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A generalization of the incompleteness theorem

by

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The aim of this paper is to prove the following generalization of the Gödel incompleteness theorem (cf. [1] and [9]):

Let a formula Φ with one numerical free variable be called free for a system S if for every n formulas $\Phi(\varDelta_0)$, $\Phi(\varDelta_1)$, ..., $\Phi(\varDelta_n)$ are completely independent (i. e., every conjunction formed of some of the these formulas and of the negations of the remaining ones is consistent; $\Phi(\varDelta_j)$ denotes here the formula obtained from Φ by substituting the j-th numeral for the variable of Φ). We shall prove that *free formulas exist for certain systems S and some of their extensions*. In fact we shall prove for a class of formal systems S a slightly more general result: *given a family of extensions of S satisfying certain very general assumptions, there exists a formula which is free for every extension of this family.*

The following circumstance deserves perhaps mentioning and justifies to a certain extent the length of the paper. Our considerations prove the existence of free formulas not only for systems based on the usual rules of proof but also for systems based on the rule ω. Thus they furnish another illustration of the parallelism noted already in [2] between these two kinds of systems. Our discussion of systems based on the rule ω rests on the remark due to J. R. Shoenfield that the decomposition of the $\varPi_1^1$ sets into constituents (cf. Kleene [5], theorem I, p. 417) can on several occasions be exploited in the same way as the recursive enumerability of the $\varSigma_1^0$ sets. Thus our paper can be considered as a test of this useful heuristic principle. From a result noted at the end of the paper it follows that no similar phenomenon occurs for $\varPi_2^1$ sets.

In view of these remarks the author hopes that his paper in spite of its rather special subject may throw some light on a more important and broader topic, to wit the constructive analogue of the theory of projective sets.

1. We consider a consistent theory T with standard formalization and infinite sequence $\varDelta_0$, $\varDelta_1$, ... of its terms without free variables. The Gödel number of a formula Φ will be denoted by $\ulcorner\Phi\urcorner$. A k-ary relation R (i. e., a subset of $N_0^k = N_0 \times ... \times N_0$ where N_0 is the set of integers $\geqslant 0$)

is *weakly representable* in T if there is a formula Φ with k free variables such that

$$(1) \qquad (n_1, \ldots, n_k) \, \epsilon \, R \equiv \, \vdash \Phi(\Delta_{n_1}, \ldots, \Delta_{n_k}) \, .$$

R is *strongly representable* in T if besides (1) the equivalence

$$(2) \qquad (n_1, \ldots, n_k) \, \acute{\epsilon} \, R \equiv \, \vdash \sim\Phi(\Delta_{n_1}, \ldots, \Delta_{n_k})$$

is true for arbitrary $n_1, \ldots, n_k$.

A function $f: N_0^k \to N_0$ is *representable* in T if there is a formula Φ with $k+1$ free variables such that

$$\vdash \Phi(\Delta_{n_1}, \ldots, \Delta_{n_k}, x) \equiv x = \Delta_{f(n_1, \ldots, n_k)} \, .$$

A relation R is *weakly* or *strongly representable relatively to a set* $Q \subseteq N_0^k$ if (1) and (2) hold for arbitrary $(n_1, n_2, \ldots, n_k)$ in Q.

We shall assume that

I. *Every primitive recursive function is strongly representable in T.*

II. *There are primitive recursive functions* $Neg, Imp, Con, Alt, Sb, Ex,$ δ *such that* $Neg(\ulcorner\Phi\urcorner) = \ulcorner\sim\Phi\urcorner$, $\ulcorner\Phi\urcorner Imp \ulcorner\Psi\urcorner = \ulcorner\Phi \supset \Psi\urcorner$, $\ulcorner\Phi\urcorner Con \ulcorner\Psi\urcorner$ $= \ulcorner\Phi \,\&\, \Psi\urcorner$, $\ulcorner\Phi\urcorner Alt \ulcorner\Psi\urcorner = \ulcorner\Phi \vee \Psi\urcorner$, $Sb(i, j, \ulcorner\Phi\urcorner) = \ulcorner Subst(x_i/\Delta_j)\Phi\urcorner$, $Ex(i, j, \ulcorner\Phi\urcorner) = \ulcorner(\mathbf{E}x_i, x_j)\Phi\urcorner$, $\delta(n) = \ulcorner\Delta_n\urcorner$.

We shall also assume that there are: a set P, called the *set of proofs* of T, and a quaternary relation $\ll$ which satisfy the following conditions:

III. *The relation* $(m, n) \ll (p, q)$ *is reflexive, transitive, and well-founded in* $P \times N_0$.

IV. *There is a formula* $\Pi(x)$ *which weakly represents* P *in* T *and a formula* $M(x, y; u, v)$ *with the free variables indicated which strongly represents* $\ll$ *in* T *relatively to* $P \times N_0 \times P \times N_0$. *Moreover, these formulas satisfy the conditions:*

(i) $\vdash \Pi(x) \,\&\, \Pi(z) \,\&\, M(x, y; z, t) \,\&\, M(z, t; x, y) \supset y = t;$

(ii)   *if* $p \, \epsilon \, P$, *then* $\vdash \Pi(x) \supset [M(x, y; \Delta_p, \Delta_q) \vee M(\Delta_p, \Delta_q; x, y)];$

(iii)   *if* $p_0 \, \epsilon \, P$ *and* $\vdash \Phi(\Delta_p, \Delta_q)$ *for every pair* (p, q) *in* $P \times N_0$ *such that* $(p, q) \ll (p_0, q_0)$, *then* $\vdash \Pi(x) \,\&\, M(x, y; \Delta_{p_0}, \Delta_{q_0}) \supset \Phi(x, y)$.

