CATEGORICITY, AMALGAMATION, AND TAMENESS

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ABSTRACT. **Theorem.** For each $2 \leq k < \omega$ there is an $L_{\omega_1,\omega}$-sentence ϕ_k such that:

(1) ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
(2) ϕ_k is not $\aleph_{k-2}$-Galois stable;
(3) ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
(4) ϕ_k has the disjoint amalgamation property;
(5) For $k > 2$,
   (a) ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame; indeed, syntactic first-order types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
   (b) ϕ_k is $\aleph_m$-Galois stable for $m \leq k-3$;
   (c) ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

We adapt an example of [9]. The amalgamation, tameness, stability results, and the contrast between syntactic and Galois types are new; the categoricity results refine the earlier work of Hart and Shelah and answer a question posed by Shelah in [17].

Considerable work (e.g. [14, 15, 16, 7, 8, 6, 18, 12, 11]) has explored the extension of Morley's categoricity theorem to infinitary contexts. While the analysis in [14, 15] applies only to $L_{\omega_1,\omega}$, it can be generalized and in some ways strengthened in the context of abstract elementary classes.

Various locality properties of syntactic types do not generalize in general to Galois types (defined as orbits under an automorphism group) in an AEC [5]; much of the difficulty of the work stems from this difference. One such locality properties is called *tameness*. Roughly speaking, $\boldsymbol{K}$ is (μ, κ)-tame if distinct Galois types over models of size κ have distinct restrictions to some submodel of size μ. For classes with arbitrarily large models, that satisfy amalgamation and tameness, strong categoricity transfer theorems have been proved [7, 8, 6, 13, 4, 10]. In particular these results yield categoricity in every uncountable power for a *tame* AEC in a countable language (with arbitrarily large models satisfying amalgamation and the joint embedding property) that is categorical in any single cardinal above $\aleph_2$ ([6]) or even above $\aleph_1$ ([13]).

In contrast, Shelah's original work [14, 15] showed (under weak GCH) that categoricity up to $\aleph_\omega$ of a sentence in $L_{\omega_1,\omega}$ implies categoricity in all uncountable cardinalities. Hart and Shelah [9] showed the necessity of the assumption by constructing sentences ϕ_k which were categorical up to some $\aleph_n$ but not eventually

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categorical. These examples were thus a natural location to look for examples of categoricity and failure of tameness.

The example expounded here is patterned on the one in Hart-Shelah, [9]: our analysis of their example led to the discovery of some minor inaccuracies (the greatest categoricity cardinal is $\aleph_{k-2}$ rather than $\aleph_{k-1}$). Although the properties we assert could be proved with more complication for the original example, we present a simpler example. In Section 1 we describe the example and define the sentences ϕ_k. In Section 2 we introduce the notion of a solution and prove lemmas about the amalgamation of solutions. From these we deduce in Section 5 positive results about tameness. In some sense, the key insight of this paper is that the amalgamation property holds in all cardinalities (Section 3) while the amalgamation of solutions is very cardinal dependent. We prove in Section 4 that this example is a model-complete AEC. We show in Section 6 that ϕ_k is not Galois stable in $\aleph_{k-2}$ and deduce the non-tameness. From the instability we derive in Section 7 the failure of categoricity in all larger cardinals, thus answering the question posed by Shelah as Problem 6.12 in [17].

Baldwin and Shelah [5] showed under often satisfied conditions ($\boldsymbol{K}$ admits intersections i.e. is closed under arbitrary intersections) amalgamation does not affect tameness. That is, for any tameness spectrum realized by an AEC $\boldsymbol{K}$ which admits intersections, there is another which has the amalgamation property but the same tameness spectrum. But this construction destroys categoricity so those examples do not address the weaker conjecture that the amalgamation property together with categoricity in a finite number of cardinals implies $(\aleph_0, \infty)$-tameness. We refute that conjecture here. Baldwin, Kueker and VanDieren [2] showed that if $\boldsymbol{K}$ is an $(\aleph_0, \infty)$-tame AEC with arbitrarily large models that is Galois-stable in κ it is Galois stable in κ^+; our results show the tameness hypothesis was essential.

This paper and [5] provide the first examples of AEC that are not tame. In both papers the examples are built from abelian groups. But while [5] obtains non-tameness from phenomena that are closely related to the Whitehead conjecture and so to non-continuity results in the construction of groups, this paper shows the failure can arise from simpler considerations.

1. THE BASIC STRUCTURE

This example is a descendent of the example in [3] of an $\aleph_1$-categorical theory which is not almost strongly minimal. That is, the universe is not in the algebraic closure of a strongly minimal set. Here is a simple way to describe such a model. Let G be a strongly minimal group and let π map X onto G. Add to the language a binary function $t : G \times X \to X$ for the fixed-point free action of G on $\pi^{-1}(g)$ for each $g \in G$. That is, we represent $\pi^{-1}(g)$ as $\{ga : g \in G\}$ for some a with $\pi(a) = g$. Recall that a strongly minimal group is abelian and so this action of G is strictly 1-transitive. This guarantees that each fiber has the same cardinality as G and π guarantees the number of fibers is the same as $|G|$. Since there is no interaction among the fibers, categoricity in all uncountable powers is easy to check.

Let $k \geq 2$ be a natural number.

Notation 1.1. The formal language for this example contains unary predicates I, K, G, G^*, H, H^*; a binary function e_G taking $G \times K$ to H; a function π_G mapping G^* to K, a function π_H mapping H^* to K, a 4-ary relation t_G on $K \times G \times G^* \times G^*$, a 4-ary relation t_H on $K \times H \times H^* \times H^*$. Certain other projection functions are in the language but not expressly described. These symbols form a vocabulary L'; we form the vocabulary L by adding a $(k+1)$-ary relation Q on $(G^*)^k \times H^*$.

We start by describing the L'-structure $M(I)$ constructed from any set I with at least k elements. Typically, the set I will be infinite; but it is useful to have all the finite structures as well. We will see that the L'-structure is completely determined by the cardinality of I. So we need to work harder to get failure of categoricity, and this will be the role of the predicate Q.

The structure $M(I)$ is a disjoint union of sets I, K, H, G, G^* and H^*. Let $K = [I]^k$ be the set of k-element subsets of I. H is a single copy of Z_2. Let G be the direct sum of K copies of Z_2. So G, K, and I have the same cardinality. We include K, G, and Z_2 as sorts of the structure with the evaluation function e_G: for $\gamma \in G$ and $k \in K$, $e_G(\gamma, k) = \gamma(k) \in Z_2$. So in $L'_{\omega_1,\omega}$ we can say that the predicate G denotes exactly the set of elements with finite support of ${}^K Z_2$.

Now, we introduce the sets G^* and H^*. The set G^* is the set of affine copies of G indexed by K. First, we have a projection function π_G from G^* onto K. Thus, for $u \in K$, we can represent an element x of $\pi_G^{-1}(u)$ in the form $(u, x') \in G^*$. Alternatively, we say that $x \in G^*_u$. We refer to the set $\pi_G^{-1}(u)$ as the G^*-*stalk, or fiber over* u. Then we encode the affine action by the relation $t_G \subset K \times G \times G^* \times G^*$ which is the graph of a regular transitive action of G on G^*_u. That is, for all $x = (u, x')$, $y = (u, y')$ there is a unique $\gamma \in G$ such that $t_G(u, \gamma, x, y)$ holds. (Of course, this can be expressed in $L'_{\omega,\omega}$.)

As a set, $H^* = K \times Z_2$. As before if $\pi_H(x) = v$ holds x has the form (v, x'), and we denote by H^*_v the preimage $\pi_H^{-1}(v)$. Finally, for each $v \in K$, $t_H \subset K \times Z_2 \times H^* \times H^*$ is the graph of a regular transitive action of Z_2 on the stalk H^*_v.

$(*)$: We use additive notation for the action of G (H) on the stalks of G^* (of H^*).

(1) For $\gamma \in G$, denote the action by $y = x + \gamma$ whenever it is clear that x and y come from the same G^*-stalk. It is also convenient to denote by $y - x$ the unique element $\gamma \in G$ such that $y = \gamma + x$.

(2) For $\delta \in H$, denote the action by $y = x + \delta$, whenever it is clear that x and y come from the same H^*-stalk. Say that $\delta = y - x$.

If I is countably infinite, let ψ_k^1 be the Scott sentence for the countably infinite L'-structure $M(I)$ based on I that we have described so far. This much of the structure is clearly categorical (and homogeneous). Indeed, suppose two such models have been built on I and I' of the same cardinality. Take any bijection between I and I'. To extend the map to G^* and H^*, fix one element in each partition class (stalk) in each model. The natural correspondence (linking those selected in corresponding classes) extends to an isomorphism. Thus we may work with a canonical L'-model; namely with the model that has copies of G (without the group structure) as the stalks G_u^* and copies of Z_2 (also without the group structure) as the stalks H_v^*. The functions t_G and t_H impose an affine structure on the stalks.

Notation 1.2. *The L-structure is imposed by a $(k+1)$-ary relation Q on $(G^*)^k \times H^*$, which has a local character. We will use only the following list of properties of Q, which are easily axiomatized in $L_{\omega_1,\omega}$:*

(1) *Q is symmetric, with respect to all permutations, for the k elements from G^*;*
(2) *$Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1}))$ implies that $u_1, \ldots, u_{k+1}$ form all the k element subsets of a $k+1$ element subset of I. We call $u_1, \ldots, u_{k+1}$ a* compatible $(k+1)$-tuple*;*
(3) *using the notation introduced at (*) Q is related to the actions t_G and t_H as follows:*
   (a) *for all $\gamma \in G$, $\delta \in H$*

$$Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1})) \Leftrightarrow \neg Q((u_1, x_1 + \gamma), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1}))$$

if and only if $\gamma(u_{k+1}) = 1$;

   (b)

$$Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1})) \Leftrightarrow \neg Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1} + \delta))$$

if and only if $\delta = 1$.

Let ψ_k^2 be the conjunction of sentences expressing (1)–(3) above, and we let $\phi_k := \psi_k^1 \wedge \psi_k^2$.

It remains to show that such an expansion to $L = L' \cup \{Q\}$ exists. We do this by explicitly showing how to define Q on the canonical L'-structure. In fact, we describe $2^{|I|\cdot|K|}$ such structures parameterized by functions ℓ.

Fact 1.3. *Let $M = M(I)$ be an L'-structure described above. Let $K := [I]^k$. Let $\ell : I \times K \to 2$ be an arbitrary function.*

For each compatible $(k+1)$-tuple $u_1, \ldots, u_{k+1}$, such that $u_1 \cup \cdots \cup u_{k+1} = \{a\} \cup u_{k+1}$ for some $a \in I$ and $u_{k+1} \in K$, define an expansion of M to L by

$$M \models Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1}))$$

if and only if $x_1(u_{k+1}) + \cdots + x_k(u_{k+1}) + x_{k+1} = \ell(a, u_{k+1}) \bmod 2$. Then M satisfies the properties (1)–(3) of Notation 1.2.

