First- or Second-Order Logic?
Quine, Putnam and the Skolem-paradox*

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The Löwenheim-Skolem theorem has been the earliest of the great theorems of metalogic. In 1922, Skolem recognized the (apparent) paradox connected with it.[1] The (LS) theorem and the paradox had a long and important later history in model theory; but for philosophy, it had had a half century long dream of a sleeping beauty[2] until it was awakened by two fundamental papers of contemporary metaphysics: by Quine's "Ontological Relativity"[3] and by Putnam's "Models and Reality"[4]. The theorem itself says (in its original and simplest form) that if a set of propositions of some first-order language has a model, then it has a countable (i.e. finite or countable infinite) model, too. Consequently, if we know that a first-order theory has no finite models and we suppose that it is consistent (and therefore, by the completeness theorem, it has a model) it follows that our theory should have a countable infinite model, too. The paradox can be made clear by a special case. The first order theory of real numbers includes a special case of Cantor's theorem about power sets: we can prove within the theory that the set of all real numbers, i. e. the set of individual objects the theory is about is not countable. But by the Löwenheim-Skolem theorem we know that the theory has a model in which the universe, the set of all "real numbers" of the model is countable.

This is not a real antinomy but just an apparent paradox, of course; but it has the important moral for the mathematician that some of our concepts are not absolute but model-dependent: a set that is countable according the countability-concept of one model may be uncountable according to the concept of another model. In this case, the basic set is countable in the model of ZFC we live in but uncountable in the model we can build up following the demonstration of the LS theorem. (ZFC is the Zermelo-Fraenkel set theory with the axiom of choice.) The theorem is significant because it is a negative categoricity result: it shows that by first-order theories we can't characterize the objects we scrutinize (the models of our theories) up to isomorphism if our theory has uncountable models.

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* The research leading to this paper was supported by OTKA, projects No. T 34861 and TS 40899. I thank to Hajnal Andréka, István Németi Ildikó Sain (Budapest), and the members of the Logic Seminar of the Philosophical Institute of the Erlangen University for their comments and suggestions.

[1] The theorem was published in that general form which is relevant for us here by Th. Skolem, "Logisch-kombinatorische Untersuchungen …", *Videnskapsselskapets skrifter, I. Matematisk-naturvidenskabelig klasse* 4(1920), pp. 1–36. The paradox was presented by him in the paper "Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre", in: *Matemetikerkongressen in Helsingfors den 4–7. Juli 1922*, Akademiska Bokhandeln Helsinki, 1922, pp. 217–232.

[2] One of the rare discussions of the theorem is to be found in :*Academic Freedeom, Logic and Religion* (ed. Morton White, University of Pennnsylvania Press, Philadelphia, 1953). where the two speakers of a symposium about the ontological significance of the LS theorem, Berry and Myhill aggree that the theorem have *no* such significance. (Professor Christian Thiel (Erlangen) called my attention to this discussion.)

[3] In: *Ontological Relativity and Other Essays*, Columbia University Press, New York, 1969, pp. 26-68.

[4] *Journal of Symbolic Logic* 45(1980), pp. 464-482.

All that I'm saying now is formulated in the language of the naïve Platonist mathematician (NPM): e.g. that we live in a model of ZFC as formulated in the metalanguage (although we can't know in which one), that our theories have objects, etc. NPM believes that the real world is *identical* with some model $<V, \in>$ of MetaZFC and therefore, truth for some proposition φ means that $<V, \in> \models \varphi$. For NPM, the problem with the Skolem paradox is not that we may have rival theories contradicting to each other and we can't have any rational reason by which we could choose among them. For him, the problem is that we may have substantially different models for the very same theory and we can formulate the difference in the metalanguage but we can't express it in the language of the theory (just that's why the rival models are equally warranted models). The problem of equally warranted rival theories is well known for him at least from the discovery of non-Euclidean geometries. NPM knows that the case of the continuum hypothesis (CH) just repeats the situation with the different geometries: we have no reason to decide either for ZFC+CH or for ZFC+¬CH. Such rival theories give alternative pictures about the world (if we suppose with NPM that set theory is about the world in some way); but the novelty following from (this simple form of) the LS theorem is that neither of the alternative theories can describe the world in the sense that it would imply all the substantial properties of the world.

