Ackermann's Function Is Not Primitive Recursive

Ackermann's function was defined in Example 3.3.1[1] by the three recursion equations

$$\begin{aligned}
&\text{(i) } H(0, m) &&= m + 1\\
&\text{(ii) } H(n + 1, 0) &&= H(n, 1)\\
&\text{(iii) } H(n + 1, m + 1) &&= H(n, H(n + 1, m))
\end{aligned}$$

In Exercise 3.3.3 it was shown that the function is total. Example 5.1.6 and Exercise 5.2.1 show, in effect, that it is partial recursive.[2] It follows that it is recursive. Our present goal is to show that Ackermann's function is not primitive recursive. The proof that we are about to give goes back at least to J. Glenn Brookshear, *Formal Languages, Automata, and Complexity* (Benjamin/Cummings: Redwood City, 1989), and before that to Peter J. Denning, Jack B. Dennis, and Joseph E. Qualitz, *Machines, Languages, and Computation* (Prentice–Hall: Englewood Cliffs, New Jersey, 1978).

Theorem. Ackermann's function $H(n, m)$ is not primitive recursive.

Our proof proceeds by way of six lemmas. The first five lemmas are quite easy. We merely state them and leave their proofs as exercises. Most of the work goes into proving the sixth lemma, from which Theorem follows immediately.

Lemma 1. For all $n \in \mathcal{N}$ we have that both

$$H(1, n) = n + 2 \qquad \text{and} \qquad H(2, n) = 2n + 3$$

Lemma 2. For all $n, m \in \mathcal{N}$ we have that $H(n, m) \geq m + 1$.

Lemma 3. For all $n, m \in \mathcal{N}$ we have that

(a) $H(n, m) < H(n, m + 1)$
(b) $H(n, m + 1) \leq H(n + 1, m)$
(c) $H(n, m) < H(n + 1, m)$

[1] All such references are to R. Gregory Taylor, *Models of Computation and Formal Languages* (Oxford University Press: New York, 1998).
[2] Essentially, Example 5.1.6 shows that Ackermann's function is *register-machine-computable*, where the register machine is an alternative machine model of computation introduced in § 5.1. It turns out, however, that the register-machine-computable functions are precisely the partial recursive functions.

Note that Lemma 3(a) entails that $H(n, m)$ is strictly monotone increasing in its second argument whereas Lemma 3(c) entails that it is strictly monotone increasing in its first argument.

Lemma 4. For all $n_1, n_2, m \in \mathcal{N}$ we have that

$$H(n_1, m) + H(n_2, m) < H(max(n_1, n_2) + 4, m)$$

Lemma 5. For all $n, m \in \mathcal{N}$ we have that

$$H(n, m) + m < H(n + 4, m)$$

See the solutions to Exercise 3.3.2 and to Exercises 1 through 4 below for proofs of Lemmas 3.1 through 3.5.

The bulk of our proof consists in proving

Lemma 6. Let $f(n_1, n_2, \ldots, n_k)$ be any k-ary primitive recursive function with $k \geq 0$. Then there exists natural number J such that, for arbitrary k-tuple $(n_1, n_2, \ldots, n_k)$, we have that

$$f(n_1, n_2, \ldots, n_k) < H(J, \Sigma(n_1, n_2, \ldots, n_k))$$

where we are writing $\Sigma(n_1, n_2, \ldots, n_k)$ for the sum $n_1 + n_2 + \ldots + n_k$. (We set $\Sigma(n_1, n_2, \ldots, n_k) = 0$ if $k = 0$.)

As mentioned earlier, the interest of Lemma 6 lies in the fact that Theorem 3.13 is an easy consequence. So, before proving Lemma 6 itself, we show that its truth implies that Ackermann's function is not a primitive recursive function.

Our demonstration is indirect. We start by assuming, for the sake of obtaining a contradiction, that $H(n, m)$ is primitive recursive. But then

$$f(n) =_{\text{def.}} \mathbf{Comp}[A, p_1^1, p_1^1](n) = H(n, n)$$

is also primitive recursive by closure under composition. By Lemma 6 and letting k=1, it follows that there exists natural number J such that, for arbitrary n, we have

$$f(n) < H(J, \Sigma(n)) = H(J, n)$$

Perversely, we set n=J so that $f(J) < H(J, J)$. But by definition $f(J) = H(J, J)$, which is a patent contradiction. We must conclude that $H(n, m)$ is not primitive recursive after all.