Indeed, it is straightforward to check that the expanded structure M satisfies the properties.

We describe the interaction of G and Q a bit more fully. Using symmetry in the first k components, we obtain the following property that was used by Hart and Shelah to define Q in [9].

Fact 1.4. *For all $\gamma_1, \ldots, \gamma_k \in G$ and all $\delta \in H$ we have*

$$\begin{aligned}&Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1}))\\ &\qquad \Leftrightarrow Q((u_1, x_1 + \gamma_1), \ldots, (u_k, x_k + \gamma_k), (u_{k+1}, x_{k+1} + \delta))\end{aligned}$$

if and only if $\gamma_1(u_{k+1}) + \cdots + \gamma_k(u_{k+1}) + \delta = 0 \bmod 2$.

In order to consider finite L-structures with L'-reducts of the form $M(I)$ for some of our inductive proofs, we introduce the following terminology.

Definition 1.5. *We call an L-structure N a* full structure for *ϕ_k if $N \restriction L'$ is isomorphic to an $M(I)$ for some I and $N \models \psi_k^2$.*

Let χ_k be the disjunction of the sentences describing $M(I)$ for each finite set I. Let $\hat{\phi}_k$ be $\phi_k \vee (\psi_k^2 \wedge \chi_k)$. Then we can write "the L-structure N is a full structure for ϕ_k" more shortly as $N \models \hat{\phi}_k$.

An L-substructure A of $M \models \phi_k$ is called a full substructure *if $A \models \hat{\phi}_k$.*

Remark 1.6. *(1) For infinite N, full structure is the same as being a model of ϕ_k; $\hat{\phi}_k$ includes structures built on a finite I.*

(2) The need for the notion of a full *substructure can be explained, for example, by the fact that a subset $\{a_0, a_1, a_2\}$ of $I(M)$ together with a single element $x \in G^*_{a_0,a_1}$ is a substructure, but not a full substructure, of $M \models \phi_2$. We want to close such a substructure under almost all the Skolem functions, excluding the ones that add elements of the "spine" I.*

In the next section, we show that ϕ_k is categorical in $\aleph_0, \ldots, \aleph_{k-2}$. So in particular ϕ_k is a complete sentence for all k. (See Chapter 7 of [1] for an account of completeness of sentences in $L_{\omega_1,\omega}$.)

Now we obtain abstract elementary classes $(\boldsymbol{K}_k, \prec_{\boldsymbol{K}})$ where $\boldsymbol{K}_k$ is the class of models of ϕ_k and for $M, N \models \phi_k$, $M \prec_{\boldsymbol{K}} N$ if $M \prec_{L_{\omega_1,\omega}} N$. We show in Section 4 that $M \subset N$ implies $M \prec_{L_{\omega_1,\omega}} N$ for models of ϕ_k.

We freely use various notions from the general theory of AEC, such as Galois type, below. All are defined in [1]. For convenience we repeat the three most used definitions.

Definition 1.7. *The AEC $\boldsymbol{K}$ has the* disjoint amalgamation property *if for any $M_0 \prec M_1, M_2$, there is a model $M \models \phi_k$ with $M \succ M_0$ and embeddings $f_i : M_i \to M$, $i = 1, 2$ such that $f_1(M_1) \cap f_2(M_2) = f_1(M_0) = f_2(M_0)$. If we omit the requirement on the intersection of the images, we have the* amalgamation property.

Under assumption of amalgamation (disjointness is not needed) and joint embedding one can construct monster models, i.e., strongly model homogeneous models $\mathbb{M}$ of an appropriate large size. (See [1] for the definitions and the construction.) Joint embedding is clear in our context and we prove amalgamation in Section 3. Using monster models, one can give the following simple definition of a Galois type.

Definition 1.8. *Let $\boldsymbol{K}$ be an AEC with amalgamation. Let $M \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The* Galois type *of a over M ($\in \mathbb{M}$) is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.*

The set of all Galois types over M is denoted ga-S(M).

In a class with amalgamation we can check whether two points have the same Galois type by the following criterion: For $M \prec_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} N_2 \in \boldsymbol{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, the Galois type a over M in N_1 is the same as the Galois type b over M in N_2 if there exist strong embeddings f_1, f_2 of N_1, N_2 into some N^* which agree on M and with $f_1(a) = f_2(b)$.

Definition 1.9. *We say $\boldsymbol{K}$ is ω-Galois stable if for any countable $M \in \boldsymbol{K}$,* $|\text{ga-S(M)}| = \aleph_0$.

Definition 1.10. *We say $\boldsymbol{K}$ is (χ, μ)*-tame *if for any $N \in \boldsymbol{K}$ with $|N| = \mu$, for all $p, q \in$* ga-S(N)*, if $p \restriction N_0 = q \restriction N_0$ for every $N_0 \leq N$ with $|N_0| \leq \chi$, then $p = q$.*

2. Solutions and categoricity

As we saw in Fact 1.3, the predicate Q can be defined in somewhat arbitrary way. Showing categoricity of the L-structure amounts to showing that any model M, of an appropriate cardinality, is isomorphic to the model where all the values of ℓ are chosen to be zero; we call such a model a *standard model*. This motivates the following definition:

Definition 2.1. *Fix a model or a full structure M. A* solution *for M is a selector f that chooses (in a compatible way) one element of the fiber in G^* above each element of K and one element of the fiber in H^* above each element of K. Formally, f is a pair of functions (g, h), where $g : K(M) \to G^*(M)$ and $h : K(M) \to H^*(M)$ such that $\pi_G g$ and $\pi_H h$ are the identity and for each compatible $(k+1)$-tuple $u_1, \ldots, u_{k+1}$:*

$$Q(g(u_1), \ldots, g(u_k), h(u_{k+1})).$$

Notation 2.2. *As usual $k = \{0, 1, \ldots k-1\}$ and we write $[A]^k$ for the set of k-element subsets of A.*

We will show momentarily that if M and N have the same cardinality and have solutions f_M and f_N then $M \cong N$. Thus, in order to establish categoricity of ϕ_k in $\aleph_0, \ldots, \aleph_{k-2}$, it suffices to find a solution in an arbitrary model of ϕ_k of cardinality up to $\aleph_{k-2}$. Our approach is to build up the solutions in stages, for which we need to describe selectors over subsets of $I(M)$ (or of $K(M)$) rather than all of $I(M)$.

Definition 2.3. *We say that (g, h)* is a solution for the subset W of $K(M)$ *if for each $u \in W$ there are $g(u) \in G^*_u$ and $h(u) \in H^*_u$ such that if $u_1, \ldots, u_k, u_{k+1}$ are a compatible $(k+1)$-tuple from W, then*

$$Q(g(u_1), \ldots, g(u_k), h(u_{k+1})).$$

If (g, h) is a solution for the set W, where $W = [A]^k$ for some $A \subset I(M)$, we say that (g, h) is a solution over A.

Remark 2.4. *Let $k \geq 2$, and let M be a model of $\hat{\phi}_k$. If $A \subset I(M)$ has k elements, then there is a solution over A. Indeed, $[A]^k$ is a singleton, so there are no restrictions coming from the predicate Q.*

Definition 2.5. *The models of ϕ_k have the* extension property for solutions *over sets of size λ (or over finite sets) if for every $M \models \phi_k$, any solution (g, h) over a set A with $|A| = \lambda$ (or A finite), and every $a \in I(M) - A$ there is a solution (g', h') over the set $A \cup \{a\}$, extending (g, h).*

One can treat the element $g(u)$ as the image of the element $(u, 0)$ under the isomorphism between the standard model and M, where 0 represents the constantly zero function in the stalk G^*_u. Not surprisingly, we have the following:

Lemma 2.6. *If M and N are models of ϕ_k of the same cardinality and have solutions f_M and f_N then $M \cong N$.*

Moreover, suppose $\boldsymbol{K}$ has solutions and has extension of solutions for models of cardinality less than $|M|$. If g is an isomorphism between full substructures (or submodels) M', N' of M and N with $|M'| < |M|$ and $|N'| < |N|$, then the isomorphism $\hat{g}$ between M and N can be chosen to extend g. Finally, if

$f_{M'}$ is a solution on M' which extends to a solution f_M on M, then $\hat{g}$ maps them to a similar extending pair on N' and N.

Proof. We prove the 'moreover' clause; the first statement is a special case when g is empty and the 'finally' is included in the proof. Say, g maps M' to N'. Without loss of generality, $M \restriction L' = M(I)$, $N \restriction L' = M(I')$. Let α be a bijection between I and I' which extends $g \restriction I$. Extend naturally to a map from $K(M)$ to $K(N)$ and from $G(M)$ to $G(N)$, which extends g on M'. By assumption there is a solution $f_{M'}$ on M'. It is clear that g maps $f_{M'}$ to a solution $f_{N'}$ on N'; by assumption $f_{N'}$ extends to a solution on N. (Note that if we do not have to worry about g, we let α be an arbitrary bijection from I to I' and let $\alpha(f_M(u))$ be $f_N(\alpha(u))$.) For $x \in G^*(M - M')$ such that $M \models \pi_G(x) = u$, there is a unique $a \in G(M)$ with $a = x - f_M(u)$ (the operation makes sense because a and $f_M(u)$ are in the same stalk).

Let $\alpha(x)$ be the unique $y \in N - N'$ such that

$$N \models t_G(\alpha(u), \alpha(a), f_N(\alpha(u)), y)$$

i.e., $y = \alpha(a) + f_N(\alpha(u))$ in the stalk $G^*_{\alpha(u)}(N)$.

Do a similar construction for H^* and observe that Q is preserved. $\square_{2.6}$

We temporarily specialize to the case $k = 2$.

Claim 2.7. *The models of $\hat{\phi}_2$ have the extension property for solutions over finite sets.*

Proof. Let $A := \{a_0, \ldots, a_{n-1}\}$, let (g, h) be a solution over A, and suppose a is not in A. For each $v = \{a, a_i\}$, let y_v be an arbitrary element of H^*_v. Now extend h to the function h' with domain $[A \cup \{a\}]^2$ by defining $h'(v) := y_v$.

It remains to define the function g' on each $\{a, a_i\}$, and we do it by induction on i.

For $i = 0$, pick an arbitrary *starting point*[1] $x \in G^*_{a,a_0}$. Let $\gamma_0 \in G$ be such that for $j = 1, \ldots, n-1$:

$$\gamma_0(a, a_j) = 1 \text{ if and only if } M \models \neg Q((\{a, a_0\}, x), g(a_0, a_j), h'(a, a_j)).$$

It is clear that $\gamma \in G(M)$ and that letting $g'(\{a, a_0\}) := (\{a, a_0\}, x + \gamma_0)$, we have a partial solution.

