Abstract model theory for extensions of modal logic

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Stanford, May 13, 2008

Largely based on joint work with Johan van Benthem and Jouko Väänänen

Abstract model theory

- Abstract model theory (AMT) studies logics and their properties on an abstract level.

Lindström's theorem

Every proper extension of FO lacks either Löwenheim-Skolem or Compactness.

Lindström's theorem (another version)

Every 'effective' proper extension of FO lacks Löwenheim-Skolem or is highly undecidable (Π^1_1-hard) for validity.

- Löwenheim-Skolem: every set of formulas with an infinite model has a countable model
- Compactness: if every finite subset of a set of formulas is satisfiable, then the set itself is satisfiable.

The limited scope of AMT

- In traditional Abstract Model Theory, most attention is on extensions of FO.
- Many results depend on coding arguments that seem to require on the expressive power of full first-order logic.
- From a CS perspective, modal logic would be a more useful starting point than FO.

Topic of this talk:

Abstract model theory for extensions of modal logic.

- Lindström's theorem will be the guiding example.

Modal logic

Modal logic: a language for describing points in relational structures.

- Syntax: $\phi \ ::= \ p \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi$
- Semantics:

$$
\begin{array}{lll}
M, w \models p & \textit{iff} & w \in P^M \\
M, w \models \neg\phi & \textit{iff} & M, w \not\models \phi \\
M, w \models \phi \wedge \psi & \textit{iff} & M, w \models \phi \text{ and } M, w \models \psi \\
M, w \models \Diamond\phi & \textit{iff} & \text{there is a } v \text{ such that } R^M wv \text{ and } M, v \models \phi
\end{array}
$$

- Modal formulas can be translated to first-order formulas $\phi(x)$ in one free variable.
  For example, $p \wedge \Diamond q$ corresponds to $Px \wedge \exists y.(Rxy \wedge Qy)$.

Modal logic (ii)

Modal logic can be seen as an elegant variable-free notation for a fragment of FO.

Theorem (Van Benthem)

A first-order formula $\phi(x)$ —in the appropriate signature— is equivalent to the translation of a modal formula iff it is invariant for bisimulations.

Virtues of modal logic

Basic modal logic is/has

- Decidable (in fact, satisfiability is PSpace-complete)
- Finite model property
- Compactness
- Craig interpolation
- Bisimulation invariance
- ...

Examples of well-behaved extensions of ML:

- Modal μ-calculus
  Has all the good properties except Compactness
- Modal logic with counting modalities
  Has all the good properties except Bisimulation invariance

Extending modal logic

- The basic modal logic has little expressive power, but it is very well behaved, both complexity-wise and model theoretically.
- How far can we extend it while preserving the good properties?
- But first, what does it mean to "extend" modal logic?
- Three kinds of extensions:
  - axiomatic extensions,
  - language extensions, and
  - signature extensions.

1. Axiomatic extensions

- Often in applications of modal logic, we want to reason about a restricted class of structures, e.g, linear orders (flows of line) or finite trees (XML documents).
- This means adding axioms to the logic.
- The good properties of the basic modal logic may or may not survive.
- This has been explored extensively for some decades now.
- As an illustration, consider universal Horn conditions.

1. Axiomatic extensions (ct'd)

Definition

A universal Horn sentence is a FO sentence of the form

$$\forall\vec{x}(\phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi)$$

or

$$\forall\vec{x}(\phi_1 \wedge \cdots \wedge \phi_n \rightarrow \bot)$$

where $\phi_1, \ldots, \phi_n, \psi$ are atomic formulas.

Typical examples:

Transitivity	$Rxy \wedge Ryz$	$\rightarrow$	Rxz
Symmetry	Rxy	$\rightarrow$	Ryx
Totality	$\top$	$\rightarrow$	Rxy
Irreflexivity	Rxx	$\rightarrow$	$\bot$

1. Axiomatic extensions (ct'd)

Universal Horn classes form a sufficiently simple setting so that one may a reasonably complete picture.

Theorem (Marx and Venema, 1997)

Let K be any class of frames defined by universal Horn sentences. The modal logic of K has Craig interpolation.

Theorem (Hemaspaandra and Schnoor, 2008)

Le K be any class of frames defined by universal Horn sentences. Satisfiability of modal formulas on K is either (i) in NP or (ii) PSpace-hard (or worse).

(This dichotomy theorem comes with a concrete criterion, which is likely to yield a decision procedure.)

1. Axiomatic extensions (ct'd)

Remaining questions:

- Can we characterize, or even decide, which universal Horn sentences express modally definable properties?
- Can we characterize, or even decide, which universal Horn classes yield finitely axiomatizable modal logics?
- Can we characterize, or even decide, which universal Horn classes yield decidable modal logics?

2. Expressive extensions

- Instead of restricting the class of structures, we can make the language more expressive.
- Examples:
  - adding the global modality,
  - adding counting modalities,
  - adding second order ("propositional") quantifiers
  - adding fixed point operators (the modal μ-calculus)

2. Expressive extensions (ct'd)

- For many extensions of ML, analogues of the usual theorems (completeness, interpolation, complexity, etc.) have been proved.
- There are also some results of a more general nature.
- We will discuss this in more detail.