Let $\{A_j\}$, $j = 0, 1, 2, \ldots$, be a family of sets each of which consists of formulas of T. We shall say that $\{A_j\}$ is a *representable family of consistent extensions of* T if, for each j,

(a) $\vdash \Phi$ *implies* $\Phi \, \epsilon \, A_j;$

(b) $\Phi \, \epsilon \, A_j$ *implies* $\sim\Phi \, \acute{\epsilon} \, A_j;$

(c) $\Phi \, \epsilon \, A_j$ *and* $\Phi \supset \Psi \, \epsilon \, A_j$ *imply* $\Psi \, \epsilon \, A_j;$

(d) $(x_k)\Phi \, \epsilon \, A_j$ *implies* $\Phi(\Delta_n) \, \epsilon \, A_j$ *for* $k, n \, \epsilon \, N_0;$

(e)   *there is a ternary relation* C *such that* $\Phi \, \epsilon \, A_j \equiv (\mathbf{E}p)[(p \, \epsilon \, P) \,\&\, C(p, j, \ulcorner\Phi\urcorner)];$

(f)   *there are formulas* $\Gamma(x, y, z)$, $\Gamma^*(x, y, z)$ *with the free variables indicated which strongly represent relatively to* $P \times N_0^2$ *the relations* $C(p, j, n)$ *and* $C^*(p, j, n) \equiv C(p, j, Neg(n));$

(g) $\vdash \Pi(x) \,\&\, \Pi(x') \,\&\, M(x, y; x', y') \,\&\, M(x', y'; x, y)$
       $\supset [\Gamma(x, y, z) \equiv \Gamma(x', y', z)];$
       $\vdash \Pi(x) \,\&\, \Pi(x') \,\&\, M(x, y; x', y') \,\&\, M(x', y'; x, y)$
       $\supset [\Gamma^*(x, y, z) \equiv \Gamma^*(x', y', z)].$

We shall denote by $\lessdot$ the relation $(m, n) \ll (p, q) \,\&\, (m, n)$non-$\gg(p, q)$ and by $\bar{M}(x, y; z, t)$ the formula $M(x, y; z, t) \,\&\, \sim M(z, t; x, y)$. $\bar{M}$ strongly represents $\lessdot$ relatively to the set $P \times N_0 \times P \times N_0$.

2. In this section we generalize Rosser's proof [9] and obtain

THEOREM 1. *If* $\{A_j\}$ *is a representable family of consistent extensions of* T, *then there is a closed formula* Θ *such that for any* j *neither* Θ *nor* $\sim\Theta$ *is in* A_j.

Proof. Let $\sigma(\ulcorner\Phi\urcorner) = Subst(y/\Delta_{\ulcorner\Phi\urcorner})\Phi$ and let $\Sigma(x, y)$ be a formula which strongly represents σ in T. Consider the relation $R(l, m, n)$ defined thus:

$$C(l, m, n) \supset (\mathbf{E}p, q)\{(p \, \epsilon \, P) \,\&\, [(p, q) \lessdot (l, m)] \,\&\, C^*(p, q, n)\}$$

and the formula $\Phi(u, v, y)$:

$$(3) \qquad \Gamma(u, v, y) \supset (\mathbf{E}s, t)[\Pi(s) \,\&\, \bar{M}(s, t; u, v) \,\&\, \Gamma^*(s, t, y)] \, .$$

We shall show that Φ strongly represents R relatively to $P \times N_0^2$. Indeed, if $l \, \epsilon \, P$ and $R(l, m, n)$, then either non-$C(l, m, n)$ or there are p, q such that $p \, \epsilon \, P$, $(p, q) \lessdot (l, m)$ and $C^*(p, q, n)$. In the former case $\vdash \sim\Gamma(\Delta_l, \Delta_m, \Delta_n)$ by (f) and in the latter $\vdash \Pi(\Delta_p) \,\&\, \bar{M}(\Delta_p, \Delta_q; \Delta_l, \Delta_m)$ $\,\&\, \Gamma^*(\Delta_p, \Delta_q, \Delta_n)$. Thus in both cases $\vdash \Phi(\Delta_l, \Delta_m, \Delta_n)$.

Next assume that $l \, \epsilon \, P$ and non-$R(l, m, n)$. It follows that $C(l, m, n)$ and non-$C^*(p, q, n)$ for every pair (p, q) such that $p \, \epsilon \, P$ and $(p, q) \lessdot (l, m)$. Using (f) and IV (iii) we infer that $\vdash \sim\Phi(\Delta_l, \Delta_m, \Delta_n)$. Thus we have proved

$$(4) \qquad l \, \epsilon \, P \supset [R(l, m, n) \equiv \, \vdash \Phi(\Delta_l, \Delta_m, \Delta_n)] \, ,$$

$$(5) \qquad l \, \epsilon \, P \supset [\text{non-}R(l, m, n) \equiv \, \vdash \sim\Phi(\Delta_l, \Delta_m, \Delta_n)] \, .$$

Let Ψ be the formula $(u, v, z)[\Pi(u) \,\&\, \Sigma(y, z) \supset \Phi(u, v, z)]$ and Θ the formula $Subst(y/\Delta_{\ulcorner\Psi\urcorner})\Psi$. Hence $\ulcorner\Theta\urcorner = \sigma(\ulcorner\Psi\urcorner)$, $\vdash \Sigma(\Delta_{\ulcorner\Psi\urcorner}, z)$ $\equiv (z = \Delta_{\ulcorner\Theta\urcorner})$ and we obtain

$$(6) \qquad \vdash \Theta \equiv (u, v)[\Pi(u) \supset \Phi(u, v, \Delta_{\ulcorner\Theta\urcorner})] \, .$$

Using (3) we obtain by elementary logical transformations

$$(7) \qquad \vdash \Theta \equiv (u, v)\{\Pi(u) \,\&\, \Gamma(u, v, \Delta_{\ulcorner\Theta\urcorner})$$
$$\supset (\mathbf{E}s, t)[\Pi(s) \,\&\, \bar{M}(s, t; u, v) \,\&\, \Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner})] \, ,$$

(8) $\vdash \sim\Theta \equiv (\mathbf{E}u, v)\{\Pi(u) \,\&\, \Gamma(u, v, \Delta_{\ulcorner\Theta\urcorner})$
$\&\, (s, t)[\Pi(s) \,\&\, \bar{M}(s, t; u, v) \supset \sim\Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner})]\}\,.$

Let us assume that there are integers j such that $\Theta \,\epsilon\, \Lambda_j$, i.e., $C(p, j, \ulcorner\Theta\urcorner)$ for some p in P. Let (p_0, j_0) be a minimal pair (with respect to $\ll$) such that $C(p_0, j_0, \ulcorner\Theta\urcorner)$. Hence

(9) $p_0 \,\epsilon\, P\,, \qquad C(p_0, j_0, \ulcorner\Theta\urcorner)\,,$

(10) $\vdash \Pi(\Delta_{p_0})\,, \qquad \vdash \Gamma(\Delta_{p_0}, \Delta_{j_0}, \Delta_{\ulcorner\Theta\urcorner})\,,$