In Quine's "Ontological relativity" the LS theorem appears in the context of the indeterminacy of reference. His whole argument shows that semantic questions can't be put in an absolute way but only with reference to some background theory. So propositions like '*s* refers to Socrates', 'the extension of *A* consists of the animals', 'φ is true' are all context-dependent propositions in the sense that they depend on the background theory – but we can't fix the reference of the terms of the background theory in an absolute way, either. The LS theorem reinforces this consequence in that it shows that terms of a mathematical theory like 'number', 'set', 'countable' etc. may have very different interpretations if the background theory changes.

Putnam's "Models and Reality" connects the problem of equally warranted models with the problem of equally warranted theories. Let us suppose that we can extend our knowledge to an ideal theory T_I, i.e. to a theory that fits to all observational and theoretical constraints we may accept at any time. We can't get rid on this way from the problem that T_I may have different models, and consequently from that T_I may have different, even contradictory extensions by propositions undecidable in T_I. (E.g. if we extend it by the continuum hypothesis (CH) or by its negation.) Putnam concludes in "Models and Reality" that with respect of T_I , the question of the truth of such a proposition is *meaningless.*[5] I have just mentioned and not explained the argument this time; it is known in the recent literature as Putnam's model theoretic argument. His proposal is to give up realist semantics for a verificationist one, but not to give up realist mathematics for the intuitionistic one: we should understand the

[5] Professor Putnam has remarked to this point that he had changed his mind in this respect. See his "Paradox revisited II: Sets – A Case of All or None", in: Sher, G. – R. Tieszen (eds.): *Between Logic and Intuition.* Essays in Honor of Charles Parsons, Cambridge University Press, Cambridge, 2000, pp. 16–256. esp. pp.23-24., and "Wittgenstein, Realism and Mathematics" (manuscript), where he writes on p. 2.: " … crucial difficulties for the view that mathematical truth cannot transcend provability can be discovered from an examination of the ways in which mathematics is used in mathematical physics." I thank to Professor Putnam for this remark and for the sending of the manuscript.

language of “normal” mathematics as referring to operations, procedures, etc, and referring to objects not as they are in the Platonist heaven of mathematics but as they are given to us by procedures.

Some people, e.g. van Fraassen[6] interpreted Putnam’s model theoretic argument in a way that every proposition should be true which is modeled by any acceptable model (i.e. by any model that is coherent with our observational and operational constraints); therefore, the model-theoretic argument as a *reductio* of the metaphysical realist view should end with an explicit contradiction: CH is true because some models qualify it as true, but ¬CH is also true according to some other models (and it is demonstrable that no extension of the constraints by knowable true propositions can help). This interpretation has an apparent similarity to NPM’s view in that it uses the word ‘true’ in an unqualified way (contrary to Quine’s ‘true to a background theory’ or to Putnam’s internal realism). But the similarity is just apparent, because for NPM it is not the case that both CH and ¬CH are true, but that the propositions

$$\mathbf{M}_1 \models \text{CH}$$

and

$$\mathbf{M}_2 \models \neg\text{CH}$$

are both true (being $\mathbf{M}_1$ and $\mathbf{M}_2$ models of ZFC in which CH is true resp. false). $\mathbf{M}_1$ and $\mathbf{M}_2$ are inside models of ZFC relative to our real world $\langle V, \in \rangle$; they can’t be identical with the real world but it is possible that one of them is isomorphic with $\langle V, \in \rangle$. If $\mathbf{M}_1$ is isomorphic with the world then CH is true, if $\mathbf{M}_2$ is such, then ¬CH is true, and if none of them, then we can’t say anything again. Therefore, the Skolem argument even together with the belief that we can use the word ‘true’ in an absolute sense doesn’t lead necessarily to a contradiction. The Skolem paradox is not an antinomy in formal logic – neither for Putnam nor for NPM.