It remains to us now to prove Lemma 6, which requires some work and which will involve appeals to Lemmas 1 through 5.

Proof of Lemma 6. Our proof proceeds by induction on the number of times that **Comp** and **Pr** are used in the definition of the primitive recursive function $f(n_1, n_2, \ldots, n_k)$, or $f(\vec{n})$, mentioned in the hypothesis of the statement of Lemma 6. The base case amounts to supposing that $f(\vec{n})$ is one of our initial functions. Afterward, two inductive cases correspond to **Comp** and **Pr**, respectively.

Base Case Function $f(\vec{n})$ is either the successor function $succ(n)$, a constant-0 function C_0^k, or a projection function p_j^k.

Subcase (1) Function $f(\vec{n})$ is $succ(n)$ so that k=1. Letting J=1, we have that, by (i) and Lemma 3(c),

$$f(n) = succ(n) = n + 1 = H(0, n) < H(1, n) = H(J, n) = H(J,\ \Sigma(\vec{n}))$$

Subcases (2) and (3) are left as easy exercises. See Exercises 5 and 6 below.

Inductive Case Function $f(\vec{n})$ is not an initial function.

Subcase (1) Suppose that $f(\vec{n})$ is of the form

$$\mathbf{Comp}[h, g_1, g_2, \ldots, g_m](\vec{n}) = h(g_1(\vec{n}), g_2(\vec{n}), \ldots, g_m(\vec{n}))$$

where h, g_1, g_2, $\ldots$, g_m are all primitive recursive functions of appropriate "arities." By induction hypothesis, there exists a natural number J_0 with $h(\vec{n}) < H(J_0, \Sigma(\vec{n}))$ for arbitrary $\vec{n} \in \mathcal{N}^m$. Similarly, there exist natural numbers J_1, J_2, $\ldots$, J_m such that, for $1 \le i \le m$, $g_i(\vec{n}) < H(J_i, \Sigma(\vec{n}))$ for arbitrary $\vec{n} \in \mathcal{N}^k$. Then

$f(\vec{n}) = h(g_1(\vec{n}), g_2(\vec{n}), \ldots, g_m(\vec{n}))$

$< H(J_0, \sum_{i=1}^{m} g_i(\vec{n}))$ by choice of J_0

$< H(J_0, \sum_{i=1}^{m} A(J_i, \Sigma(\vec{n})))$ by choice of the J_i and strict monotonicity

$< H(J_0, H(J^*, \Sigma(\vec{n})))$, where $J^* =_{\text{def.}} max(J_1,\ J_2,\ \ldots,\ J_m) + 4\cdot(m-1)$

by Lemma 4 applied $m-1$ times

$< H(J_0, H(J^*+1, \Sigma(\vec{n})))$ by strict monotonicity

$\le H(J_0, H(max(J_0, J^*)+1, \Sigma(\vec{n})))$ by strict monotonicity

$\le H(max(J_0, J^*), H(max(J_0, J^*)+1, \Sigma(\vec{n})))$ by strict monotonicity

$= H(max(J_0, J^*) + 1,\ \Sigma(\vec{n})+1)$ by (iii)

$\le H(max(J_0, J^*) + 2,\ \Sigma(\vec{n}))$ by Lemma 3(b)

Thus we see that, setting $J = max(J_0, J^*) + 2$, it follows that

$$f(\vec{n}) < H(J, \Sigma(\vec{n}))$$

and we are done.

Subcase (2) We suppose that $f(\vec{n})$ is a $(k+1)$-ary function of the form $\mathbf{Pr}[h,g](\vec{n})$ where h and g are primitive recursive functions of appropriate "arities." That is, we assume that $f(\vec{n})$ is defined by

$$f(n_1, \ldots, n_k, 0) = h(n_1, \ldots, n_k)$$
$$f(n_1, \ldots, n_k, m+1) = g(n_1, \ldots, n_k, m, f(n_1, \ldots, n_k, m))$$

By induction hypothesis, there exists natural number J_h with $h(\vec{n}) < H(J_h, \Sigma(\vec{n}))$ for all $\vec{n} \in \mathcal{N}^k$. Similarly, there exists natural number J_g with $g(\vec{n}, m, p) < H(J_g, \Sigma(\vec{n}, m, p))$ for all $\langle \vec{n}, m, p\rangle \in \mathcal{N}^{k+2}$. Furthermore, by strict monotonicity, we may assume that $J_g > 2$. Our goal is now to show that there exists a J such that

(*) $$f(\vec{n}, m) < H(J,\ \Sigma(\vec{n}, m)) = H(J,\ \Sigma(\vec{n}) + m)$$

for all $\langle \vec{n}, m\rangle \in \mathcal{N}^{k+1}$. It is sufficient to let J be any number larger than $max(J_h, J_g) + 4$. One then proves the desired inequality at (*) by induction on m. The proof is not really so hard and so we leave it as an exercise. Q.E.D.

