# Lindenbaum's Lemma As An Axiom For Infinitary Logic

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# Topics:

- Deduction systems for logics of coalgebras of certain measurable polynomial functors on the category $\mathbf{Meas}$ of measurable spaces.

- The role of Lindenbaum's Lemma in these infinitary logics.

# Lindenbaum's Lemma: (1920's)

*Every consistent set of sentences can be enlarged to form a consistent and complete system.*

Published by Tarski in 1930.

# Stated for Finitary Consequence Operators:

if $\varphi \in Cn(X)$, then $\varphi \in Cn(Y)$ for some finite $Y \subseteq X$.

- implies: the union of a chain of consistent sets is consistent

Henkin 1953 –1955:
Stone's representation of Boolean algebras is equivalent (without choice) to the Gödel-Malcev completeness theorem.

# What is a coalgebra ?

Let $T : \mathcal{C} \to \mathcal{C}$ be a functor.

$T$-coalgebra $(A, \alpha)$: $\alpha$ is a $\mathcal{C}$-arrow of the form

$$A \xrightarrow{\alpha} TA$$

$A$ = state set

$\alpha$ = transition structure

# Morphism of $T$-Coalgebras

$$(A, \alpha) \xrightarrow{f} (B, \beta)$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ TA & \xrightarrow{Tf} & TB \end{array}$$

$$i.e. \quad \beta \circ f = Tf \circ \alpha$$

# Example 1: Directed Graphs / Modal frames

$(A, R)$ with $R \subseteq A \times A$.

Define $\alpha(x) = \{y : xRy\} \subseteq A$.

Then

$$A \xrightarrow{\alpha} \mathcal{P}A$$

is a $\mathcal{P}$-coalgebra, where

$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$$

is the powerset functor.

$$xRy \text{ iff } y \in \alpha(x).$$

# Example 2: Input-Output Automata

$TA = A^I \times O^I$, for some fixed sets

$I$ (= inputs) and $O$ (= outputs).

A $T$-coalgebra

$$A \longrightarrow A^I \times O^I$$

is a pair of functions

$$A \longrightarrow A^I , \qquad A \longrightarrow O^I ,$$

or equivalently

$A \times I \longrightarrow A$ state transition function, and

$A \times I \longrightarrow O$ output function.

# Other examples: data structures

lists, streams, stacks, trees,...

algebra constructs

coalgebra deconstructs

# The Category Meas

Objects: measurable spaces

$$\mathbb{X} = (X, \mathcal{A}_{\mathbb{X}}),$$

where $\mathcal{A}_{\mathbb{X}}$ is a $\sigma$-algebra of measurable subsets of $X$.

Arrows $(X, \mathcal{A}) \xrightarrow{f} (X', \mathcal{A}')$:

measurable functions $X \xrightarrow{f} X'$, i.e. $A \in \mathcal{A}'$ implies $f^{-1}(A) \in \mathcal{A}$

# Polynomial functors $T : \mathbf{Meas} \to \mathbf{Meas}$

are constructed from

- the identity functor $Id : \mathbb{X} \longmapsto \mathbb{X}$

  and

- constant functors (every $\mathbb{Y}$) $\longmapsto \mathbb{X}$

  by forming

- products $T_1 \times T_2 : \mathbb{X} \longmapsto T_1\mathbb{X} \times T_2\mathbb{X}$,

- coproducts $T_1 + T_2 : \mathbb{X} \longmapsto T_1\mathbb{X} + T_2\mathbb{X}$,

  and

- exponential functors $T^E : \mathbb{X} \longmapsto (T\mathbb{X})^E$ with fixed exponent $E$.

# Measurable polynomial functors:

Constructible using also

$$\Delta T : \mathbb{Y} \longmapsto \Delta(T\mathbb{Y})$$

where $\Delta\mathbb{X}$ is the space of all probability measures on $\mathbb{X}$.

Measure: $\mu : \mathcal{A}_{\mathbb{X}} \to [0, \infty]$ is countably additive with $\mu(\emptyset) = 0$.

- Countably additive:
  $\mu(\bigcup_n A_n) = \sum_0^\infty \mu(A_n)$ if $A_n$'s pairwise disjoint.
- Probability measure: $\mu(X) = 1$.

The $\sigma$-algebra on $\Delta\mathbb{X}$ is generated by the sets

$$\beta^p(A) = \{\mu \mid \mu(A) \geqslant p\}$$

where $A \in \mathcal{A}_{\mathbb{X}}$ and $p \in [0, 1] \cap \mathbb{Q}$.

**Theorem [L. Moss and I. Viglizzo, *Inform. & Comp. 2006*]**

*For any measurable polynomial $T$, there exists a final $T$-coalgebra $\mathbb{X}_{\text{final}}$*

for each $T$-coalgebra $\mathbb{Y}$ there is a unique morphism

$$\mathbb{Y} \xrightarrow{!} \mathbb{X}_{\text{final}}$$

Motivation: "universal type spaces" in game-theoretic economics.