The key point of the difference between NPM and Putnam lies somewhere else. For Putnam_{1980},[7] for Dummett[8] and for several other philosophers propositions whose truth or falsity is unknowable for reasons of principle are meaningless – and CH is a very good example for such propositions. NPM says it is all philosophical bullshit – he knows very well what does he mean by CH. On this point, I need to stress that I’m not identical or even isomorphic with NPM – I rather suspend my judgment. I just want to quote some argument from NPM in favor of his view.

If a sentence of mathematics is meaningless then it can’t have a truth-value. But NPM has good reasons to believe that at least some of his undecidable propositions have a truth-value in the absolute sense. Let ‘ P_2 ’ denote the set of second-order Peano axioms, ‘$\Rightarrow_2$’ the (semantic) consequence relation of standard second-order logic and Con(ZFC) the formula of first-order arithmetic that expresses the consistency of ZFC by a fixed but arbitrary Gödel numbering. The sentence

(A) $P_2 \Rightarrow_2 \text{Con (ZFC)}$

[6] “Elgin on Lewis’s Putnam’s Paradox”, *Journal of Philosophy* XCIV(1997), pp. 85-87.
[7] I. e. Putnam in “Models and Reality”; but cf. again footnote 5.
[8] For his view see *The Logical Basis of Metaphysics* (Duckworth, London, 1991, e.g. p. 316.)

is for NPM either true or false like any correct mathematical sentence that we can formulate within the language of MetaZFC – it is just a presupposition connected with the use of this language. But in this case, NPM's conviction that (A) should have a truth-value has some clear intuitive grounds – much more clear ones at least than in the case of CH or even in the case of the axiom of choice. Namely, Con(ZFC) is an arithmetical proposition, although not a very simple one: it contains quantifications that prevent us to verify it by simple calculation. (More exactly, it is a Π_2 proposition.) P_2 is a *categorical* system of axioms, and this fact entails that we can take any model $\mathbf{M}_{P_2}$ of it (in fact, P_2 has only one model up to isomorphism) and put the question whether this model makes Con(ZFC) true or not; this question will be equivalent with the question of truth of (A). It seems that this question can be translated into a question about a potentially infinite series of simple signs on an infinite tape. We can say in a sufficiently clear sense that (A) is in principle – although not in practice – decidable.

But the example of (A) differs e.g. from the problem of twin primes that Dummett uses as his example in the *Logical Basis of Metaphysics.* The truth-value of (A) is unknowable in a definite sense because of the Second Incompleteness Theorem (assumed that MetaZFC is consistent). To see this, let us translate the axioms P_2 into truths of first-order ZFC on the usual way; let us denote their conjunction by P_2^*. Con(ZFC) will be translated into some proposition Con(ZFC)* of first-order set theory, too. The truth-value of Con(ZFC)* must depend on the interpretation of the non-logical constant of the language of set theory ($\in$), because otherwise this formula (or its negation) would be a semantical consequence of ZFC in first-order logic, and therefore, by completeness, it would be deducible in ZFC. Consequently, the truth of Con(ZFC)*, and hence the truth of (A) depends on set-theoretic properties of our world $<V, \in>$ that are undecidable in (Meta)ZFC – otherwise, we could prove the consistency of ZFC (or its contrary).

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So NPM is a metaphysical realist: he believes (for good or wrong reasons) that there are unanswerable but meaningful questions – like the question of the truth-value of (A). However, we shall see from this very example why does he agree with Quine and Putnam in another important question connected with the LS theorem – and this is the alternative of first- and second-order logic. Maybe he has even better arguments for choosing first-order logic.