Exercises

1. Prove Lemma 2.

2 (a) Prove Lemma 3(a).
  (b) Prove Lemma 3(b).
  (c) Prove Lemma 3(c).

3. Prove Lemma 4.

4. Prove Lemma 5.

5. Prove Subcase (2) of the Base Case in the proof of Lemma 6.

6. Prove Subcase (3) of the Base Case in the proof of Lemma 6.

7. Complete the proof of Subcase (2) of the Inductive Case in the proof of Lemma 6.

Solutions to Exercises for the Document Entitled "Ackermann's Function Is Not Primitive Recursive"

© R. Gregory Taylor

1. We present an argument by double induction.

   Main base case: Suppose that $n=0$. Then, by (i), $H(0,m)$ is defined as $m+1$ for $m\geq 0$.
   Main inductive case: Suppose that $n=k+1$. Within this case, we carry out an induction on A's second argument m, assuming as main induction hypothesis that $H(k,j)\geq j+1$ for all $j\geq 0$.
   Subordinate base case: Suppose that $m=0$. Then, by (ii), $H(k+1,0)$ is defined as $H(k,1)$, which, in turn, is assumed to be ≥ 2 by the main induction hypothesis. But from $H(k+1,0)\geq 2$ it follows that $H(k+1,0)\geq 1$.
   Subordinate inductive case: Suppose that $m=j+1$ and assume as subordinate induction hypothesis that $H(k+1,j)\geq j+1$. Then, by (iii), $H(k+1,j+1)$ is defined as $H(k,H(k+1,j))$. Working from the inside, $H(k+1,j)\geq j+1$ by the subordinate induction hypothesis, so that $H(k,H(k+1,j))\geq j+2$ by the main induction hypothesis. So we see that $H(k+1,\mathbf{j+1})=H(k,H(k+1,j))\geq \mathbf{j+2}$. Q.E.D.

2 (a) We present a proof of (a) by induction on n. First, if $n=0$, then we have by (i) that

$$H(0,m) = m+1 < m+2 = H(0,m+1)$$

Letting $n=k+1$, we have

$$\begin{aligned} H(k+1,m) &< H(k+1,m) + 1 & \\ &\leq H(k,H(k+1,m)) & \text{by Lemma 2} \\ &\leq H(k+1,m+1) & \text{by (iii)} \end{aligned}$$

(b) We present an argument by induction on second argument m. Letting $m=0$, we have

$$H(n,0+1) = H(n,1) = H(n+1,0) \qquad \text{by (ii)}$$

For the inductive case, let $m=j+1$ and assume as induction hypothesis that $H(n,j+1) \leq H(n+1,j)$. By Lemma 2, we have that $(j+1)+1 \leq H(n,j+1)$. It then follows, by the monotonicity of $H(n,m)$ in its second argument, that

$$H(n,(j+1)+1) \leq H(n,H(n,j+1))$$

Next, by the induction hypothesis and monotonicity in the second argument again, we have that

$$H(n,H(n,j+1)) \leq H(n,H(n+1,j))$$

Putting these together, we see that

$$\begin{aligned} H(n,(j+1)+1) &\leq H(n,H(n+1,j)) & \\ &= H(n+1,j+1) & \text{by (iii)} \end{aligned}$$

and we are done.

(c) This follows immediately from (a) and (b).