Moss-Viglizzo Construction: model-theoretic

# Syntax for fixed $T$

(cf. B. Jacobs, *Many-sorted coalgebraic modal logic*, 2001)

- Ingredient: any functor involved in formation of $T$, or $Id$.
- Ing $T$: the graph of ingredients of $T$ (multi-edged labelled directed)

1. $S_1 \times S_2 \overset{\mathsf{pr}_j}{\rightsquigarrow} S_j$ for $j \in \{1,2\}$;
2. $S_1 + S_2 \overset{\mathsf{in}_j}{\rightsquigarrow} S_j$;
3. $S^E \overset{\mathsf{ev}_e}{\rightsquigarrow} S$ for all $e \in E$;
4. $\Delta S \overset{\geqslant p}{\rightsquigarrow} S$ for $p \in [0,1]_{\mathbb{Q}}$;
5. $Id \overset{\mathsf{next}}{\rightsquigarrow} T$.

# Many-sorted formulas

Notation: $\varphi : S$ means $\varphi$ is a formula of sort $S \in \mathsf{lng}\, T$

1. $\bot_S : S$
2. If $\varphi_1 : S$ and $\varphi_2 : S$, then $\varphi_1 \to \varphi_2 : S$
3. $A : \mathbb{X}$ if $A \in \mathcal{A}_{\mathbb{X}}$ or $A$ is a singleton subset of $\mathbb{X}$
4. If $S \overset{\kappa}{\rightsquigarrow} S'$ in $\mathsf{lng}\, T$ with $\kappa \neq (\geqslant p)$, and $\varphi : S'$, then $[\kappa]\varphi : S$
5. If $\Delta S \in \mathsf{lng}\, T$ and $\varphi : S$, then $[\geqslant p]\varphi : \Delta S$ for any $p \in [0, 1] \cap \mathbb{Q}$

Probability modality: $[\geqslant p]\varphi$ is read “the probability is at least $p$ that $\varphi$”.

# Satisfaction relation

For a $T$-coalgebra $(\mathbb{X}, \alpha)$, define

$$\alpha, x \models_S \varphi$$

for $x \in S\mathbb{X}$, and $\varphi : S$. Put $[\![\varphi]\!]_S^\alpha = \{x \mid \alpha, x \models_S \varphi\}$.

$$\alpha, x \not\models_S \bot_S$$

$$\alpha, x \models_S \varphi_1 \to \varphi_2 \quad \text{iff} \quad \alpha, x \models_S \varphi_1 \text{ implies } \alpha, x \models_S \varphi_2$$

$$\alpha, x \models_{\mathbb{Y}} A \quad \text{iff} \quad x \in A$$

$$\alpha, x \models_{S_1 \times S_2} [\mathsf{pr}_j]\varphi \quad \text{iff} \quad \alpha, \pi_j(x) \models_{S_j} \varphi$$

$$\alpha, x \models_{S_1 + S_2} [\mathsf{in}_j]\varphi \quad \text{iff} \quad x = in_j(y) \text{ implies } \alpha, y \models_{S_j} \varphi$$

$$\alpha, f \models_{S^E} [\mathsf{ev}_e]\varphi \quad \text{iff} \quad \alpha, f(e) \models_S \varphi$$

$$\alpha, x \models_{Id} [\mathsf{next}]\varphi \quad \text{iff} \quad \alpha, \alpha(x) \models_T \varphi$$

$$\alpha, \mu \models_{\Delta S} [\geqslant p]\varphi \quad \text{iff} \quad \mu([\![\varphi]\!]_S^\alpha) \geqslant p$$

# Semantic Consequence Relations

- Local
  $\Gamma \models^{\alpha}_{S} \varphi$ : $\quad \alpha, x \models_S \Gamma$ implies $\alpha, x \models_S \varphi$, $\quad$ all $x \in S\mathbb{X}$.
- Global
  $\Gamma \models_{S} \varphi$ : $\quad \Gamma \models^{\alpha}_{S} \varphi$ $\quad$ for all $T$-coalgebras $\alpha$
- $Conseq^{\alpha}_{T} = \{ \models^{\alpha}_{S} \mid S \in \mathsf{lng}\, T \}$
- $Conseq_{T} = \{ \models_{S} \mid S \in \mathsf{lng}\, T \}$

# Role of Proof Theory

to give a syntactic characterisation of the many-sorted system

$$Conseq_T = \{ \models_S \mid S \in \mathsf{lng}\, T \}.$$

Should depend only on

- syntactic shape of formulas
- basic set-theoretic properties of sets of formulas.

Answer: $Conseq_T$ is the least Lindenbaum deduction system for $T$

# Strategy

1. Axiomatically define the notion of a $T$-deduction system :

$$D = \{ \vdash_S^D \mid S \in \mathsf{Ing}\, T \}$$

2. Define $D$ to be Lindenbaum if every $\vdash_S^D$-consistent set of formulas can be enlarged to a $\vdash_S^D$-maximal one.