LS is the single negative theorem of metalogic in which the fact that we work in first-order logic plays an apparently substantial role. The incompleteness theorems infect by incompleteness any theory whose language can express first-order Peano arithmetic. The Church-Turing theorem infects by undecidability every theory the decidability of which can be reduced to the stop problem of Turing machines – and the case is similar with Tarski's theorem about the undefinability of truth. But with LS as a negative categoricity result, this is not the case. If we extend our first-order logic to second-order logic,[9] the impossibility of a categorical arithmetic disappears; even

[9] A terminological digression: by second-order logic I mean here standard second-order logic with semantic consequence relation defined on the natural way – unlike Putnam, who in "Models and Reality"

the second-order version of ZFC is very close to being categorical. It leaves only one question open: how large is the world?[10]

The categoricity of second-order arithmetic may awake the illusion that accepting P_2 **,** we fix the truth-value of any arithmetical proposition, or in other words, a propositon is a truth of arithmetic if and only if it is a semantical consequence of P_2 , and this is in principle a finaél answer to any question concerning arithmetic. But this is just an illusion, as a little extension of the above argumentation shows. In fact, (A) reproduces the Skolem paradox in a funny way. Being P_2 a complete, even a categorical system of axioms, it determines the truth-value of any arithmetical formula, inclusive that of Con(ZFC). Therefore, Con(ZFC) should have an absolute truth-value. How is it possible, then, that Con(ZFC) may have different truth-values in different models as we observed above?

Well, Con(ZFC) has an absolute truth-value *relative to the given world* $<V, \in>$. In other words, categoricity implies that in any model of P_2 , Con(ZFC) will have the same truth-value – in any model of P_2 *constructed within the same world of sets.* But if God changes the world $<V, \in>$ around us for another model of ZFC, the truth-value of ZFC may get changed; and as we learned above, God can do this alteration without that we observe it on any – intellectual or perceptual – way. We could have a categorical arithmetic but we don't fix the truth-value of each arithmetical proposition by that – because we can't fix the world of sets we live in.

Therefore, by changing our logic to the second-order one we don't get rid of the discomfort that the impossibility of a categorical arithmetic, the unprovability of consistency etc. causes – we just import the problems into the logical framework. On the very end, the logical consequence relation $\Rightarrow_2$ depends from $<V, \in>$ on a way that we can't control – it is $<V, \in>$-dependent whether P_2 does entail Con(ZFC) or not.

Let us accept a rather common but suitable picture about what should we expect from logic – to tell the truth, the picture will involve some realist commitment. Pictures are not good as philosophical arguments; but sometimes they are comfortable tools to illuminate our wishes, fears, expectations etc. The world is large and difficult, full of unsolved, maybe unsolvable problems, full of information and we can comprehend a very little part of it, it is full of phenomena we can't keep under our control. But we have logic to elaborate the little information we can reach, and we can keep our logic pretty well under control – as completeness and compactness shows. A change from first-order to second-order involves the loss of this control for just apparent advantages.

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identifiessecond-order logic with Henkin-style second-order logic that allows partial models. For me – as well as for NPM – logic is a language together with the rules for its interpretation; the language does not determinate in itself how it should be interpreted. Consequence should be characterized primarily on semantic way, with reference to models for the language. From a Platonist point of view, it is not a crucial problem in itself that the consequence relation gained on this way is not finitely axiomatizable; we know by the definition what logical consequence is and it is just another question that we don't have a complete method to reach the consequences. (Cf. S. Shapiro: *Foundations witjout Foundationalism*, Oxford University Press, Oxford etc., 1991, passim.) Second-order logic with interpretation rules that don't allow partial models – i.e. standard second-order logic – is therefore a possible candidate for our logical framework – not a good one, however.

[10] See E. Zermelo: "Über Grenzzahlen und Mengenbereiche", *Fundamenta Mathematicae*, 16(1930), pp. 29-47.