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[1]For an argument in Section 4, we will need to choose this point more carefully; we will use the term "starting point" then.

Suppose that $g'(\{a, a_j\})$, $j < i$, have been defined. Pick an arbitrary *starting point* $x \in G^*_{a,a_i}$. Let $\gamma_i \in G(M)$ be such that for $j \in \{0, \ldots, n-1\} \setminus \{i\}$

$$\gamma_i(a, a_j) = 1 \text{ if and only if } M \models \neg Q((\{a, a_i\}, x), g(a_i, a_j), h'(a, a_j)).$$

Also let $\gamma'_i \in G(M)$ be such that for $j < i$

$$\gamma'_i(a_i, a_j) = 1 \text{ if and only if } M \models \neg Q((\{a, a_j\}, x), g'(a, a_j), h(a_i, a_j)).$$

Now letting $g'(\{a, a_i\}) := (\{a, a_i\}, x + \gamma_i + \gamma'_i)$ yields a well-defined solution on $A \cup \{a\}$. □

Corollary 2.8. *The sentence ϕ_2 is $\aleph_0$-categorical, and hence is a complete sentence.*

Proof. Let M be a countable model. Enumerate $I(M)$ as $\{a_i \mid i < \omega\}$. As we pointed out in Remark 2.4, a solution exists over the set $\{a_0, a_1\}$ (any elements in the stalks $G^*_{a_0,a_1}$ and $H^*_{a_0,a_1}$ work). By the extension property for solutions over finite sets we get a solution defined over the entire $I(M)$. Hence ϕ_2 is countably categorical by Lemma 2.6. □

We see that extension for solutions over finite sets translates into existence of solutions over countable sets. This is part of a general phenomenon that we describe below. We return to the general case $k \geq 2$.

Definition 2.9. *Let M with $M \restriction L' = M(I)$ be a model of $\hat{\phi}_k$. Let A be a subset of $I(M)$ of size λ, and consider an arbitrary n-element set $\{b_0, \ldots, b_{n-1}\} \subset I$. Suppose that, for each $(n-1)$-element subset w of $n = \{0, \ldots, n-1\}$, we have a solution (g_w, h_w) over $A \cup \{b_l \mid l \in w\}$ such that the solutions are compatible (i.e., $(\bigcup_w g_w, \bigcup_w h_w)$ is a function).*

We say that M has n-amalgamation for solutions over sets of size λ *if for every such set A, there is a solution (g, h) over $A \cup \{b_0, \ldots, b_{n-1}\}$ that simultaneously extends all the given solutions $\{(g_w, h_w) \mid w \in [n]^{n-1}\}$.*

For $n = 0$ the given system of solutions is empty, thus 0-amalgamation over sets of size λ is existence for solutions over sets of size λ. For $n = 1$, the initial system of solutions degenerates to just $(g_\emptyset, h_\emptyset)$, a solution on A; so the 1-amalgamation property corresponds to the extension property for solutions. Generally, the number n in the statement of n-amalgamation property for solutions refers to the "dimension" of the system of solutions that we are able to amalgamate.

Remark 2.10. *Immediately from the definition we see that n-amalgamation for solutions of certain size implies m-amalgamation for solutions of the same size for any $m < n$. Indeed, we can obtain m-amalgamation by putting $n - m$ elements of the set $\{b_0, \ldots, b_{n-1}\}$ inside A.*

Using Remark 2.4, we see that 2-amalgamation for solutions of size λ implies extension, and existence, of solutions of the same size.

Lemma 2.11. *The models $\hat{\phi}_k$ have the $(k-1)$-amalgamation property for solutions over finite sets.*

Proof. Enumerate $A = \{a_0, \dots, a_{r-1}\}$. We are given that $(\bigcup_w g_w, \bigcup_w h_w)$ is a function (where the union is over all $w \in [k-1]^{k-2}$). Moreover, it is a solution over $W = \bigcup_w \mathrm{dom}(g_w)$, $(\mathrm{dom}\, g_w = [A \cup \{b_i : i \in w\}]^k)$, since if $u_1, \dots u_{k+1}$ is a compatible $(k+1)$-tuple of k-tuples from W, then each u_i is in $\mathrm{dom}(g_w) = \mathrm{dom}(h_w)$ for at least one $w \in [k-1]^{k-2}$. Denote the function $\bigcup_w g_w$ by g_{-1}.

It is clear that in order to extend to a solution on $A \cup \{b_0, \dots, b_{k-2}\}$, we only need to define the values (g, h) on the stalks $\{a_i, b_0, \dots, b_{k-2}\}$ for all $i < r$. For each $i < r$, let $h(a_i, b_0, \dots, b_{k-2})$ be an arbitrary element of $H^*_{a_i, b_0, \dots, b_{k-2}}$. We need to check that (g_{-1}, h) is still a solution.

Remark 2.12. *Hart and Shelah assert that categoricity holds up to $\aleph_{k-1}$; we show in Theorem 7.1 that this statement is incorrect. The Hart–Shelah argument breaks down at this very point. Their formulation of the analog of Lemma 2.11 asserts essentially the k, not $k-1$, amalgamation property for solutions over finite sets. However, they did not make the compatibility requirement in Definition 2.9; and did not check that the function obtained after defining h is a partial solution. In fact, in their setting without the compatibility condition it need not be a solution, and there may not be a way of defining h to make (g_{-1}, h) a solution. We present an example of the failure of 2-amalgamation for solutions over finite sets for models of ϕ_2 at the end of this proof.*

As we will see in Lemma 2.14, $(k-1)$-amalgamation for solutions over finite sets translates into existence of solutions, and hence categoricity, in $\aleph_{k-2}$. This is the reason for subscript of the categoricity cardinal being off by one in [9].

It is clear that (g_{-1}, h) is a function with values in the appropriate stalks. To check that it is a solution, we need to make sure that we have not introduced new values that violate the predicate Q. This is easy: for each $a_i \in A$, any compatible $k+1$ tuple containing the k element set $\{a_i, b_0, \dots, b_{k-2}\}$ has to contain a k element set of the form $\{a_j, b_0, \dots, b_{k-2}\}$ for some $j \neq i$. Since the value g_{-1} at $\{a_j, b_0, \dots, b_{k-2}\}$ is not defined, there are simply no new compatible $k+1$ tuples to worry about.

Finally, we need to define g on the stalks of the form $\{a_i, b_0, \dots, b_{k-2}\}$. We do it by induction on $i < n$, building an increasing chain of functions g_i, $i < n$, with g_0 extending g_{-1}. Let $\{w_s \mid s < k-1\}$ be an enumeration of all the $k-2$ element subsets of $k-1$; let $\boldsymbol{b}_{w_s}$ denote the sequence $\{b_i \mid i \in w_s\}$ and let $\boldsymbol{c}_{s,j} = \langle a_0, a_j, \boldsymbol{b}_{w_s} \rangle$.

For $i = 0$, pick an arbitrary starting point $x \in G^*_{a_0,b_0,\dots,b_{k-2}}$. Let $\gamma_0 \in G$ be such that for $j = 1, \dots, n-1$

$$\gamma_0(a_j, b_0, \dots, b_{k-2}) = 1 \text{ if and only if}$$
$$M \models \neg Q((\{a_0, b_0, \dots, b_{k-2}\}, x), g_{-1}(\boldsymbol{c}_{0,j}), \dots, g_{-1}(\boldsymbol{c}_{k-1,j}), h(a_j, b_0, \dots, b_{k-2})).$$

Now we can extend the function g_{-1} to the function g_0 by letting $g_0(a_0, b_0, \dots, b_{k-2}) := (\{a_0, b_0, \dots, b_{k-2}\}, x + \gamma_0)$. It is clear that (g_0, h) is a solution from its definition.

For arbitrary i, suppose that the solution (g_{i-1}, h) has been defined so that

$$\operatorname{dom}(g_{i-1}) = \operatorname{dom}(g_{-1}) \cup [\{a_0, \dots, a_{i-1}, b_0, \dots, b_{k-2}\}]^k.$$

We need to extend g_{i-1} to a function g_i, with domain $\operatorname{dom}(g_{-1}) \cup [\{a_0, \dots, a_i, b_0, \dots, b_{k-2}\}]^k$, by defining $g_i(a_i, b_0 \dots, b_{k-2})$. The strategy will be the same as before: we pick an arbitrary starting point and work to resolve all possible conflicts with the predicate Q.

Let $\boldsymbol{d}_{s,j}$ denote $\langle a_i, a_j, \boldsymbol{b}_{w_s} \rangle$. Pick an arbitrary starting point $x \in G^*_{a_i,b_0,\dots,b_{k-2}}$. Let $\gamma_i \in G$ be such that for $j \in \{0, \dots, n-1\} \setminus \{i\}$

$$\gamma_i(a_j, b_0, \dots, b_{k-2}) = 1 \text{ if and only if } M \models$$
$$\neg Q((\{a_i, b_0, \dots, b_{k-2}\}, x), g_{-1}(\boldsymbol{d}_{0,j}), \dots, g_{-1}(\boldsymbol{d}_{k-1,j}), h(a_j, b_0, \dots, b_{k-2}))$$

and $\gamma_i(u) = 0$ if $u \in \operatorname{dom}(g_{-1}) \cup [\{a_0, \dots, a_i, b_0, \dots, b_{k-2}\}]^k$ is not of this form.

For each $(k-2)$-element set w of $k-1$, let $\gamma_i^w \in G$ be such that for $j < i$

$$\gamma_i^w(a_i, a_j, \boldsymbol{b}_w) = 1 \text{ if and only if } M \models$$
$$\neg Q((\{a_i, b_0, \dots, b_{k-2}\}, x), g_{i-1}(a_j, b_0, \dots, b_{k-2}), .., g_{-1}(\boldsymbol{d}_{s,j}), .., h(a_i, a_j, \boldsymbol{b}_w)),$$

and $\gamma_i^w(u) = 0$ if $u \in \operatorname{dom}(g_{-1}) \cup [\{a_0, \dots, a_i, b_0, \dots, b_{k-2}\}]^k$ is not of this form,

where $\boldsymbol{d}_{s,j}$ ranges over all sequences $\langle a_i, a_j, \boldsymbol{b}_{w_s} \rangle$ with w_s a $(k-2)$-element subset of $k-1$ except $w_s = w$. The role of γ_i^w is to avoid the conflict with the values already defined by g_{i-1}. Notice that we have finitely many conditions to meet, so γ_i as well as γ_i^w are all finite support functions in G.

Now we let

$$g_i(a_i, b_0, \dots, b_{k-2}) := \Big(\{a_i, b_0, \dots, b_{k-2}\}, x + \gamma_i + \sum_{w \in [k-1]^{k-2}} \gamma_i^w \Big).$$

From the definition, (g_i, h) is a solution. □

We now give an examples explicitly showing that, for models of ϕ_2, the solutions over finite sets do not have 2-amalgamation.