3. Observe that
   - Each local system $Conseq_T^\alpha = \{ \models_S^\alpha \mid S \in \mathsf{Ing}\, T \}$ is a Lindenbaum deduction system.
   - The global system $Conseq_T = \{ \models_S \mid S \in \mathsf{Ing}\, T \}$ is a Lindenbaum deduction system, hence extends the least one.

4. If $D$ is Lindenbaum, construct a "canonical" $T$-coalgebra $(\mathbb{X}^D, \alpha^D)$ such that

$$\Gamma \models_S^{\alpha^D} \varphi \quad \text{iff} \quad \Gamma \vdash_S^D \varphi,$$

i.e. $Conseq_T^{\alpha^D} = D$.

# Conclusion

1. The local semantic consequence systems $Conseq_T^{\alpha}$ are exactly the Lindenbaum $T$-deduction systems.

2. The global semantic consequence system $Conseq_T$ is the least Lindenbaum $T$-deduction system.

When $D$ is $Conseq_T$, $(\mathbb{X}^D, \alpha^D)$ is a final $T$-coalgebra.

# Rules for Deduction Systems

- Assumption Rule: $\varphi \in \Gamma \cup Ax_S$ implies $\Gamma \vdash_S \varphi$.
- Modus Ponens: $\{\varphi, \varphi \to \psi\} \vdash_S \psi$.
- Cut Rule: If $\Gamma \vdash_S \psi$ for all $\psi \in \Sigma$ and $\Sigma \vdash_S \varphi$, then $\Gamma \vdash_S \varphi$.
- Deduction Rule: $\Gamma \cup \{\varphi\} \vdash_S \psi$ implies $\Gamma \vdash_S \varphi \to \psi$.
- Constant Rule: If $\mathbb{X} \in \mathsf{Ing}\, T$, $\{\neg\{c\} \mid c \in X\} \vdash_{\mathbb{X}} \bot_{\mathbb{X}}$.
- Definite Box Rule: For each edge $S \overset{\kappa}{\rightsquigarrow} S'$ in $\mathsf{Ing}\, T$ with $\kappa$ definite, $\Gamma \vdash_{S'} \psi$ implies $\{[\kappa]\varphi \mid \varphi \in \Gamma\} \vdash_S [\kappa]\psi$ .
- Archimedean Rule: If $\Delta S \in \mathsf{Ing}\, T$, $\{\, [\geqslant q]\varphi \mid q < p\} \vdash_{\Delta S} [\geqslant p]\varphi$.
- Countable Additivity Rule:
  $\{\varphi_0, \ldots, \varphi_n, \ldots\} \vdash_S \psi$ implies
  $\{[\geqslant p](\varphi_0 \wedge \cdots \wedge \varphi_n) \mid n < \omega\} \vdash_{\Delta S} [\geqslant p]\psi$.

# Failure of Lindenbaum's Lemma

- $\mathbb{N}$ = the constant functor for the discrete space $\omega = \{0, 1, 2, \dots\}$
- $T$ = the exponential functor $\mathbb{N}^{\mathbb{R}}$
- Ing $T$ looks like $Id \overset{\text{next}}{\rightsquigarrow} T \overset{\text{ev}_r}{\rightsquigarrow} \mathbb{N}$
- $(r \mapsto n)$ is the formula $[\text{next}][\text{ev}_r]\{n\}$, of sort $Id$
- A $T$-coalgebra $(\mathbb{X}, \alpha)$ has a transition function $\alpha : X \to \omega^{\mathbb{R}}$, with

$$\alpha, x \models_{Id} (r \mapsto n) \text{ iff } \alpha(x)(r) = n.$$

- $\Gamma_{\mathbb{R}} = \{(r \mapsto n) \to \neg(s \mapsto n) \mid r, s \in \mathbb{R},\ r \neq s,\ n \in \omega\}.$
- $\Gamma_{\mathbb{R}}$ is unsatisfiable:

  if $\alpha, x \models_{Id} \Gamma_{\mathbb{R}}$, then $\alpha(x) : \mathbb{R} \to \omega$ is injective ☹

- There is a $T$-deduction system $D$ for which $\Gamma_{\mathbb{R}}$ is $\vdash^{D}_{Id}$-consistent.
- If $D$ was Lindenbaum, then $\Gamma_{\mathbb{R}} \subseteq x$ with $x$ being $\vdash^{D}_{Id}$-maximal.

  But then $\alpha^{D}, x \models_{Id} \Gamma_{\mathbb{R}}$ ☹

# Reference

Deduction Systems for Coalgebras Over Measurable Spaces.

*J. Logic & Computation*, to appear

`http://www.mcs.vuw.ac.nz/˜rob`