3. Signature extensions

- Sometimes, we want to describe more general type of mathematical structures.
- Two examples:
  - k-ary modalities (with $k+1$-ary accessability relations)
  - neighborhood models (in particular, topological spaces).
- Many positive results about modal logic generalize to other types of structures.
- There is a quite general perspective based on coalgebra.

3. Signature extensions (ct'd)

- Recall that a Kripke model is a structure of the form $M = (D, R, V)$ with $R \subseteq D \times D$ and $V : D \to 2^{PROP}$.
- Equivalently, we can write $M = (D, f)$ where $f : D \to \wp(D) \times \wp(PROP)$.
- Generalizing from this, let $\tau(X)$ be any term generated by the following inductive definition:

$$\tau(X) ::= X \mid A \mid \tau + \tau' \mid \tau \times \tau' \mid \tau^B \mid \wp(\tau)$$

  with A any set and B any finite set.
- Each such τ gives rise to a functor on Set, called a Kripke polynomial functor (KPF).
- A "τ-coalgebra" is a pair $M = (D, f)$ with $f : D \to \tau(D)$.
- Kripke models are coalgebras for one particular functor.

3. Signature extensions (ct'd)

- So, Kripke models are coalgebras of a specific functor.
- Another simple example: ternary Kripke models
- Further generalizations of the class of KPFs are possible, covering also, e.g., neighborhood models.
- With each KPF τ we can associate a "basic modal language".
- Various results for modal logic (e.g., decidability, finite axiomatization, Goldblatt-Thomason theorem) generalize to arbitrary KPFs. [Rößiger; Jacobs; Kurz & Rosicky]

There are many cross inter-relations between the three types of extensions of ML.

For some beautiful examples, see

> *Marcus Kracht and Frank Wolter (1997).* ***Simulation and Transfer Results in Modal Logic – A Survey****.*
> *Studia Logica 59: 149–177.*

From now on, we will focus on expressive extensions of modal logic.

Lindström's theorem

Lindström's theorem again

Every logic properly extending FO lacks either Löwenheim-Skolem or Compactness.

- Here, a "logic" means a language for defining classes of first-order structures.
  $\Rightarrow$ In this talk, we restrict attention to relational structures with only unary and binary relations.
- Some further regularity conditions are assumed:
  - Invariance for isomorphism
  - Closure under the Boolean operations
  - Closure under uniformly replacing one relation symbol by another (of the same arity)
  - Closure under relativisation by a unary relation symbol

A modal Lindström theorem

Lindström theorem for modal logic (Van Benthem 2006)

Every proper extension of ML lacks either Compactness or Bisimulation invariance.

This is a strengthening of usual bisimulation characterization:

Proof of Bisimulation Characterization from Lindström Theorem

Let L be the bisimulation invariant fragment of FO.

Then L extends ML, is bisimulation invariant and is compact.

But then L cannot be a *proper* extension of modal logic.

QED

Note: we are now concerned with languages for describing classes of pointed structures.

Modal Lindström theorem (ii)

Proof of the modal Lindström theorem (sketch)

- Suppose $L \supseteq ML$ satisfies Compactness and bisimulation invariance, and let $\phi \in L$. [TO PROVE: $\phi \in ML$]
- Pick a new proposition letter (unary predicate) p, and

$$\Sigma = \{p, \Box p, \Box\Box p, \ldots\}$$

- By bisimulation invariance, $\Sigma \models \phi \leftrightarrow \phi^p$
- By Compactness there is a $k \in \mathbb{N}$ such that $\bigwedge_{n \leq k} \Box^n p \models (\phi \leftrightarrow \phi^p)$
- This shows that ϕ only "sees" nodes at distance $\leq k$ from the starting node.
- ML can express all bisimulation invariant properties of finite depth. Hence, $\phi \in ML$.

Graded modal logic

Another example:

- Graded modal logic is modal logic with counting modalities

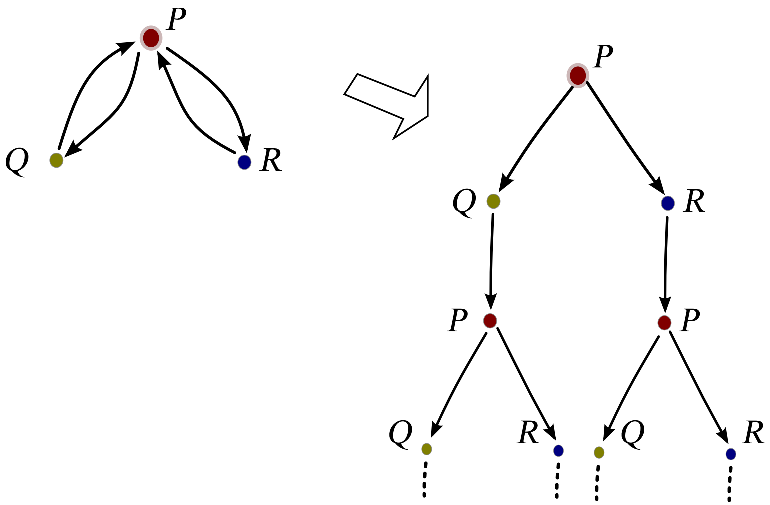
  $w \models \Diamond^{\geq k}\phi$ if w has at least k R-successors satisfy ϕ

- Graded modal formulas are not bisimulation invariant . . .