Example 2.13. *Let M be the standard countable model of ϕ_2 (i.e., a model where the values $\ell(a,u)$ are all zero). Take four points $a,b,c,d \in I$. Define functions (g_1,h_1) on $[\{a,b,c\}]^2$ and (g_2,h_2) on $[\{a,b,d\}]^2$ such that*

(1) *for $u \in \mathrm{dom}(h_i)$, $i=1,2$, the values $h_i(u)$ are zeros in the stalks H^*_u;*
(2) *for $u \in \mathrm{dom}(g_i)$, $i=1,2$, the values $g_i(u)$ are zero functions in the stalks* G^*_u except
(3) *$g_2(bd)$ is the function in $G^*_{b,d}$ with the support containing exactly one element $\{c,d\} \in K$. That is, $g_2(bd)(u)=1$ if and only if $u=\{c,d\}$.*

In particular, both (g_1,h_1) and (g_2,h_2) are solutions on their domains and they agree on $\{a,b\}$.

However,

$$M \models Q(g_1(ac), g_2(ad), \delta) \wedge \neg Q(g_1(bc), g_2(bd), \delta),$$

for any $\delta \in H^*_{c,d}$. Thus, the h-part of the solution cannot be defined on $H^*_{c,d}$. This shows that, using the notation of the above proof, the function (g_{-1},h) need not be a solution when we amalgamate two solutions over finite sets for $k=2$.

There are several reasonable ways to try to vary the definition of solution to obtain 2-amalgamation of finite solutions for ϕ_2. Ultimately, none of them work because models of ϕ_2 fail to have extension property for countable solutions; this is used in Section 6 to construct many Galois types over a countable model.

Lemma 2.14. *Let $M \models \phi_k$ for some $k \geq 2$ and let $n \leq k-2$. If M has $(n+1)$-amalgamation for solutions over sets of size less than $\lambda \geq \aleph_0$, then M has n-amalgamation for solutions over sets of size λ.*

Proof. Let $A = \{a_i \mid i < \lambda\}$ be a subset of $I(M)$, let $\{b_0,\dots,b_{n-1}\}$ be distinct points in $I(M) \setminus A$ and let

$$\{(g_w,h_w) \mid w \in [n]^{n-1}, \mathrm{dom}(g_w) = \mathrm{dom}(h_w) = [A \cup \{b_l \mid l \in w\}]^k\}$$

be a system of compatible solutions. We need to simultaneously extend the system of solutions.

By induction on $i < \lambda$, we are building an increasing continuous chain of solutions (g^i,h^i) such that

(1) $\mathrm{dom}(g^i) = \mathrm{dom}(h^i) = [\{a_j \mid j < i\} \cup \{b_0,\dots,b_{n-1}\}]^k$, and
(2) (g^{i+1},h^{i+1}) extends simultaneously (g^i,h^i) as well as for all $w \in [n]^{n-1}$,

$$(g_w,h_w) \restriction [\{a_j \mid j < i+1\} \cup \{b_l \mid l \in w\}]^k.$$

To define (g^0,h^0), consider for $w \in [n]^{n-1}$ the system of solutions $(g_w,h_w) \restriction [\{b_l \mid l \in w\}]^k$. Since $(n+1)$-amalgamation for solutions implies

n-amalgamation for solutions over a fixed set and we have $(n+1)$-amalgamation for solutions over the empty set, we get a simultaneous extension (g^0, h^0).

At limit stages, we take unions, and at the successor step we simultaneously extend (g^i, h^i) and $(g_w, h_w) \upharpoonright [\{a_j \mid j < i+1\} \cup \{b_l \mid l \in w\}]^k$, for all $w \in [n]^{n-1}$. Clearly, all the restrictions of (g_w, h_w) are pairwise compatible, and for each $w \in [n]^{n-1}$ the intersection $\mathrm{dom}(g^i, h^i) \cap \mathrm{dom}(g_w, h_w)$ is equal to $[\{a_j \mid j < i\} \cup \{b_l \mid l \in w\}]^k$, where their definitions coincide. So by $(n+1)$-amalgamation property for solutions of size less than λ there is the required common extension (g^{i+1}, h^{i+1}). Finally, $\bigcup_{i<\lambda}(g^i, h^i)$ is the needed solution. □

Corollary 2.15. *Every model of $\hat{\phi}_k$ of cardinality at most $\aleph_{k-2}$ admits a solution. Thus, the sentence ϕ_k is categorical in $\aleph_0, \ldots, \aleph_{k-2}$.*

Proof. Let $M \models \phi_k$. By Lemma 2.11, M has $(k-1)$-amalgamation for solutions over finite sets. So M has $(k-2)$-amalgamation for solutions over countable sets, $(k-3)$-amalgamation for solutions over sets of size $\aleph_1$, and so on until we reach 0-amalgamation for solutions over sets of size $\aleph_{k-2}$ (provided M is large enough). Since for $m < n$ and any λ, the n-amalgamation property for solutions over sets of cardinality λ implies m-amalgamation solutions over sets of cardinality λ, we have 0-amalgamation, that is, existence of solutions for sets of size up to and including $\aleph_{k-2}$.

Now Lemma 2.6 gives categoricity in $\aleph_0, \ldots, \aleph_{k-2}$. □

Corollary 2.16. *For all $k \geq 2$, the sentence ϕ_k is $L_{\omega_1,\omega}$-complete.*

The following further corollary will be useful in applications.

Corollary 2.17. *Let $M \models \hat{\phi}_k$ for some $k \geq 2$ and $n \leq k-2$. Suppose M has 2-amalgamation for solutions over sets of cardinality $\lambda \geq \aleph_0$ (or over finite sets). If $A_0 \subset A_1, A_2 \subset M$ have cardinality λ (or are finite) and $(g^1, h^1), (g^2, h^2)$ are solutions of A_1, A_2 respectively that agree on A, there is a solution (g, h) on $A_1 \cup A_2$ extending both of them.*

Proof. It suffices to show that a one point extension can be amalgamated with an extension of cardinality λ. For this, enumerate $A_1 - A_0$ as $\{a_0, a_1, a_2, \ldots\}$ and say $A_2 - A_0$ is $\{b\}$. Now successively apply 2-amalgamation of solutions to amalgamate $(g^2, h^2) \upharpoonright A_0 \cup \{b\}$ with $(g^1, h^1) \upharpoonright A_0 \cup \{a_0\}$ over A_0, with $(g^1, h^1) \upharpoonright A_0 \cup \{a_0, a_1\}$ over $A_0 \cup \{a_0\}$, etc. □

3. Disjoint amalgamation for models of $\hat{\phi}_k$

In contrast to the previous section, where we studied amalgamation properties of solutions, this section is about (the usual) amalgamation property for the class of models of $\hat{\phi}_k$. The amalgamation property is a significant assumption for the behavior and even the precise definition of Galois types, so it is important to establish that the class of models of our ϕ_k has it. We claim that the class has the disjoint amalgamation property in every cardinality. Note that the argument also establishes the joint embedding property.

Theorem 3.1. *Fix $k \geq 2$. The class of models of $\hat{\phi}_k$ has the disjoint amalgamation property.*

Proof. Let $M_i = M_i(I_i)$, $i = 0, 1, 2$, where of course $I_0 \subset I_1, I_2$; K_0, K_1, K_2 are the associated sets of k-tuples. We may assume that $I_1 \cap I_2 = I_0$. Otherwise take a copy I'_2 of $I_2 \setminus I_0$ disjoint from I_1, and build a structure M'_2 isomorphic to M_2 on $I_0 \cup I'_2$.

We are building a model $M \models \hat{\phi}_k$ on the set $I_1 \cup I_2$ making sure that it is a model of $\hat{\phi}_k$ and that it embeds M_1 and M_2, where the embeddings agree over M_0. We start by building the L'-structure on $I_1 \cup I_2$. So let $I = I(M) := I_1 \cup I_2$; the set $K = [I]^k$ can be thought of as $K_1 \cup K_2 \cup \partial K$, where ∂K consists of the new k-tuples.

Let G be the direct sum of K copies of Z_2, notice that it embeds $G(M_1)$ and $G(M_2)$ in the natural way over $G(M_0)$. We will assume that the embeddings are identity embeddings.

Let G^* be the set of K many affine copies of G, with the action by G and projection to K defined in the natural way. Let H^* be the set of K many affine copies of Z_2, again with the action by Z_2 and the projection onto K naturally defined.

For $i = 1, 2$, we now describe the embeddings f_i of $G^*(M_i)$ and $H^*(M_i)$ into G^* and H^*. Later, we will define the predicate Q on M in such a way that f_i become embeddings of L-structures.

For each $u \in K_0$, choose arbitrarily an element $x_u \in G^*_u(M_0)$. Now for each $x' \in G^*_u(M_1)$, let γ be the unique element in $G(M_1)$ with $x' = x_u + \gamma$. Let $f_1(x') := (u, \gamma)$. Similarly, for each $x' \in G^*_u(M_2)$, let $\delta \in G(M_2)$ be the element with $x' = x_u + \delta$. Define $f_2(x') := (u, \delta)$. Note that the functions agree over $G^*_u(M_0)$: if $x' \in G^*_u(M_0)$, then the element $\gamma = x' - x_u$ is in $G(M_0)$. In particular, $f_1(x_u) = f_2(x_u) = 0$, the constantly zero function.

For each $u \in K_i \setminus K_0$, $i = 1, 2$, choose an arbitrary $x_u \in G^*_u(M_i)$, and for each $x' \in G^*_u(M_i)$ define $f_i(x') := (u, x' - x_u)$. This defines the embeddings $f_i : G^*(M_i) \to G^*(M)$.

Embedding $H^*(M_i)$ into $H^*(M)$ is even easier: for each $v \in K_1$, pick an arbitrary $y_v \in H^*_v(M_1)$, and let $f_1(y_v) := (v, 0)$, $f_1(y_v + 1) := (v, 1)$. For each $v \in K_2$, if $v \in K_1$, define f_2 to agree with f_1. Otherwise choose an arbitrary $y_v \in H^*_v(M_2)$, and let $f_2(y_v) := (v, 0)$, $f_2(y_v + 1) := (v, 1)$.

This completes the construction of the disjoint amalgam for L'-structures. Now we define Q on the structure M so that f_i, $i = 1, 2$ become L-embeddings. The expansion is described in terms of the function ℓ that we discussed in Fact 1.3.

Let $u_1, \ldots, u_k, v$ be a compatible $(k+1)$-tuple of elements of K; $u_1 \cup \cdots \cup u_k \cup v = \{a\} \cup v$ for some $a \in I$.