(11)   if $p \,\epsilon\, P\,, \quad (p, j) \ll (p_0, j_0)$ then non-$C(p, j, \ulcorner\Theta\urcorner)\,.$

Since $\Theta \,\epsilon\, \Lambda_{j_0}$, it follows by (9), (10), and (7) that

$(\mathbf{E}s, t)[\Pi(s) \,\&\, \bar{M}(s, t; \Delta_{p_0}, \Delta_{j_0}) \,\&\, \Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner})] \,\epsilon\, \Lambda_{j_0}\,.$

If we had $\vdash \sim\Gamma^*(\Delta_p, \Delta_j, \Delta_{\ulcorner\Theta\urcorner})$ for every (p, j) such that $p \,\epsilon\, P$ and $(p, j) \ll (p_0, j_0)$, then we should obtain by IV (iii) applied to the formula $\sim\Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner}) \vee M(\Delta_{p_0}, \Delta_{j_0}; s, t)$

$\vdash (s, t)[\Pi(s) \,\&\, \bar{M}(s, t; \Delta_{p_0}, \Delta_{j_0}) \supset \sim\Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner})]$

and Λ_{j_0} would be inconsistent. Hence there is a (p_1, j_1) such that

(12) $p_1 \,\epsilon\, P\,, \qquad (p_1, j_1) \ll (p_0, j_0)$

and non $\vdash \sim\Gamma^*(\Delta_{p_1}, \Delta_{j_1}, \Delta_{\ulcorner\Theta\urcorner})$, i.e.,

(13) $\vdash \Gamma^*(\Delta_{p_1}, \Delta_{j_1}, \Delta_{\ulcorner\Theta\urcorner})\,.$

This gives $C^*(p_1, j_1, \ulcorner\Theta\urcorner)$, i.e., $C(p_1, j_1, \ulcorner\sim\Theta\urcorner)$ and hence $\sim\Theta \,\epsilon\, \Lambda_{j_1}$. Using (8) and IV (ii) we obtain

$\vdash \sim\Theta \equiv (\mathbf{E}u, v)[M(u, v; \Delta_{p_1}, \Delta_{j_1}) \,\&\, \Upsilon(u, v)]$
$\vee (\mathbf{E}u, v)[M(\Delta_{p_1}, \Delta_{j_1}; u, v) \,\&\, \Upsilon(u, v)]$

where $\Upsilon(u, v)$ is the formula

$\Pi(u) \,\&\, \Gamma(u, v, \Delta_{\ulcorner\Theta\urcorner}) \,\&\, (s, t)[\Pi(s) \,\&\, \bar{M}(s, t; u, v) \supset \sim\Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner})]\,.$

Since $\vdash \Pi(\Delta_{p_1})$ by (12) we easily see that

$\vdash M(\Delta_{p_1}, \Delta_{j_1}; u, v) \,\&\, \Upsilon(u, v) \supset [\sim\Gamma^*(\Delta_{p_1}, \Delta_{j_1}, \Delta_{\ulcorner\Theta\urcorner})$
$\vee M(\Delta_{p_1}, \Delta_{j_1}; u, v) \,\&\, M(u, v; \Delta_{p_1}, \Delta_{j_1}) \,\&\, \Gamma(u, v, \Delta_{\ulcorner\Theta\urcorner})]$

and hence by (g)

$M(\Delta_{p_1}, \Delta_{j_1}; u, v) \,\&\, \Upsilon(u, v) \supset [\sim\Gamma^*(\Delta_{p_1}, \Delta_{j_1}, \Delta_{\ulcorner\Theta\urcorner}) \vee \Gamma(\Delta_{p_1}, \Delta_{j_1}, \Delta_{\ulcorner\Theta\urcorner})]\,.$

By (13) and the consistency of Λ_{j_1} we have $\vdash \sim\Gamma(\Delta_{p_1}, \Delta_{j_1}, \Delta_{\ulcorner\Theta\urcorner})$ and thus using (13) we obtain

$\vdash \sim[M(\Delta_{p_1}, \Delta_{j_1}; u, v) \,\&\, \Upsilon(u, v)]\,.$

This proves that

$\vdash \sim\Theta \equiv (\mathbf{E}u, v)[M(u, v; \Delta_{p_1}, \Delta_{j_1}) \,\&\, \Upsilon(u, v)]\,,$

and so the right-hand side of this equivalence belongs to Λ_{j_1}. Using IV (iii) we infer that it is not for every pair (p, j) with $p \,\epsilon\, P$ and $(p, j) \ll (p_1, j_1)$ that the formula $\vdash \sim\Upsilon(\Delta_p, \Delta_j)$ holds. Hence there is a pair (p_2, j_2) such that $p_2 \,\epsilon\, P$ and $(p_2, j_2) \ll (p_1, j_1)$ and

(14)   non $\vdash \sim\Gamma(\Delta_{p_2}, \Delta_{j_2}, \Delta_{\ulcorner\Theta\urcorner})$
$\supset (\mathbf{E}s, t)[\Pi(s) \,\&\, \bar{M}(s, t; \Delta_{p_2}, \Delta_{j_2}) \,\&\, \Gamma^*(s, t, \Delta_{\ulcorner\Theta\urcorner})]\,.$

This is a contradiction since, by (12), $(p_2, j_2) \ll (p_0, j_0)$, and hence, by (11), $\vdash \sim\Gamma(\Delta_{p_2}, \Delta_{j_2}, \Delta_{\ulcorner\Theta\urcorner})$.

Let us now assume that $\sim\Theta \,\epsilon\, \Lambda_j$ for some j, i.e., that $C(p_1, j_1, \ulcorner\sim\Theta\urcorner)$ for some pairs (p_1, j_1) with p_1 in P. This gives us formula (13) and we can argue as above and infer that there is a pair (p_2, j_2) with p_2 in P such that (14). This proves that non $\vdash \sim\Gamma(\Delta_{p_2}, \Delta_{j_2}, \Delta_{\ulcorner\Theta\urcorner})$ and hence $C(p_2, j_2, \ulcorner\Theta\urcorner)$, which contradicts our former result.