  . . . but they are invariant for . . .

  tree unraveling.

Tree unraveling



Graded modal logic (ii)

- **Question:** Is GML maximal with respect to tree unraveling invariance and compactness?
- **Answer:** No, take the extension with $\Diamond_{\aleph_1}$ ("uncountably many successors…").
- **Revised question** Is GML maximal with respect to tree unraveling invariance, compactness and Löwenheim-Skolem?
- **Answer: Yes!** (Van Benthem, tC and Väänänen, 2007)
- The proof is more difficult than for ML.
- Again, we obtain a preservation theorem as a **corollary**: GML is the tree-unraveling-invariant fragment of FO.

'Classical' AMT revisited

An AMT for languages below FO may not only help us understand modal languages better, it may also help us understand FO logic and its extensions better.

A. **The scope of Lindström's characterization**

B. **Restricted classes of structures, in particular trees**

The scope of Lindström's theorem

- Lindström's theorem singles out FO as being special only within the class of logics extending FO.
- For all we know, there might be other logics incomparable to FO and equally well behaved.
- Within how big a class of logics can we characterize FO in terms of Löwenheim-Skolem and Compactness?
- We give a positive and a negative result.

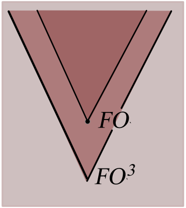
A positive result

For $k \in \mathbb{N}$, let FO^k be the *k*-variable fragment of FO.

Theorem (Van Benthem, tC and Väänänen, 2007)

For any logic L extending FO^3, the following are equivalent:

1. L has the Compactness and Löwenheim-Skolem properties
2. L is contained in FO.



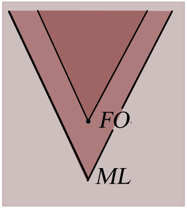
Proof: a careful analysis of the classic Lindström argument: the encoding only needs three variables.

A negative result

Let $ML^\bullet$ be modal logic extended with the following operator:
$w \models \bullet\phi$ iff w has infinitely many reflexive successors satisfying ϕ

Theorem (Van Benthem, tC and Väänänen, 2007)

$ML^\bullet$ is not contained in FO, but nevertheless satisfies Compactness, Löwenheim-Skolem, invariance for potential isomorphisms, PSPACE-decidability, finite axiomatizability, and Craig interpolation.



- **Open question:** Is every extension of FO^2 satisfying Compactness and Löwenheim-Skolem contained in FO?

Special classes of structures

What about special classes of structures?

- finite stuctures (relational DBs)
- trees (XML documents, nested words, computation trees, . . .)

Note:

- Compactness fails for FO on these structures.
- Löwenheim-Skolem is meaningless on finite structures.

Two questions:

(1) Find other properties that characterize FO (but which??)
(2) Consider fragments of FO that still well behaved.

Here: a positive result along (2) for trees.

Graded modal logic again

Take GML as a language for defining properties of (nodes in) trees.

- Very limited expressive power:
  - Formulas can only look downwards from any node
  - Formulas can only look finitely deep into the subtree
  - can only count successors up to a fixed finite number (viz. the largest index of a modal operator in the formula)
- On trees (possibly infinite but well-founded) GML has Compactness and Löwenheim-Skolem

Lindström Theorem for GML on trees (vB, tC and V, 2007)

On trees, GML is maximal w.r.t. Compactness and Löwenheim-Skolem.

More AMT on trees

Perhaps FO is not the most natural logic for describing trees.

The most well-behaved logic for describing trees seems to be Monadic Second-Order Logic (MSO).

- Evaluating MSO formulas in trees, or even structures of bounded tree-width, is in PTime (data-complexity) — Courcelle's theorem.
- Satisfiability for MSO formulas on trees, or even structures of bounded tree-width, is decidable – Rabin's theorem.
- MSO defines precisely the regular tree languages.

More AMT on trees (ct'd)

Question 1: Find a Lindström-style characterization of MSO on trees

Question 2: Is MSO finitely generated on trees? I.e., is there a finite set of generalized quantifiers $Q_1, \ldots, Q_n$ such that $FO(Q_1, \ldots, Q_n)$ and MSO have the same expressive power on trees?

Concerning Question 2,

- MSO is not finitely generated on arbitrary graphs (Hella 92)
- $FO(monTC) \subsetneq MSO$ on trees (tC and Segoufin 08).

Conclusion and outlook

The general theme: AMT for logics below/incomparable to FO.

Three more concrete lines of research:

1. Characterizing logics below/incomparable to FO.

   E.g., can we characterize the μ-calculus in terms of bisimulation and the finite model property?

2. The scope of Lindström's theorem

   Lindstrom's criteria characterize FO within the class of extensions of FO^3. How about extensions of FO^2?

3. Characterizing logics on specific classes of structures

   Largely open.