Case 1. $u_1, \ldots, u_k, v \in K_1$ (or all in K_2). This is the most restrictive case. Each of the stalks $G^*_{u_i}(M_1)$ contains an element x_{u_i} defined at the previous stage; and the stalk H^*_v has the element $y_v \in M_1$. Define

$$\ell(a, v) := 0 \text{ if } M_1 \models Q((u_1, x_{u_1}), \ldots, (u_k, x_{u_k}), (v, y_v)),$$

and $\ell(a, v) := 1$ otherwise.

Case 2. At least one of the $u_1, \ldots, u_k, v$ is in ∂K. Then the predicate Q has not been defined on these $k+1$ stalks, and we have the freedom to define it in any way. So choose $\ell(a, v) := 0$ for all such compatible $(k+1)$-tuples.

Now define Q on M from the function ℓ as in Fact 1.3.

It is straightforward to check that f_1 and f_2 become L-embeddings into the L-structure M that we have built. □

It would be interesting to investigate the higher-dimensional amalgamation properties in the family of classes given by ϕ_k for $k \geq 2$. This would require a good understanding of independence in these structures, and goes beyond the scope of this paper.

4. Model completeness

In this section we show that the class of models of ϕ_k is model-complete in an almost classical sense. Namely, we show that if $M, N \models \phi_k$ and $M \subset N$, then $M \prec_{L_{\omega_1,\omega}} N$. An essential step in the proof involves showing that for each finite set $A \subset M$, there is a complete, modulo ϕ_k, existential formula isolating the $L_{\omega_1,\omega}$-type of A.

The notion of completeness for a sentence of $L_{\omega_1,\omega}$ is rather more subtle than in the first order case, (there is no obvious canonical choice of a "complete theory in $L_{\omega_1,\omega}$" attached to a structure M). The definitions and an explanation appear in Chapter 7 of [1].

Full substructures of models of ϕ_k will play an important role. Let us make the notion of a full substructure more explicit.

Fact 4.1. *Let $k \geq 2$ and $M \models \phi_k$. If a subset A of the universe of M is the universe of a* full substructure of *M in the sense of Definition 1.5, then*

(1) *A is an L-substructure of M;*
(2) *$G(A)$ is the set of all finite support functions in $G(M)$ whose support is contained in $K(A)$;*
(3) *for all $u \in K(A)$ and for some $x \in G^*_u(M)$, we have $G^*_u(A) = \{x + \gamma \mid \gamma \in G(A)\}$; and*
(4) *for all $v \in K(A)$ and for some $y \in H^*_u(M)$, we have $H^*_v(A) = \{y + \delta \mid \delta \in \mathbb{Z}_2\}$.*

Lemma 4.2. *For any set $A \subset M \models \hat{\phi}_k$, there is a minimal full substructure M_A containing A. Moreover, if A is a finite set, then M_A is also finite.*

Proof. The full structure M_A is constructed as follows. First add to A all the elements of $G(M)$ of the form $\gamma = x - y$, where $x, y \in G^*_u(M) \cap A$, and take the closure of the resulting set under all the projections to obtain a set X. Let $I_A := X \cap I(M)$, $K_A := [I_A]^k$. Then add elements to G, G^*, and H^* to satisfy the conditions (2)–(4) in Remark 4.1. Namely, form the set X' by adding the needed functions to $G(X)$; for any $u \in K_A$ such that $G^*_u(M) \cap X$ is empty, add a single element from the fiber $G^*_u(M)$, and for any $v \in K_A$ add, if necessary, both elements in the fiber $H^*_v(M)$. Finally, close X' under the action by the group $G(X')$. □

Note that there may be many minimal full substructures over A contained in M. The goal of the following few claims is to show that a minimal full substructure of M containing A is unique up to isomorphism over A, justifying our notation M_A. A key point is that if N and M are models of ϕ_k and A is imbedded in both M and N, the structure M_A need not be isomorphic to N_A over A.

Claim 4.3. *If M_A, M'_A are minimal full substructures of M containing $A \subset M$, then the following sets are equal:*

$$I(M_A) = I(M'_A),\ K(M_A) = K(M'_A),\ G(M_A) = G(M'_A),\ H^*(M_A) = H^*(M'_A).$$

In addition, for each $u \in K(M)$, if $G^*_u(M) \cap A \neq \emptyset$, then $G^*_u(M_A) = G^*_u(M'_A)$.

Proof. It is clear that the set I_A constructed in the previous lemma is the minimal one that works. So in fact we have $I(M_A) = I(N_A) = I_A$. The equalities $K(M_A) = K(N_A)$ and $G(M_A) = G(N_A)$ follow from the equality of I's.

By the definition of a full structure, both $H^*(M_A)$ and $H^*(N_A)$ must contain a double cover of $K(M_A) = K(N_A)$. There is a unique such double cover inside M, so $H^*(M_A) = H^*(N_A)$ follows.

Finally, by the definition of a full structure, if $x \in G^*_u(M) \cap A$, then $G^*_u(M_A)$ (and $G^*_u(N_A)$) must be the orbit of x under the action by $G(M_A)$ (and $G(N_A)$). Since the groups are the same, the orbits are in fact equal. □

Thus, the only possible non-uniqueness occurs in G^*-stalks that are not "populated" by elements from A. Indeed, in that case we have a complete freedom to choose a *starting point* in the stalk.

Lemma 4.4. *If $M \models \phi_k$, $A \subset M$ is a set of cardinality at most $\aleph_{k-2}$, and M_A, M'_A are minimal full substructures of M containing A, then M_A, M'_A are isomorphic over A.*

Proof. We claim that it is enough to show that there are solutions (g, h) on M_A and (g', h') on M'_A such that $h = h'$ everywhere and $g(u) = g'(u)$ for all u such that $G^*_u \cap A \neq \emptyset$. Indeed, then the map which is the identity on $I(M_A)$, $K(M_A)$, $G(M_A)$, $H^*(M_A)$ and all the stalks $G^*_u(M_A)$ such that $G^*_u \cap A \neq \emptyset$; and, for the remaining G^*-stalks, sends $g(u)+\gamma$ to $g'(u)+\gamma$, for all $\gamma \in G(M_A)$, is the desired isomorphism.

Now we show that such solutions can be constructed. Start by defining h and h' arbitrarily, but to be the same. For the g-part, follow the existence of solutions construction, picking the same element $x \in A$ as the *starting point* when dealing with the stalk $G^*_u \cap A \neq \emptyset$. If $G^*_u \cap A = \emptyset$, choose one starting point in each stalk and extend the isomorphism to the whole stalk using the action by G. □

Claim 4.5. *Let $M \models \phi_k$, and let M_0 be a finite full substructure of M. Let ψ_{M_0} be the quantifier-free first order formula describing the quantifier-free diagram of M_0. Then ψ_{M_0} is a complete $L_{\omega_1,\omega}$-formula modulo ϕ_k.*

Proof. It suffices to note that $\phi_k \wedge \psi_{M_0}[c_0, \dots, c_{l-1}]$ is a complete $L_{\omega_1,\omega}$ sentence. Indeed, the realizations of $c_0, \dots, c_{l-1}$ form a full finite structure, which must have a solution for any k. We also know that for any $k \geq 2$ the models of ϕ_k have the extension property for solutions over finite subsets of I.

Thus, by Lemma 2.6, $\phi_k \wedge \psi_{M_0}[c_0, \dots, c_{l-1}]$ is ω-categorical and hence complete. □

Claim 4.6. *Let $M \models \phi_k$, $n \geq k$, and let $I_0 = \{a_i \mid i < n\}$ be a subset of $I(M)$. Let*

$$\psi_{I_0} := \bigwedge_{i<n} I(x_i) \wedge \bigwedge_{i \neq j} x_i \neq x_j.$$

Then ψ_{I_0} is a complete $L_{\omega_1,\omega}$-formula modulo ϕ_k.

Proof. Let M_0 be a minimal full substructure of M containing I_0. Let ψ_{M_0} be the complete formula from the previous claim, let $x_0, \dots, x_{n-1}$ be the list of variables that correspond to the elements of I_0, and let $y_0, \dots, y_{m-1}$ be the remaining variables of ψ_{M_0}.

Since minimal full substructures over finite subsets of I are unique up to isomorphism (by the existence of solutions over finite subsets of I),

$$\phi_k \models \psi_{I_0} \rightarrow \exists y_0 \dots y_{m-1} \psi_{M_0}.$$

By Claim 4.5, $\exists y_0 \dots y_{m-1} \psi_{M_0}$ is a complete formula modulo ϕ_k. This completes the proof. $\square$

The $L_{\omega_1,\omega}$-type of an arbitrary subset A of M is also isolated. In contrast with subsets of $I(M)$ or finite full structures the formula isolating the type is not quantifier-free, but existential.

Claim 4.7. *Let $M \models \phi_k$, and let A be a finite subset of M. Then there is a complete, modulo ϕ_k, existential formula ψ_A that isolates the type of A.*

Proof. As in the previous claim, we take M_A a minimal full substructure of M containing A, and the formula $\psi_A := \exists y_0 \dots y_{m-1} \psi_{M_A}$, where we quantify over the elements in $M_A - A$. This formula is as needed. $\square$

Corollary 4.8. *Suppose $M \subset N$, where $M, N \models \phi_k$. Then $M \prec_{L_{\omega_1,\omega}} N$.*

Proof. Take $\boldsymbol{a} \in M$. Let $A := \boldsymbol{a}$, and let M_A, N_A be minimal full substructures of M and N over A in M and N respectively. It is enough to show that $N \models \exists y_0 \dots y_{m-1} \psi_{N_A}[\boldsymbol{a}]$ implies $M \models \exists y_0 \dots y_{m-1} \psi_{N_A}[\boldsymbol{a}]$. Since existential formulas are upwards persistent, we have $N \models \exists y_0 \dots y_{m-1} \psi_{M_A}[\boldsymbol{a}]$, and since ψ_{M_A} and ψ_{N_A} are complete modulo ϕ_k, by Claim 4.5 we have

$$\phi_k \models \exists y_0 \dots y_{m-1} \psi_{N_A}(\mathbf{x}) \leftrightarrow \exists y_0 \dots y_{m-1} \psi_{M_A}(\mathbf{x}).$$

Thus, since $M \models \phi_k$, we get $M \models \exists y_0 \dots y_{m-1} \psi_{N_A}[\boldsymbol{a}]$. $\square$

Corollary 4.9. *Let $M \models \phi_k$. For any $N \supset M$, all $b \in I(N) - I(M)$ satisfy the same syntactic type over M.*

Proof. Let $M \subset N \models \phi_k$ and let $b \in I(N) - I(M)$. For any full finite substructure $A \subset M$, by Claim 4.7 there is a formula ψ_{Ab} that generates the type of Ab. If we replace the constant for b by a variable x to get $\psi_{Ab}(x)$ the type,

$$\{\psi_{Ab}(x) \mid A \subset_{fin} M\}$$

generates $\mathrm{tp}(b/M)$. It remains to note that, by the extension of solutions over finite substructures, the formulas $\psi_{Ab}(x)$ depend only on A. $\square$

The significance of Corollary 4.9 will be clear in Section 6, where we show that the unique syntactic type of a spine element over a model of ϕ_k of size $\aleph_{k-2}$ splits into $2^{\aleph_{k-2}}$ distinct Galois types over that model.