Theorem 1 is thus proved. From its proof we also obtain

THEOREM 1$^{\mathrm{bis}}$. *If Γ and Γ^* are arbitrary formulas satisfying* (f) *and* (g), *then the formula Θ defined by* (3) *and* (6) *is undecidable in any Λ_j.*

3. In this section we shall add two more assumptions to our assumptions I-IV concerning the theory T:

V. *For every primitive recursive function $f(n_1, ..., n_k)$ there is a formula $\Phi(x_1, ..., x_k, y)$ which strongly represents f and satisfies the condition*

$\vdash \Phi(x_1, ..., x_k, y) \,\&\, \Phi(x_1, ..., x_k, y') \supset y = y'\,.$

VI. *For every $k > 0$ there is a formula $H^{(k)}(x, t_1, ..., t_k)$ with the variables indicated such that:*

(i) *there is a primitive recursive function* [1] *$\sigma_k(j_1, j_2)$ for which*

$\vdash H^{(k)}(\Delta_{\sigma_k(j_1, j_2)}, t_1, ..., t_k) \equiv [H^{(k)}(\Delta_{j_1}, t_1, ..., t_k) \vee H^{(k)}(\Delta_{j_2}, t_1, ..., t_k)]\,,$

(ii) *there is a primitive recursive function $\tau(j)$ such that*

$\vdash \Pi(t_1) \supset \{H^{(4)}(\Delta_{\tau(j)}, t_1, ..., t_4)$
$\equiv (\mathbf{E}u, v)[\Pi(u) \,\&\, \bar{M}(u, v; t_1, t_2) \,\&\, H^{(4)}(\Delta_j, u, v, t_3, t_4)]\}\,,$

(iii) *there is a primitive recursive function* [1] *$\zeta_k(j, m, n)$ such that, if $k \geqslant 2$,*

$\vdash H^{(k-2)}(\Delta_{\zeta_k(j, m, n)}, t_1, ..., t_{k-2}) \equiv H^{(k)}(\Delta_j, t_1, ..., t_{k-2}, \Delta_m, \Delta_n)\,.$

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[1] This function need not be recursive in k although in the examples which we shall discuss later this is actually the case.

Let $\{A_j\}$ be a representable family of consistent extensions of T and h an integer such that for every $j \leqslant lh(h)$ the integer $(h)_j$ is the Gödel number of a closed formula $F_{h,j}$. Let $F_{h,j} = F_{h,0}$ for $j > lh(h)$ and let $A_j^{(h)} = \{\Psi: F_{h,K(j)} \supset \Psi \in A_{L(j)}\}$ where K, L are functions inverse to the pairing function $J(m,n) = \frac{1}{2}(m+n)(m+n-1)+n$.

LEMMA 1. *If neither $F_{h,j}$ nor $\sim F_{h,j}$ belong to A_k $(j, k = 0, 1, 2, ...)$, then $\{A_j^{(h)}\}$ is a representable family of consistent extensions of T.*

Proof. Conditions (a)-(d) are obvious. To prove (e) we denote by $\bar{C}_h$ the relation

$$C\big(p, L(j), (h)_{K(j)} \, Imp \, n\big) \,\&\, \big(K(j) \leqslant lh(h)\big)$$
$$\vee\, C\big(p, L(j), (h)_0 \, Imp \, n\big) \,\&\, \big(K(j) > lh(h)\big)$$

and easily verify that (e) is satisfied. Finally, to prove (f) and (g) we denote by $\bar{\Gamma}_h$ the formula

$$(\mathbf{E}u, v, w)[\Gamma(x, u, v) \,\&\, A(y, u) \,\&\, B(\varDelta_h, y, z, v) \,\&\, D(y, \varDelta_h)$$
$$\vee\, \Gamma(x, u, w) \,\&\, B(\varDelta_h, \varDelta_0, z, w) \,\&\, \sim D(u, \varDelta_h)],$$

where A, B, D are formulas which strongly represent the functions $L(j), (h)_{K(j)} \, Imp \, n$ and the relation $K(j) \leqslant lh(h)$ and which satisfy the conditions

$$\vdash A(y, u) \,\&\, A(y, u') \supset u = u', \qquad \vdash B(s, y, z, v) \,\&\, B(s, y, z, v') \supset v = v'.$$

By $\bar{\Gamma}_h^*$ we denote a similar formula with Γ replaced by Γ^*. It is easy to verify that $\bar{\Gamma}_h, \bar{\Gamma}_h^*$ satisfy (f) and (g) (in the proof of (g) we use IV (i)).

LEMMA 2. *If there are integres r_1, r_2 such that (with the notation used in the proof of lemma 1) the formulas $H^{(4)}(\varDelta_{r_1}, x, y, z, \varDelta_h)$ and $H^{(4)}(\varDelta_{r_2}, x, y, z, \varDelta_h)$ strongly represent relatively to $P \times N_0^2$ the relations non-$\bar{C}_h$ and $\bar{C}_h^*$ and satisfy condition (g), then there is a primitive recursive function $\vartheta(h)$ such that if $(h)_j$ is the Gödel number of a closed formula $F_{h,j}$ for $j \leqslant lh(h)$ and neither $F_{h,j}$ nor $\sim F_{h,j}$ belong to A_k $(j \leqslant lh(h),\ k = 0, 1, 2, ...)$, then the following formula Θ_h*

$$(u, v)[\Pi(u) \supset H^{(2)}(\varDelta_{\vartheta(h)}, u, v)]$$

is undecidable in every $A_j^{(h)}$, $j = 0, 1, 2, ...$

Proof. Consider the following formulas $\Phi_h(t_1, t_2, y)$, $\Psi_h(y)$, Θ_h:

$$H^{(4)}(\varDelta_{r_1}, t_1, t_2, y, \varDelta_h) \vee (\mathbf{E}u, v)[\Pi(u) \,\&\, \bar{M}(u, v, t_1, t_2)$$
$$\&\, H^{(4)}(\varDelta_{r_2}, u, v, y, \varDelta_h)],$$

$$(t_1, t_2, z)[\Pi(t_1) \,\&\, \Sigma(y, z) \supset \Phi_h(t_1, t_2, z)],$$
$$(t_1, t_2, z)[\Pi(t_1) \,\&\, \Sigma(\ulcorner \Psi_h \urcorner, z) \supset \Phi_h(t_1, t_2, z)].$$

