{\ell}}, \ldots, b_k^{\frac{1}{\ell}})) \approx (\mathcal{Z}/m\mathcal{Z})^{Nk}.$$

2. *Let F be a countable algebraically closed field and F_0 an algebraically closed subfield. Given any $b_1, \ldots b_k \in \mathbb{A}(F)$ that are multiplicatively independent over $\mathbb{A}(F_0)$, there is an integer ℓ such that for any integer m:*

$$\mathrm{Gal}(F_0(\mathbb{A}_{ml}, b_1^{\frac{1}{m\ell}}, \ldots, b_k^{\frac{1}{m\ell}}) : F_0(\mathbb{A}_l, b_1^{\frac{1}{\ell}}, \ldots, b_k^{\frac{1}{\ell}})) \approx (\mathcal{Z}/m\mathcal{Z})^{Nk}.$$

3. *Given any $b_1, \ldots b_k$ that are multiplicatively independent over $\prod_{s \subset n} \mathbb{A}(F_s)$, there is an integer ℓ such that for any integer m:*

$$\mathrm{Gal}(k_a^B(\mathbb{A}_{ml}, b_1^{\frac{1}{m\ell}}, \ldots, b_k^{\frac{1}{m\ell}}) : k_a^B(\mathbb{A}_l, b_1^{\frac{1}{\ell}}, \ldots, b_k^{\frac{1}{\ell}})) \approx (\mathcal{Z}/m\mathcal{Z})^{Nk}.$$

Now Zilber's exploits the analysis by Keisler [Kei71] and Shelah ([She75, She83a, She83b] to deduce algebraic facts about semi-abelian varieties from $\aleph_1$-categoricity.

Theorem 37 (Zilber [Zil00, Zil03]) *If $T_A + \Lambda = Z^N$ is $\aleph_1$-categorical then*

1. *For any finite extension k of k_0, $\mathbb{A}(k)$ has only finite many torsion elements.*

2. *Condition 36.1 holds.*

Stronger algebraic facts are in fact equivalent to categoricity in uncountable powers. Categoricity is deduced by analyzing the algebraic facts about $(\mathcal{C}, \cdot)$ used in the proof of Theorem 35 and finding analogs that must hold of $\mathbb{A}$ to obtain categoricity in all powers. Of course, this includes excellence. The converse requires Shelah's Theorem 34; the new work is the translation of excellence into these specific algebraic conditions.

The condition of Theorem 37.2 is an immediate consequence of Mordell-Weil (Dirchelet's theorem for $\mathbb{A} = (\mathcal{C}, \cdot)$. The other conditions below (and some related corollaries to categoricity) are known to apply to some semi-abelian varieties and not to others by work of Serre, Bashkamov and general techniques of number theory. In particular, the existence of complex multiplication on the variety $\mathbb{A}$ affects when $T_A + \Lambda = Z^N$ can be categorical.

Theorem 38 (Zilber [Zil00, Zil03]) *($2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$) $T_A + \Lambda = Z^N$ is $\aleph_1$-categorical in all uncountable cardinals if and only*

1. *a model theoretic version of Theorem 37.1: For any finite extension k of k_0, $\mathbb{A}(k)$ has only finite many torsion elements.*

2. *All the conditions of Notation 36 hold.*

9 Tameness and Future Work

Our analysis here has been strictly in the context of sentences in $L_{\omega_1,\omega}$ (and $L_{\omega_1,\omega}(Q)$). But much recent work explores categoricity in the context of tame abstract elementary classes with the amalgamation property. (Abstract elementary classes generalize the semantic properties of elementary classes without an explicit syntax.) With amalgamation, one is able to obtain a notion of Galois type (corresponding to orbits in the monster model). With this notion one (see for example) [She99] can develop a theory of stability and categoricity that provides for a general study of categoricity. See [Balb, Bal04, Gro02] for an overview and [Bala] for a detailed exposition. Grossberg and VanDieren saw that a key aspect of the analysis in [She99] was the situation when Galois types are determined by their restrictions to small sets. They label this notion tame in [GVb]. They began the analysis of categorical AEC's under the additional hypotheses of the existence of arbitrarily large models and the amalgamation property. This result in a sequence of strong categoricity transfer theorems for tame AEC: [GVa, GVc, Les05, BL00, GVV, Van, HV]. One line of work is to try to apply this study of tame AEC to the algebraic situation of Section 8. Happily, the classes studied in Section 8 can easily be seen (by Zilber's methods) to have arbitrarily large models and the amalgamation property. Less happily, tameness remains in doubt. But the work of Villaveces-Zambrano [VZ05] and Grossberg-Kolesnikov [GK] provides hope for further progress here.

The analysis of number theoretic problems using infinitary logic provides exciting opportunities for continuing the almost 100 year interaction between model theory and number theory.

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