3. The proof may be structured as a string of inequalities. To start,

$H(n_1,m) + H(n_2,m) \leq H(max(n_1,n_2),m) + H(max(n_1,n_2),m)$	by Lemma 3(c)
$= 2 \cdot H(max(n_1,n_2),m)$	
$< 2 \cdot H(max(n_1,n_2),m) + 3$	
$= H(2,H(max(n_1,n_2),m))$	by Lemma 1
$< H(2,H(max(n_1,n_2)+3,m))$	by strict monotonicity in both arguments (Lemmas 3(a) and 3(c))
$\leq H(max(n_1,n_2)+2,H(max(n_1,n_2)+3,m))$	by Lemma 3(c)
$= H(max(n_1,n_2)+3,m+1)$	by (iii)
$\leq H(max(n_1,n_2)+4,m)$	by Lemma 3(b)

4. This follows rather directly from Lemma 4. We may write

$H(n,m) + m < H(n,m) + m + 1$	
$= H(n,m) + H(0,m)$	by (i)
$< H(n+4,m)$	

5. We are supposing that $f(\vec{n})$ is a constant function. There are two sub-subcases to consider. First, suppose that $f(\vec{n})$ is $C_0^k(\vec{n})$ with k=0. Letting J=0, we have

$$f(\vec{n}) = C_0^0(\vec{n}) = 0 < 1 = H(0,0) = H(J, \Sigma(\vec{n}))$$

by (i) and the fact that $\Sigma(\vec{n})$=0.

On the other hand, suppose that $f(\vec{n})$ is $C_0^k(\vec{n})$ with k>0. Again, we set J=0 so that

$$f(\vec{n}) = C_0^k(\vec{n}) = 0 \leq \Sigma(\vec{n}) < \Sigma(\vec{n})+1 = H(0,\Sigma(\vec{n})) = H(J, \Sigma(\vec{n}))$$

by (i).

6. Suppose that $f(\vec{n})$ is a projection function $p_j^k(\vec{n})$ with $k \geq 1$ and $1 \leq j \leq k$. Letting J=0, we may write

$$f(\vec{n}) = p_j^k(\vec{n}) = n_j \leq \Sigma(\vec{n}) < \Sigma(\vec{n})+1 = H(0,\Sigma(\vec{n})) = H(J, \Sigma(\vec{n}))$$

by (i).

7. First, suppose that m=0. Then we have that

$f(\vec{n},0) \leq f(\vec{n},0) + \Sigma(\vec{n})$	
$= h(\vec{n}) + \Sigma(\vec{n})$	by the definition of $f(\vec{n},0)$
$< H(J_h,\Sigma(\vec{n})) + \Sigma(\vec{n})$	by choice of J_h
$< H(J_h+4,\Sigma(\vec{n}))$	by Lemma 5
$< H(J,\Sigma(\vec{n}))$	by definition of J and strict monotonicity
$= H(J,\Sigma(\vec{n})+0)$	

Next, suppose that m=k+1 and assume as induction hypothesis that $f(\vec{n},k) < H(J, \Sigma(\vec{n})+k)$. Then

$f(\vec{n},k+1) = g(\vec{n},k,f(\vec{n},k))$	by definition of $f(\vec{n},k+1)$
$< H(J_g,\Sigma(\vec{n})+k+f(\vec{n},k))$	by choice of J_g
$< H(J_g,\Sigma(\vec{n})+k+1+f(\vec{n},k))$	
$= H(J_g,H(0,\Sigma(\vec{n})+k)+f(\vec{n},k))$	by (i)
$< H(J_g,H(0,\Sigma(\vec{n})+k)+H(J,\Sigma(\vec{n})+k))$	by hypothesis of induction

	and strict monotonicity
$< H(J_g,H(J,\Sigma(\vec{n})+k)+H(J,\Sigma(\vec{n})+k))$	by strict monotonicity since $J>0$
$= H(J_g,2\cdot[H(J,\Sigma(\vec{n})+k)])$	
$< H(J_g,2\cdot[H(J,\Sigma(\vec{n})+k)]+3)$	
$= H(J_g,H(2,H(J,\Sigma(\vec{n})+k))$	by Lemma 1
$< H(J_g,H(J_g+1,H(J,\Sigma(\vec{n})+k))$	by strict monotonicity since $J_g>2$
$= H(J_g+1,H(J,\Sigma(\vec{n})+k)+1)$	by (iii)
$\leq H(J_g+2,H(J,\Sigma(\vec{n})+k))$	by Lemma 3(b)
$< H(J-1,H(J,\Sigma(\vec{n})+k))$	by strict monotonicity since $J>max(J_h,J_g)+4$
$= H(J,\Sigma(\vec{n})+(k+1))$	by (iii) since $J \neq 0$