Arguing as in the proof of (6) we obtain

$$\vdash \Theta_h \equiv (t_1, t_2)[\Pi(t_1) \supset \Phi_h(t_1, t_2, \lrcorner \ulcorner \Theta_h \urcorner)].$$

By theorem 1bis and lemma 1 neither Θ_h nor $\sim \Theta_h$ belongs to $A_j^{(h)}$, $j = 0, 1, 2, ...$ Thus it remains to construct a primitive recursive function $\vartheta(h)$ such that

$$\vdash \Pi(t_1) \supset [\Phi_h(t_1, t_2, \varDelta_{\ulcorner \Theta_h \urcorner}) \equiv H^{(2)}(\varDelta_{\vartheta(h)}, t_1, t_2)].$$

Using VI (i), (ii) we obtain

$$\Pi(t_1) \supset [\Phi_h(t_1, t_2, y) \equiv H^{(4)}(\varDelta_{\sigma_4(r_1, \tau(r_2))}, t_1, t_2, y, \varDelta_h)],$$

whence by VI (iii)

$$\Pi(t_1) \supset [\Phi_h(t_1, t_2, \varDelta_{\ulcorner \Theta_h \urcorner}) \equiv H^{(2)}(\varDelta_{\zeta_4(\sigma_4(r_1, \tau(r_2)), \ulcorner \Theta_h \urcorner, h)}, t_1, t_2)].$$

Thus it is sufficient to take $\vartheta(h) = \zeta_4\big(\sigma_4(r_1, \tau(r_2)), \ulcorner \Theta_h \urcorner, h\big)$. This function is primitive recursive since so is $\ulcorner \Theta_h \urcorner$ by II.

We put as usual $\Phi^0 = \Phi$, $\Phi^1 = \sim \Phi$ for any formula Φ.

Let us call a formula $Y(x)$ with one free variable *free* for the family $\{A_j\}$ of extensions of T if for an arbitrary zero-one sequence $i_0, i_1, ..., i_{n-1}$ the formula $Y^{i_0}(\varDelta_0) \,\&\, ... \,\&\, Y^{i_{n-1}}(\varDelta_{n-1})$ does not belong to any A_j.

THEOREM 2. *If $\{A_j\}$ is a representable family of consistent extensions of T satisfying the assumptions of lemma 2, then there is a formula free for this family.*

Proof. Let Φ_0 be the formula $(x)(x = x)$ and let $\delta, \eta, \varepsilon$ be primitive recursive functions defined thus:

$$\delta(0) = \ulcorner \Phi_0 \urcorner, \quad \eta(0, 0) = \delta(0), \quad \eta(0, s) = 0 \quad \text{for} \quad s > 0, \quad \varepsilon(0) = 2^{\eta(0,0)},$$

$$\delta(k+1) = \ulcorner (u, v)[\Pi(u) \supset H^{(2)}(\varDelta_{\vartheta(\varepsilon(k))}, u, v)] \urcorner,$$

$$\eta(k+1, j) = \begin{cases} \eta(k, j) \, Con \, \delta(k+1) & \text{for} \quad 0 \leqslant j < 2^k, \\ \eta(k, j-2^k) \, Con \, Neg \, \delta(k+1) & \text{for} \quad 2^k \leqslant j < 2^{k+1}, \\ 0 & \text{for} \quad j \geqslant 2^{k+1}, \end{cases}$$

$$\varepsilon(k+1) = \prod_{0 \leqslant j < 2^{k+1}} p_j^{\eta(k+1,j)}.$$

We prove by induction on k that for every $k \geqslant 0$ and every j with $0 \leqslant j < 2^k$ there are formulas Ω_k and $\varXi_{k,j}$ such that $\delta(k) = \ulcorner \Omega_k \urcorner$, $\eta(k,j) = \ulcorner \varXi_{k,j} \urcorner$ and $(\varepsilon(k))_j = \ulcorner \varXi_{k,j} \urcorner$. Moreover, neither $\varXi_{k,j}$ nor $\sim \varXi_{k,j}$ belongs to any A_i $(0 \leqslant j < 2^k, \ k = 1, 2, \ldots, \ i = 0, 1, 2, \ldots)$.

Indeed, for $k = 0$ it is sufficient to take $\Omega_0 = \varXi_{0,0} = \Phi_0$ and for $k > 0$ $\Omega_k = \Theta_{\varepsilon(k-1)}$, $\varXi_{k,j} = \varXi_{k-1,j} \ \& \ \Omega_k$ for $0 \leqslant j < 2^{k-1}$, $\varXi_{k,j} = \varXi_{k-1,j} \ \& \sim \Omega_k$ for $2^{k-1} \leqslant j < 2^k$. Using lemma 2 we immediately see that neither $\varXi_{k,j}$ nor $\sim \varXi_{k,j}$ belongs to A_i $(0 \leqslant j < 2^k, \ i = 0, 1, 2, \ldots, \ k > 0)$.

For every $n > 0$ and every finite zero-one sequence $i_1, \ldots, i_n$ there is an index $j < 2^n$ such that $\varXi_{n,j} = \Phi_0 \ \& \ \Omega_1^{i_1} \ \& \ \Omega_2^{i_2} \ \& \ldots \& \ \Omega_n^{i_n}$. Indeed, this is obvious for $n = 1$ and if it is true for an integer n it is also true for the integer $n+1$ because of the definition of $\varXi_{n+1,j}$. It follows that for every finite zero-one sequence the conjunction $\Omega_1^{i_1} \ \& \ldots \& \ \Omega_n^{i_n}$ does not belong to A_i. In order to accomplish the proof it is therefore sufficient to construct a formula $Y(x)$ such that

$$(15) \qquad \vdash Y(\varDelta_k) \equiv \Theta_{\varepsilon(k)} \quad \text{for every} \quad k > 0 \, .$$

To obtain such a formula let us denote by $G(x, y)$ a formula which strongly represents the function $\vartheta(\varepsilon(n))$ and take as $Y(x)$ the formula

$$(z, u, v)[G(x, z) \ \& \ \Pi(u) \supset H^{(2)}(z, u, v)] \, .$$

We then have $\vdash Y(\varDelta_k) \equiv (u, v)[\Pi(u) \supset H^{(2)}(\varDelta_{\vartheta(\varepsilon(k))}, u, v)]$, and hence by lemma 2 we obtain (15). Theorem 2 is thus proved.