5. Tameness

Here we study the tameness properties for models of ϕ_k. We know that ϕ_k is categorical up to $\aleph_{k-2}$; so without loss of generality we may deal with the standard models of ϕ_k in powers $\aleph_0, \dots, \aleph_{k-2}$.

In Section 6 we establish that ϕ_2 has $2^{\aleph_0}$ Galois types over a countable model; and that ϕ_3 is not $(\aleph_0, \aleph_1)$-tame.

We claim that the class of models of ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame. So the first index where some tameness appears is $k = 4$. In fact, the result is stronger than tameness: the Galois type of an element over a model of size up to $\aleph_{k-3}$ is determined by its existential type (this is interesting for $k \geq 3$).

Theorem 5.1. *Let $k \geq 3$. Then the class of models of ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame. Moreover, the Galois types of finite tuples over a model of size up to $\aleph_{k-3}$ are determined by the syntactic types over that model.*

Proof. We concentrate on the second statement, the first follows easily.

Fix $k \geq 3$ and suppose that $M \models \phi_k$ is of size $\lambda \leq \aleph_{k-3}$. Let $\boldsymbol{a} \in M^a$, $\boldsymbol{b} \in M^b$ be finite tuples that have the same existential type over M, where $M \prec M^a, M^b$.

Let M_0^a, M_0^b be (finite) minimal full structures containing $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Let $M_0 := M_0^a \cap M$. We may assume, adding elements of M to $\boldsymbol{a}$ and $\boldsymbol{b}$ if necessary, that $M_0 \neq \emptyset$. It is easy to check that M_0 is a full finite substructure of M and that $M_0 = M_0^b \cap M$.

Since $\boldsymbol{a}$, $\boldsymbol{b}$ satisfy the same existential type over M_0, there is an isomorphism $f_0 : M_0^a \to M_0^b$ fixing M_0 and sending $\boldsymbol{a}$ to $\boldsymbol{b}$.

Let $\langle g_0, h_0 \rangle$ be a solution for M_0, and let $\langle g_0^a, h_0^a \rangle$ be a solution extending $\langle g_0, h_0 \rangle$ to the full substructure M_0^a. Let $\langle g_0^b, h_0^b \rangle$ be the induced solution on M_0^b:

$$\langle g_0^b, h_0^b \rangle = \langle g_0^a, h_0^a \rangle^{f_0} := \langle f_0 \circ g_0^a \circ f_0^{-1}, f_0 \circ h_0^a \circ f_0^{-1} \rangle.$$

It is easy to check that the induced solution on M_0^b extends the solution $\langle g_0, h_0 \rangle$ on M.

Let $\{M_i \mid i < \lambda\}$ be an increasing continuous chain of full substructures of M that starts with the full substructure M_0 constructed above, with $|M_i| \leq |i| + \aleph_0$, and $\bigcup_{i<\lambda} M_i = M$. Let $\{M_i^a \mid i < \lambda\}$ and $\{M_i^b \mid i < \lambda\}$ be increasing continuous chains of full substructures starting with M_0^a and M_0^b constructed above. We define M_{i+1}^a (and M_{i+1}^b) to be the disjoint amalgam of M_i^a (respectively, M_i^b) and M_{i+1} over M_i.

Using the extension property for solutions, we get a chain $\{\langle g_i, h_i \rangle \mid i < \lambda\}$ of solutions for the models M_i, with $\langle g_i, h_i \rangle \subset \langle g_j, h_j \rangle$ for $i < j$. Using 2-amalgamation for solutions (which holds for $\mu \leq \aleph_{k-4}$) and Corollary 2.17, we get increasing chains of solutions $\langle g_i^a, h_i^a \rangle$ and $\langle g_i^b, h_i^b \rangle$, $i < \lambda^+$, where $\langle g_{i+1}^a, h_{i+1}^a \rangle$ has domain M_{i+1}^a and is gotten by extension of solutions from the 2-amalgam of solutions $\langle g_i^a, h_i^a \rangle$ and $\langle g_{i+1}, h_{i+1} \rangle$ that has domain $M_i^a \cup M_{i+1}$. Further by repeated application of the strong form of Lemma 2.6 we get an increasing sequence isomorphisms f_i from M_i^a onto M_i^b which fix M_i and map $\boldsymbol{a}$ to $\boldsymbol{b}$ and preserve the solutions. The union $\bigcup_{i<\lambda} f_i$ is the needed isomorphism between $\bigcup_{i<\lambda} M_i^a$ and $\bigcup_{i<\lambda} M_i^b$ that fixes M and sends $\boldsymbol{a}$ to $\boldsymbol{b}$. □

An alternative notion of ω-stability might count the number of syntactic types over a countable model. By Theorem 5.1 the ω-stability of ϕ_k for $k \geq 3$ does not depend on which definition of ω-stable is used. This theorem does not address ω-syntactic stability of ϕ_2; we show it is not ω-Galois stable in Theorem 6.1. A separate argument to show ϕ_2 is not ω-syntactically stable is in preparation. See [11] for related results.

Our earlier argument for tameness used the notion of superhomegenity. Although no longer needed for the main argument, we include the following results since superhomogeneity is an intriguing property in its own right. For now, let $k \geq 3$. We claim that the model of ϕ_k with power $\aleph_{k-3}$ is *superhomogeneous* in the following sense (note M_0 may have cardinality $\aleph_{k-3}$.)

Definition 5.2. *The structure M is* superhomogeneous *if for any $M_0 \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ and $\boldsymbol{a}, \boldsymbol{b} \in M$ which realize the same Galois type over M_0, there is an automorphism of M which takes $\boldsymbol{a}$ to $\boldsymbol{b}$ and fixes M_0.*

It is important that $\boldsymbol{a}$, $\boldsymbol{b}$ are finite tuples here. The lemma below fails otherwise. Forgetting the finiteness condition is also possible; the price to pay is the additional demand that M is weakly full over M_0.

Lemma 5.3. *Let M be the model of ϕ_k with power $\leq \aleph_{k-3}$. Then M is superhomogeneous.*

Proof. In fact, we show that if M has extension of solutions over subsets of smaller cardinality then M is superhomogeneous; the precise statement then follows from the proof of Corollary 2.15. Let $\boldsymbol{a}, \boldsymbol{b} \in M = M(I)$ have the same Galois type over M_0. By Theorem 5.1 there is an isomorphism f between M_0^a and M_0^b mapping $\boldsymbol{a}$ to $\boldsymbol{b}$.

Let $\langle g_0, h_0\rangle$ be a solution for M_0, and let $\langle g_1, h_1\rangle$ be a solution extending $\langle g_0, h_0\rangle$ to the model M_0^a. Then

$$\langle g_1, h_1\rangle^f := \langle f \circ g_1 \circ f^{-1}, f \circ h_1 \circ f^{-1}\rangle$$

is a solution for M_0^b that extends $\langle g_0, h_0\rangle$.

From our hypotheses, $|I(M) - I(M_0^a)| = |I(M) - I(M_0^b)|$. So we can extend the solutions $\langle g_1, h_1\rangle$ and $\langle g_1, h_1\rangle^f$, in the same number of steps, to full solutions over M. This gives the desired automorphism of M. □

6. Instability and Non-tameness

In this section we show that ϕ_k is not Galois stable in $\aleph_{k-2}$. We warm up by treating the case: $k = 2$, showing there are continuum Galois types over a countable model of ϕ_2. The proof reduces equality of types p_σ, p_τ to the equivalence relation of eventual equality between σ and τ. The argument for larger k involves a family of equivalence relations instead of just one.

More precisely, we show that for M the standard model of cardinality $\aleph_{k-2}$, the unique syntactic type over M of a new element in the spine splits into $2^{\aleph_{k-2}}$ Galois types.

Since for any u, the stalk G_u is affine (L')-isomorphic to the finite support functions from K to Z_2, without loss of generality we may assume each stalk has this form. We are working with models of cardinality $\leq \aleph_{k-2}$ so they admit solutions; thus, if we establish L'-isomorphisms they extend to L-isomorphisms. For any G^*-stalk G_u, the 0 in $(u, 0)$ denotes the identically 0-function in that stalk. But for a stalk in H^*, the 0 in $(u, 0)$ denotes the constant 0.

Claim 6.1. *Let M be the standard countable model of ϕ_2. There are $2^{\aleph_0}$ Galois types over M.*

Proof. Let E_0 be the equivalence relation of eventual equality on ${}^{\omega}2$; there are of course $2^{\aleph_0}$ equivalence classes.

Let $I(M) = \{a_0, \ldots, a_i, \ldots\}$. Pick a function $s \in {}^{\omega}2$, and define a model $M_s \succ M$ as follows. The L'-structure is determined by the set $I(M_s) = I(M) \cup \{b_s\}$. For the new compatible triples of the form $\{a_0, a_i\}$, $\{a_0, b_s\}$, $\{a_i, b_s\}$, define

$$M_s \models Q((\{a_0, a_i\}, 0), (\{a_0, b_s\}, 0), (\{a_i, b_s\}, 0))$$

if and only if $s(i) = 0$. The values of Q for any u_1, u_2, u_3 among the remaining new compatible triples is defined as:

$$M_s \models Q((u_0, 0), (u_1, 0), (u_2, 0)).$$

Note that 0 in the first two components of the predicate Q refer to the constantly zero functions in the appropriate G^*-stalks, and in the third component, 0 is a member of Z_2. A compact way of defining the predicate Q is:

$$(*) \qquad M_s \models Q((\{a_0, a_i\}, 0), (\{a_0, b_s\}, 0), (\{a_i, b_s\}, s(i))).$$

Note that by Notation 1.2, the definition of Q is determined on all of M.

Now we show that the E_0-class of s can be recovered from the structure of M_s over M. Take two models M_s and M_t and suppose that the Galois types $\text{ga-tp}(b_s/M)$ and $\text{ga-tp}(b_t/M)$ are equal. Then there is an extension N of the model M_t and an embedding $f : M_s \to N$ that sends b_s to b_t. We work to show that in this case s and t are E_0-equivalent.

First, let us look at the stalks $G^*_{a_1,a_i}$, $G^*_{a_1,b_t}$, $H^*_{a_i,b_t}$ for $i > 1$. Since f fixes M, the constantly zero function $0 \in G^*_{a_1,a_i}$ is fixed by f. Let $x \in G^*_{a_1,b_t}$ be the image of $0 \in G^*_{a_1,b_s}$ under f. Then we have

$$M_t \models Q((\{a_1, a_i\}, 0), (\{a_1, b_t\}, x), (\{a_i, b_t\}, f(0))).$$

Since x is a finite support function, and we have defined

$$M_t \models Q((\{a_1, a_i\}, 0), (\{a_1, b_t\}, 0), (\{a_i, b_t\}, 0)),$$

for co-finitely many $i > 1$ we must have $f(0) = 0$ in the stalks $H^*_{a_i,b_t}$. In other words, f preserves all but finitely many zeros in $H^*_{a_i,b_t}$. In particular, by $(*)$ for any $s : \omega \to 2$ the functions s and $f(s)$ are E_0-equivalent.

We focus now on the stalks of the form $G^*_{a_0,a_i}$, $G^*_{a_0,b_t}$, $H^*_{a_i,b_t}$, $i \geq 1$. Again, since f fixes M, the constantly zero function $0 \in G^*_{a_0,a_i}$ is fixed by f. Letting $y \in G^*_{a_0,b_t}$ be the image of $0 \in G^*_{a_0,b_s}$ under f, we get

$$M_t \models Q((\{a_0, a_i\}, 0), (\{a_0, b_t\}, y), (\{a_i, b_t\}, f[s(i)])).$$

Since y is a finite support function, there is a natural number n such that $y(a_i, b_t) = 0$ for all $i > n$. Since we have defined

$$M_t \models Q((\{a_0, a_i\}, 0), (\{a_0, b_t\}, 0), (\{a_i, b_t\}, t(i))),$$

we get $t(i) = f(s(i))$ for all $i > n$, or $f(s)$ and t are E_0-equivalent. Combining this with the previous paragraph, we get that s is E_0-equivalent to t, as desired.

□

Now we turn to the proof that many Galois types exist for a general k. We will reduce equality on Galois types indexed by elements of ${}^{\omega_k}2$ to the equivalence relation of eventual equality on ${}^{\omega_k}2$. This requires some more technical notions.

Remark 6.2. *In fact, for any $\mu \geq \aleph_k$ the relation of equality on Galois types indexed by elements of ${}^{\mu}2$ reduces to the equivalence relation E_μ on ${}^{\mu}2$, where $E_\mu(s,t)$ if and only if $|\{s(i) = t(i) \mid i < \mu\}| = \mu$.*

We do our analysis on $\aleph_k$ as that is the most important application; but the argument can be used on any $\mu \geq \aleph_k$.

Definition 6.3. *Fix a natural number n. Let E_n be the equivalence relation of eventual equality on the set of sequences ${}^{\omega_n}2$.*

Let $P_n := \omega \times \cdots \times \omega_n$. Define the family of equivalence relations F_n on the sets of sequences ${}^{P_n}2$ by induction. Let $F_0 := E_0$. Having defined the relation F_{n-1} on ${}^{P_{n-1}}2$, define F_n as follows. Two sequences $s, t \in {}^{P_n}2$ are F_n-equivalent if and only if there is a set $B_n \in \omega_n$ such that

(1) *the complement of* B_n *has cardinality less than* $\aleph_n$*;*
(2) *for all* $i^* \in B_n$ *the sequences* $s(i_0, \dots, i_{n-1}, i^*)$ *and* $t(i_0, \dots, i_{n-1}, i^*)$ *are* F_{n-1}*-equivalent.*

Claim 6.4. *The equivalence relation* E_n *is reducible to* F_n*. In particular,* F_n *has* $2^{\aleph_n}$ *equivalence classes.*

Proof. Given a sequence $s \in {}^{\omega_n}2$, define $\overline{s} \in {}^{P_n}$ by

$$\overline{s} : (i_0, \dots, i_n) \in P_n \mapsto s(i_n).$$

Clearly, $E_n(s,t)$ if and only if $F_n(\overline{s}, \overline{t})$. □

We will identify a sequence $s \in {}^{\omega_n}2$ with its image $\overline{s}$ in ${}^{P_n}2$.

Proposition 6.5. *Let* M *be the standard model of* ϕ_{k+2} *of size* $\aleph_k$*. There are* $2^{\aleph_k}$ *Galois types over* M*.*

Proof. Without loss of generality, we may assume that

$$I = I(M) = \{a_0, a_1\} \cup I_0 \cup \cdots \cup I_k,$$

where I_l is a well-ordered set of order-type ω_l, $l = 0, \dots, k$. We denote the elements of I_l by a^l_i, for $i < \omega_l$, $l < k$.

The Galois types over the model M will be coded essentially by E_k, but we will need the finer relation F_k to describe the situation. Pick a function $s \in {}^{\omega_k}2$, and define a model $M_s \succ M$ as follows. The L'-structure is determined by the set $I(M_s) = I(M) \cup \{b_s\}$. The L-structure on M_s is given as in the original definition of Q in Section 1 from the function ℓ, where:

$$\ell(a_0, \{a^0_{i_0}, \dots, a^k_{i_k}, b_s\}) = s(i_k) \text{ for all } (i_0, \dots, i_k) \in P_k,$$

and the rest of the values of ℓ are all zero. In particular,

$$\ell(a_1, \{a^0_{i_0}, \dots, a^k_{i_k}, b_s\}) = 0 \text{ for all } (i_0, \dots, i_k) \in P_k.$$

Let us note explicitly the most relevant relations. For $(i_0, \dots, i_k) \in P_k$, we introduce some special notation for $k+2$ element subsets of $\{a_0, a^0_{i_0}, \dots, a^k_{i_k}, b_s\}$. Let

$$v_{i_0 \dots i_k, s} := \{a^0_{i_0}, \dots, a^k_{i_k}, b_s\}.$$

List the remaining $k+2$ element subsets of $\{a_0, a^0_{i_0}, \dots, a^k_{i_k}, b_s\}$ as $u_{i_0 \dots i_k}$ (the subset not containing b_s), and $u_{i_0 .. \hat{i}_j .. i_k, s}$ for $j \le k$ (omitting $a^j_{i_j}$).

Similarly, let $w_{i_0 \dots i_k}$, $w_{i_0 .. \hat{i}_j .. i_k, s}$ list the $k+2$ element subsets of $\{a_1, a^0_{i_0}, \dots, a^k_{i_k}, b_s\}$ that do not contain respectively b_s and $a^j_{i_j}$ for $j \le k$. Then we have

$$M_s \models Q((u_{i_0 \dots i_k}, 0), (u_{\hat{i}_0, i_1 \dots i_k, s}, 0), \dots, (u_{i_0 \dots i_{k-1}, \hat{i}_k, s}, 0), (v_{i_0 \dots i_k, s}, s(j)))$$

and

$$M_s \models Q((w_{i_0 \dots i_k}, 0), (w_{\hat{i}_0, i_1 \dots i_k, s}, 0), \dots, (w_{i_0 \dots i_{k-1}, \hat{i}_k, s}, 0), (v_{i_0 \dots i_k, s}, 0)).$$

Now we show that the F_k-class of s can be recovered from the structure of M_s over M. Take two models M_s and M_t and suppose that the Galois types $\text{ga-tp}(b_s/M)$ and $\text{ga-tp}(b_t/M)$ are equal. Then there is an extension N of the model M_t and an embedding $f : M_s \to N$ that sends b_s to b_t. We work to show that in this case s and t are F_k-equivalent and hence E_k-equivalent.

First, let us look at the stalks $G^*_{w_{i_0 .. \hat{i}_j .. i_k, t}}$, $j \le k$ and $H^*_{v_{i_0 \dots i_k}, t}$ in M_t. Since f fixes M, the constantly zero function $0 \in G^*_{w_{i_0 \dots i_k}}$ is fixed by f.

For $j \le k$ and $i_j < \omega_j$ let $x_{i_0 .. \hat{i}_j .. i_k} \in G^*_{w_{i_0 .. \hat{i}_j .. i_k, t}}$ be the image of $0 \in G^*_{w_{i_0 .. \hat{i}_j .. i_k, s}}$ under f. Let $y_{i_0 \dots i_k} \in H^*_{v_{i_0 \dots i_k}, t}$ be the image of $0 \in H^*_{v_{i_0 \dots i_k}, s}$. We will analyze the value of $y_{i_0 \dots i_k}$ in two ways. We write $\boldsymbol{i}$ for $\langle i_0 \dots i_k \rangle$ and $\boldsymbol{i}^-$ for the first k-elements: $\langle i_0 \dots i_{k-1} \rangle$.

Since f is an embedding we have:

$$M_t \models Q((w_{\boldsymbol{i}}, 0), (w_{\hat{i}_0, i_1 \dots i_k, t}, x_{\hat{i}_0, i_i \dots i_k}), \dots, (w_{\boldsymbol{i}^-, t}, x_{\boldsymbol{i}^-}), (v_{\boldsymbol{i}, t}, y_{\boldsymbol{i}})). \ (**)$$

For each $(i_0, \dots, i_k) \in P_k$, let $\overline{f}(i_0, \dots, i_k) := y_{i_0 \dots i_k}$. Since each $y_{i_0 \dots i_k}$ is either 0 or 1, $\overline{f}$ is a function in ${}^{P_k}2$. Since f is an isomorphism, the image of any $\delta \in H^*_{i_0 \dots i_k, s}$ is the element $\delta + y_{i_0 \dots i_k} \mod 2$ in the stalk $H^*_{i_0 \dots i_k, t}$. The following claim thus implies $F_k(s, t)$, which in turn implies $E_k(s, t)$, as required. □

Claim 6.6. *The function $\overline{f}$ is F_k-equivalent to the constantly zero function on P_k.*

Proof. Since each $x_{i_0 .. \hat{i}_j .. i_k}$ is a finite support function, there is a subset $B_k \subset \omega_k$ such that the complement of B_k has cardinality smaller than $\aleph_k$ and for each $i_k \in B_k$, for all $i_0, \dots, i_{k-1} \in P_{k-1}$ none of the functions $x_{i_0 \dots i_{k-1}}$ contain i_k in any of the subsets in their support.

Fix an arbitrary $i_k^* \in B_k$. There are ω_{k-2} many functions of the form $x_{i_0 \dots i_{k-2}, i_k^*}$, each with a finite support. Therefore, there is a subset B_{k-1, i_k^*} of ω_{k-1} such that its complement has cardinality smaller than $\aleph_{k-1}$ and for each $i_{k-1} \in B_{k-1, i_k^*}$ for all $i_0, \dots, i_{k-2}$ none of the functions $x_{i_0 \dots i_{k-2}, i_k^*}$ contain i_{k-1} in any of the subsets in their support.