4. In the present and in the next sections we shall give examples of theories and families of their extensions to which the foregoing theory is applicable.

Denoting by $J(m, n)$ the pairing function $\frac{1}{2}(m+n)(m+n-1)+n$ we put $J_1(m_1) = m_1$, $J_{k+1}(m_1, \ldots, m_{k+1}) = J(J_k(m_1, \ldots, m_k), m_{k+1})$. For every k, m $(k \geqslant 1)$ there are uniquely determined integers $m_1 = K_1^{(k)}(m), \ldots, m_k = K_k^{(k)}(m)$ such that $m = J_k(m_1, \ldots, m_k)$.

Let R^* be a theory which differs from R (cf. [11], p. 52) by containing three new operation symbols $\iota, \varkappa, \lambda$ and axioms

$$(\Omega_6) \quad z = \iota(x, y) \equiv (\mathrm{E}u, v) \{[(x+y = 0) \ \& \ (z = 0) \lor (x+y \neq 0) \ \& \ (u+\varDelta_1 = x+y)] \ \& \ \varDelta_2 \cdot v = (x+y) \cdot u] \ \& \ (z = v+y)\} \, ,$$

$$(\Omega_7) \quad \varkappa(\iota(x, y)) = x \, ,$$

$$(\Omega_8) \quad \lambda(\iota(x, y)) = y \, ,$$

$$(\Omega_9) \quad x \leqslant y \lor y \leqslant x \, ,$$

$$(\Omega_{10}) \quad (x \leqslant y) \ \& \ (y \leqslant x) \supset (x = y) \, .$$

LEMMA 3. *The following formulas are provable in R^**

$$z = \iota(\varDelta_m, \varDelta_n) \equiv z = \varDelta_{J(m,n)} \, ,$$
$$z = \varkappa(\varDelta_m) \equiv z = \varDelta_{K(m)} \, ,$$
$$z = \lambda(\varDelta_m) \equiv z = \varDelta_{L(m)} \, .$$

Indeed, writing the right-hand side of (Ω_6) as $(\mathrm{E}u, v) X(u, v, x, y, z)$ we easily infer that the formula

$$X(u, v, x, y, z) \equiv (u = \varDelta_{m+n-1}) \ \& \ (v = \varDelta_{\frac{1}{2}(m+n)(m+n-1)}) \ \& \ (z = \varDelta_{J(m,n)})$$

is provable in R. This shows that the first equivalence is provable. Provability of the second and third formulas follows from (Ω_7) and (Ω_8).

LEMMA 4. *Theory R^* and every theory T (with standard formalization) in which R^* is interpretable satisfy conditions* I, II, V.

Proof. II is obvious and I is implied by V, whence it remains to prove V. Let $f(n_1, \ldots, n_k)$ be a primitive recursive function and $\Phi^*(x_1, \ldots, x_k, y)$ a formula which strongly represents f. The existence of Φ^* was proved in [11], pp. 56-60. Take as Φ the formula $\Phi^*(x_1, \ldots, x_k, y) \ \& \ (y')[(y' < y) \supset \sim \Phi^*(x_1, \ldots, x_k, y')]$. Using axiom (Ω_9) we easily see that V is satisfied.

LEMMA 5. *The set $P = N_0$, the relation $J(m, n) \leqslant J(p, q)$ and the formulas $x = x$, $\iota(x, y) \leqslant \iota(z, t)$ satisfy conditions* III, IV *for the theory R^* and its arbitrary consistent extensions.*

Proof. III is obvious. It is also obvious that the formulas $x = x$ and $\iota(x, y) \leqslant \iota(z, t)$ strongly represent the set P and the relation $J(m, n) \leqslant J(p, q)$ (cf. lemma 3). Formula IV (i) results from axioms (Ω_{10}) and (Ω_8); formula IV (ii) results from axiom (Ω_9). Let us finally assume that $\Phi(\varDelta_p, \varDelta_q)$ is provable in R^* (or in its extension T) for arbitrary p, q such that $J(p, q) \leqslant J(p_0, q_0)$. Since

$$\vdash [\iota(x, y) \leqslant \iota(\varDelta_{p_0}, \varDelta_{q_0})] \equiv [\iota(x, y) = \varDelta_0 \lor \ldots \lor \iota(x, y) = \varDelta_{J(p_0, q_0)}]$$

and since every $r \leqslant J(p_0, q_0)$ is representable as $J(p, q)$, we conclude that the formula $\iota(x, y) \leqslant \iota(\varDelta_{p_0}, \varDelta_{q_0}) \supset \Phi(x, y)$ is provable in R^* (or in T).

THEOREM 3. *All the assumptions of theorem 1 are satisfied for each recursively enumerable family of consistent sets containing axioms of R^* and closed with respect to the rules of proof.*

Proof. We proved above that I-IV are satisfied. If $\{A_j\}$ is a recursively enumerable family of sets, then there is a recursive relation $C(p, j, n)$ such that $\Phi \in A_j \equiv (\mathrm{E}p) C(p, j, \ulcorner \Phi \urcorner)$. Hence $\{A_j\}$ satisfies condition (e) and therefore (f) because every recursive relation is strongly representable in R^*. Conditions (a)-(d) follow from the consistency of A_j, from its closure with respect to the rules of proof and from the fact that it contains axioms of R^*. Finally (g) follows from axioms (Ω_7) and (Ω_8).