Iterating, we build a family of sets $B_{r, i_{r+1}^*, \dots, i_k^*}$, $r \le k$, such that for each $i_r \in B_{r, i_{r+1}^*, \dots, i_k^*}$ and for all $i_0, \dots, i_{r-1} \in P_{r-1}$, none of the functions $x_{i_0 .. i_{r-1}, i_{r+1}^* .. i_k^*}$ contain i_r in any of the subsets in their support and so that $B_{r, i_{r+1}^*, \dots, i_k^*}$ has complement of size less than $\aleph_r$. Take $i_k^* \in B_k$, $i_{k-1}^* \in B_{k-1, i_k^*}$,

$\ldots, i_0^* \in B_{0,i_1^*,..,i_k^*}$. If we can show that $y_{i_0^*\ldots i_k^*} = 0$ for each such $i_0^* \ldots i_k^*$, we show $\overline{f}$ is F_n-equivalent to the constantly zero function on P_k and finish. We write $\boldsymbol{i}_*$ for $\langle i_0^* \ldots i_k^* \rangle$ and $\boldsymbol{i}_*^-$ for the first k-elements: $\langle i_0^* \ldots i_{k-1}^* \rangle$. By definition,

$$M_t \models Q((w_{\boldsymbol{i}_*}, 0), (w_{\hat{i}_0^*, i_1^* \ldots i_k^*, t}, 0), \ldots, (w_{\boldsymbol{i}_*^-, t}, 0), (v_{\boldsymbol{i}_*^-, t}, 0)).$$

We also have $x_{i_0^* \ldots i_{k-1}^*, \hat{i}_k}[i_0^* \ldots i_k^*, t] = 0, \ldots, x_{\hat{i}_0, i_1^* \ldots i_k^*}[i_0^* \ldots i_k^*, t] = 0$, since for all $0 \leq r \leq k$ the support of the function $x_{i_0^* .. \hat{i}_r^* .. i_k^*}$ does not include any $k+1$ tuple containing i_r^*.

Thus we have

$$M_t \models Q((w_{\boldsymbol{i}_*}, 0), (w_{\hat{i}_0, i_1^* \ldots i_k^*, t}, x_{\hat{i}_0, i_1^* \ldots i_k^*}), \ldots, (w_{\boldsymbol{i}_*^-, t}, x_{\boldsymbol{i}_*^-}), (v_{\boldsymbol{i}_*, t}, 0)).$$

Comparing this display with (**), which holds for all $i_0 \ldots i_k$, we conclude that $y_{i_0^* \ldots i_k^*} = 0$. □

Continuing the notation of the last lemma, we focus on a specific conclustion.

Corollary 6.7. *Let M be the standard model of ϕ_{k+2} of size $\aleph_k$. If $\neg E_k(s,t)$, the Galois types $(b_s/M; M_s)$ and $(b_t/M; M_t)$ are distinct. That is, b_s and b_t are in distinct orbits.*

We can now conclude, working with ϕ_k rather ϕ_{k+2}:

Proposition 6.8. *The class of models of ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.*

Proof. Let s, t be sequences in ${}^{\omega_{k-2}}2$ with $\neg E_{k-2}(s,t)$. By Corollary 6.7, the Galois types of b_s, b_t over the standard model M of size $\aleph_{k-2}$ are different. But, by Corollary 5.1, the Galois type of b_s is the same as the Galois type of b_t over any submodel $N \prec M$, $\|N\| \leq \aleph_{k-3}$, as b_s and b_t have the same syntactic type over N. □

This analysis shows the exact point that tameness fails. Grossberg pointed out that after establishing amalgamation in Section 3, non-tameness at some (μ, κ) could have been deduced from eventual failure of categoricity of the example and the known upward categoricity results [6, 13]. However, one could not actually compute the value of κ without the same technical work we used to show tameness directly. In addition, failure of categoricity itself is established using the Galois types constructed in Proposition 6.5.

By analyzing the proof of Proposition 6.5, one sees the following.

Corollary 6.9. *Let $\chi_0, \ldots \chi_k$ be a strictly increasing sequence of infinite cardinals. Then there is a model of ϕ_{k+2} of cardinality χ_k over which there are 2^{χ_k} Galois types. In particular, ϕ_{k+2} is unstable in every cardinal greater than $\aleph_k$.*

7. NUMBER OF MODELS

We showed in Section 6 that ϕ_k is not Galois-stable in $\aleph_{k-2}$ and above. We have shown that the models of ϕ_k have disjoint amalgamation and it easy to see that ϕ_k has arbitrarily large models. For any Abstract Elementary class satisfying these conditions, categoricity in λ implies Galois stability in μ for $\mathrm{LS}(\boldsymbol{K}) \leq \mu < \lambda$ [16, 1]. Thus we can deduce from Corollary 2.15 and Corollary 6.9:

Theorem 7.1. (1) *Let $k \geq 3$; ϕ_k is $\aleph_m$-Galois stable for $m \leq k-3$.*
(2) *Let $k \geq 2$; ϕ_k is not $\aleph_{k-1}$-categorical.*

We will apply the instability directly to refine this result by showing that if $\mu \geq \aleph_{k-2}$ and λ is the least cardinal with $\lambda^\mu < 2^\lambda$, then ϕ_k has 2^λ non-isomorphic models of cardinality λ. Under the weak generalized continuum hypothesis ($2^\mu < 2^{\mu^+}$), we get that ϕ_k has maximal number of models in every cardinal beginning with $\aleph_{k-1}$. Without WGCH, we obtain that ϕ_k is not categorical everywhere above and including $\aleph_{k-1}$, with the maximal number of models in arbitrarily large cardinalities.

Remark 7.2. *Our ϕ_k is not the same one as in Hart-Shelah. We have simplified the construction by using only one level. However, our models are definable in theirs. So the assertion* [9] *that the Hart-Shelah ϕ is $\aleph_{k-1}$-categorical is incorrect; the correct statement is for $\aleph_{k-2}$-categoricity. We discussed the source of the miscalculation in Section 2.*

We start with a link between many Galois types in our example and failure of categoricity.

Lemma 7.3. *Let $k \geq 2$. Let $M \models \phi_k$ be of size μ, and suppose that there is a set $X = \{b_s \mid s < 2^\mu\}$ such that the Galois types (b_s/M) are pairwise distinct; let $M_s = M(I \cup \{b_s\})$. Let λ be the least cardinal with $\lambda^\mu < 2^\lambda$. Then ϕ_k has 2^λ non-isomorphic models of size λ.*

Proof. We start by noting that there are 2^λ subsets of size λ of the set X. For a subset S of the set X of size λ, let M_S be a model of size λ with the spine $I(M) \cup \{b_s \mid s \in S\}$. Namely, M_S is a minimal disjoint amalgam of the models M_s, $s \in S$.

It is now easy to see that the models M_S, $M_{S'}$ are not isomorphic over M for $S \neq S'$: any isomorphism preserves the Galois type of all the elements b_s over M; so M_S, $M_{S'}$ realize distinct sets of Galois types over M. Thus, we get 2^λ models over M. It remains to note that since $\lambda^\mu < 2^\lambda$, we must have 2^λ non-isomorphic L-structures. □

In conjunction with Proposition 6.5 we get

Corollary 7.4. *Let $k \geq 2$, $\mu \geq \aleph_{k-2}$, and let λ be the least cardinal with $\lambda^\mu < 2^\lambda$. Then ϕ_k has 2^λ non-isomorphic models of cardinality λ.*

While we know from Theorem 7.1 that categoricity fails everywhere above $\aleph_{k-2}$, using the following lemma we can avoid the heavy machinery quoted in that theorem and prove directly that categoricity fails everywhere above $2^{\aleph_{k-2}}$.

Claim 7.5. *Suppose that ϕ_k is categorical in λ. Then ϕ_k is categorical in every $\mu < \lambda$.*

Proof. We know that every model of size λ has a solution by categoricity. We also have that every model of size μ can be extended to a model of size λ.

So the proof boils down to showing the following: if M, with $\|M\| = \mu$, is a submodel of N, with $\|N\| = \lambda$, and N has a solution, then M has a solution. Let $(\overline{g}, \overline{h})$ be a solution for N. It is tempting to take the restriction of $\overline{g}$ and $\overline{h}$ to the model M, but $\overline{g}(u)$ does not have to be in M for $u \in K(M)$. Indeed, it may happen that $g(u) = (u, x)$, where the support of the function x is not contained in $K(M)$. Let us denote by $g(u) \upharpoonright K(M)$ the pair $(u, x') \in G^*(M)$, where $x'(v) = x(v)$ for all $v \in K(M)$ and $x'(v) = 0$ otherwise.

Now we make the natural definition: let $h := \overline{h} \upharpoonright K(M)$; and for $u \in K(M)$ let $g(u) := \overline{g}(u) \upharpoonright K(M)$. It is easy to check that (g, h) is a solution for M. □

Let λ be the least cardinal with $\lambda^{\aleph_{k-2}} < 2^\lambda$, then $\lambda \leq 2^{\aleph_{k-2}}$. By Corollary 7.4 ϕ_k is not categorical in λ and so by Claim 7.5, ϕ_k is not categorical in any $\kappa \geq \lambda$. So without reliance on the Ehrenfeucht-Mostowski machinery necessary to prove Theorem 7.1, we see ϕ_k is not categorical in any κ with $\kappa \geq 2^{\aleph_{k-2}}$.

We close by formally stating our most complete results on the spectra of ϕ_k.

Corollary 7.6. *Let $k \geq 2$. The sentence ϕ_k is categorical in $\aleph_0, \ldots, \aleph_{k-2}$, is not categorical in every cardinality greater than or equal to $\aleph_{k-1}$, and has 2^λ models in some λ with $\aleph_{k-2} < \lambda \leq 2^{\aleph_{k-2}}$. Moreover, for any $\mu \geq \aleph_{k-1}$ there is $\lambda > \mu$ such that ϕ_k has 2^λ models of cardinality λ.*

If in addition $2^\mu < 2^{\mu^+}$ (WGCH) for all $\mu \geq \aleph_{k-2}$, then ϕ_k has 2^{μ^+} isomorphism classes in every $\mu^+ \geq \aleph_{k-2}$.

Proof. We have already established the claims in the first paragraph.

To prove the second, suppose $2^\mu < 2^{\mu^+}$ for all $\mu \geq \aleph_{k-2}$. Let μ be greater than or equal $\aleph_{k-2}$. For all $\lambda \leq \mu$ we have $\lambda^\mu = 2^\mu \geq 2^\lambda$. So μ^+ is the least candidate for λ with $\lambda^\mu < 2^\lambda$. By our assumption, we have

$$(\mu^+)^\mu = 2^\mu < 2^{\mu^+}.$$

By Corollary 7.4 we get the maximal number of non-isomorphic models in μ^+, and by Claim 7.5 ϕ_k is not categorical everywhere above $\aleph_{k-1}$. $\square$

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