To obtain a theory to which theorem 2 is applicable we shall add to R^* several new operation symbols and axioms. The new symbols are: a unary symbol ε and binary symbols π, φ. We shall write x^y instead of $\pi(x, y)$. We also introduce the following abbreviations:

$$\iota_1(x) = x, \qquad \iota_{k+1}(x_1, ..., x_{k+1}) = \iota\big(\iota_k(x_1, ..., x_k), x_{k+1}\big),$$

$$\varkappa_1^{(1)}(x) = x, \qquad \varkappa_j^{(k+1)}(x) = \varkappa_j^{(k)}\big(\varkappa(x)\big) \quad \text{for} \quad j = 1, ..., k,$$

$$\varkappa_{k+1}^{(k+1)}(x) = \lambda(x).$$

The new axioms are

(Ω_{11}) $x \cdot y = \mathbf{0} \equiv [(x = \mathbf{0}) \lor (y = \mathbf{0})]$,

(Ω_{12}) $(\mathbf{E}z)\big[(z \leqslant x) \,\&\, \big(\varepsilon(x) \leqslant z+z\big) \,\&\, \big(x = z \cdot z + \varepsilon(x)\big)\big]$,

(Ω_{13}) $\varphi(\varDelta_0, y) = y + \varDelta_1$,

(Ω_{14}) $\varphi(\varDelta_1, y) = \varepsilon(y)$,

(Ω_{15}) $\varphi(\varDelta_{12} \cdot x + \varDelta_6, y) = \varphi\big(\varkappa(x), y\big) + \varphi\big(\lambda(x), y\big)$,

(Ω_{16}) $\varphi(\varDelta_{12} \cdot x + \varDelta_7, y) = \varphi\big(\varkappa(x), y\big) \cdot \varphi\big(\lambda(x), y\big)$,

(Ω_{17}) $\varphi(\varDelta_{12} \cdot x + \varDelta_8, y) = \varphi\big(\varkappa(x), \varphi\big(\lambda(x), y\big)\big)$,

(Ω_{18}) $\varphi(x, y) = z \supset \varphi\big(x, \varphi(\varDelta_{12} \cdot x + \varDelta_9, z)\big) = z$,

(Ω_{19}) $\varphi(x, y) = z \,\&\, t < \varphi(\varDelta_{12} \cdot x + \varDelta_9, z) \supset \varphi(x, t) \neq z$,

(Ω_{20}) $\varphi\big(\varDelta_6 \cdot \big[\varDelta_2^{z+\varDelta_1}(\varDelta_2 \cdot x + \varDelta_1)\big], y\big) = \varphi\big(x, \iota(y, z)\big)$,

(Ω_{21}) $\varphi(\varDelta_{12} \cdot x + \varDelta_{10}, y) = \mathbf{0} \equiv (\mathbf{E}t)\big\{\big[t < \iota\big(\varkappa_1^{(4)}(y), \varkappa_2^{(4)}(y)\big)\big]$
                     $\&\, \big[\varphi\big(x, \iota_4\big(\varkappa(t), \lambda(t), \varkappa_3^{(4)}(y), \varkappa_4^{(4)}(y)\big)\big) = \mathbf{0}\big]\big\}$,

(Ω_{22}) $\varphi(\varDelta_{12} \cdot x + \varDelta_{11}, y) = \mathbf{0} \equiv (t)\big[t \leqslant \varkappa(y) \supset \varphi\big(x, \iota\big(\lambda(y), t\big)\big) = \mathbf{0}\big]$,

(Ω_{23}) $\pi(x, \varDelta_0) = \varDelta_1$,

(Ω_{24}) $\pi(x, y + \varDelta_1) = \pi(x, y) \cdot x$.

Let R^{**} be a theory with the primitive symbols enumerated above and based on axioms (Ω_1)-(Ω_{24}).

In order to make the content of the axioms (Ω_1)-(Ω_{24}) more accessible we shall sketch (informally) the proof of the following

LEMMA 6. *Theory R^{**} is interpretable in P.*

We interpret $x + y$, $x \cdot y$, x^y in the usual way, $\iota(x, y)$, $\varkappa(x)$, $\lambda(x)$ as the pairing functions J, K, L, $\varepsilon(x)$ as the excess of x over the nearest square not greater than x. $\varphi(x, y)$ is interpreted as the function $U(n, m) = U_n(m)$ defined by induction. The values of $U_0(m)$, $U_1(m)$ are determined according

to (Ω_{13}), (Ω_{14}), the values of $U_2(m)$, $U_3(m)$, $U_4(m)$, $U_5(m)$ are arbitrary, the values of $U_{12n+6+j}(m)$ for $j = 0, 1, 2$ are respectively $U_{K(n)}(m) + U_{L(n)}(m)$, $U_{K(n)}(m) \cdot U_{L(n)}(m)$, and $U_{K(n)}\big(U_{L(n)}(m)\big)$. The value of $U_{12n+9}(m)$ is either the least p such that $U_n(p) = m$ or 0 if such a p does not exists. The value of $U_{12n+11}(m)$ is equal to $\sum\limits_{j=0}^{K(m)} U_n\big(J\big(L(m), j\big)\big)$ and the value of $U_{12+10}(m)$ is equal to $\prod U_n\big(J_4(u, v, K_3^{(4)}(m), K_4^{(4)}(m))\big)$ where the product is extended over pairs u, v such that $J(u, v) < J\big(K_1^{(4)}(m), K_2^{(4)}(m)\big)$. Finally, the value of $U_{6 \cdot 2^{p+1}(2n+1)}(m)$ is $U_n\big(J(m, p)\big)$ and the values of $U_{6 \cdot 2^{p+1}(2n+1)+j}(m)$ for $j = 1, 2, ..., 5$ are arbitrary.

LEMMA 7. *For every general recursive function $f(n_1, ..., n_k)$ there is an integer e such that*

$$\vdash \varphi\big(\varDelta_e, \iota_k(\varDelta_{n_1}, ..., \varDelta_{n_k})\big) = \varDelta_{f(n_1, ..., n_k)}.$$

($\vdash$ means here "provable in R^{**}").

Proof. First assume that $k = 1$ and let $\mathcal{F}$ be the family of functions $f(n)$ such that there is an e satisfying

$$(16) \qquad \vdash \varphi(\varDelta_e, \varDelta_n) = \varDelta_{f(n)}$$

for each n. Functions $S(n) = n + 1$ and $E(n) =$ excess of n over the nearest lower square belong to $\mathcal{F}$ because we can take $e = 0$ or $e = 1$. If $f, g \in \mathcal{F}$, then there are integers e, d such that (16) and the following formula (17) hold for all n:

$$(17) \qquad \vdash \varphi(\varDelta_d, \varDelta_n) = \varDelta_{g(n)},$$

whence by axioms (Ω_{15}), (Ω_{17}), and (Ω_1)

$$\vdash \varphi(\varDelta_{12J(e,d)+6}, \varDelta_n) = \varDelta_{f(n)+g(n)},$$
$$\vdash \varphi(\varDelta_{12J(e,d)+8}, \varDelta_n) = \varDelta_{f(g(n))},$$

and hence the functions $f(n) + g(n)$ and $f\big(g(n)\big)$ belong to $\mathcal{F}$.

If f is in $\mathcal{F}$ and f assumes all natural numbers as values, then using axioms (Ω_4), (Ω_5), (Ω_{18}) and (Ω_{19}) we easily obtain

$$\vdash \varphi(\varDelta_{12e+9}, \varDelta_n) = \varDelta_{f^{-1}(n)}$$

and hence $f^{-1} \in \mathcal{F}$. Thus $\mathcal{F}$ contains all general recursive functions of one argument.

If $k > 1$ and $f(n_1, ..., n_k)$ is general recursive, then so is $g(m) = f\big(K_1^{(k)}(m), ..., K_k^{(k)}(m)\big)$ and hence there is a d satisfying (17). Substituting $n = J_k(n_1, ..., n_k)$ we obtain the desired result.

Let $H^{(k)}(x, t_1, ..., t_k)$ be the formula $\varphi\big(x, \iota_k(t_1, ..., t_k)\big) = \mathbf{0}$.

LEMMA 8. _Formulas $H^{(k)}$ satisfy condition_ VI.

Proof. VI (i) is satisfied with $\sigma_k(j_1, j_2) = 12J(j_1, j_2) + 7$; in the proof we use axioms (Ω_{16}) and (Ω_{11}).

Formula VI (ii), which becomes in this case

$$\vdash H^{(4)}(\varDelta_{\tau(j)}, t_1, t_2, t_3, t_4) \equiv (Eu, v)\{[\iota(u, v) < \iota(t_1, t_2)] \\ \& \; H^{(4)}(\varDelta_j, u, v, t_3, t_4)\},$$

is but a different formulation of (Ω_{21}) for $x = \varDelta_j$; we put $\tau(j) = 12j + 10$.

From (Ω_{20}), (Ω_{23}) and (Ω_{24}) we obtain by substitution $x = \varDelta_j$, $y = \iota_{k-1}(t_1, \ldots, t_{k-1})$, $z = \varDelta_n$

$$\vdash \varphi\big(\varDelta_{6 \cdot 2^{n+1}(2j+1)}, \iota_{k-1}(t_1, \ldots, t_{k-1})\big) = \mathbf{0} \equiv \varphi\big(\varDelta_j, \iota_k(t_1, \ldots, t_{k-1}, \varDelta_n)\big) = \mathbf{0}$$

and repeating the same argument

$$\vdash \varphi\big(\varDelta_{\zeta(j,m,n)}, \iota_{k-2}(t_1, \ldots, t_{k-2})\big) = \mathbf{0} \equiv \varphi\big(\varDelta_j, \iota_k(t_1, \ldots, t_{k-2}, \varDelta_m, \varDelta_n)\big) = \mathbf{0}$$

where $\zeta(j, m, n) = 6 \cdot 2^{m+1}[12 \cdot 2^{n+1}(2j+1) + 1]$. Thus formula VI (iii) is provable.

THEOREM 4. _For every recursively enumerable family $\{A_j\}$ of consistent sets containing axioms of R^{**} and closed with respect to the rules of proof there is a formula free for that family._

Proof. Relations $\bar{C}_h$, $\bar{C}_h^*$ defined in the proof of lemma 1 are in the present case recursive in the four arguments p, j, n, h. By lemma 7 there are integers r_1, r_2 such that the formulas $H^{(4)}(\varDelta_{r_1}, x, y, z, \varDelta_h)$ and $H^{(4)}(\varDelta_{r_2}, x, y, z, \varDelta_h)$ strongly represent the relations non-$\bar{C}_h(p, j, n)$ and $\bar{C}_h(p, j, n)$. These formulas satisfy condition (g) because (with our choice of formulas $\varPi$ and M) the antecedent of both formulas in (g) becomes (by (Ω_{10})) $\iota(x, y) = \iota(x', y')$, and thus (by Ω_7) and (Ω_8)) is equivalent to $(x = x') \& (y = y')$. This proves that theorem 4 follows from theorem 2.

COROLLARY 1. _There is a formula free for the theory P and each of its recursively enumerable extensions._

The corollary follows from theorem 4 and lemma 6.

COROLLARY 2. _There is a formula free for every sub-theory of P._

Indeed, a formula free for a theory is free for an arbitrary sub-theory.

It is an open question whether for every recursively enumerable extension of R there is a formula free for that extension.

5. In this section we shall briefly discuss a theory F obtained from R^{**} by enlarging the set of axioms by all formulas $(x)A(x)$ such that $A(\varDelta_n)$ is provable in R^{**} for $n = 0, 1, 2, \ldots$ (cf. [3]). Let $Cn_F(X)$ denote the smallest set containing X and the axioms of F and closed with respect to the rules of proofs of F.

THEOREM 5. _If $\{B_j\}$ is a recursively enumerable family of sets consistent in F each of which consists of closed formulas of F, then there is a formula free for the family $A_j = Cn_F(B_j)$._

Proof. Choose P, $\ll$, $\varPi$, and M as in lemma 5. It is obvious that conditions I-V are satisfied. It is also obvious that the family $\{A_j\}$ satisfies conditions (a)-(d) and (g).

If R is a recursive subset of N^{k+3}, then the set

$$(*) \qquad \{(n_1, \ldots, n_k) : (Er)(s)(Et)[(r, s, t, n_1, \ldots, n_k) \in R]\}$$

is weakly representable in F and the set

$$(**) \qquad \{(n_0, \ldots, n_k) : (r)(Es)[(r, s, n_0, \ldots, n_k) \in R]\}$$

is strongly representable in F. Actually sets of the form $(*)$ are most general sets weakly representable in F but we shall not need this fact in our discussion.

Let W_1 be the set of Gödel numbers of formulas with one free variable x_1 and B—the provability relation for R^{**}. Both sets are obviously primitive recursive. Since

$$\varPhi \in A_j \equiv (Ep)\big([K_1^{(3)}(p) \in B_j] \& [K_2^{(3)}(p) \in W_1] \& (n)(Em) \\ [mB Sb(1, n, K_2^{(3)}(p))] \& K_3^{(3)}(p) B[Ex(1, 1, NegK_2^{(3)}(p)) \\ Alt(NegK_1^{(3)}(p) Alt \ulcorner\varPhi\urcorner)]\big),$$

it follows that $\{A_j\}$ satisfies conditions (e) and (f).

Let $H_F^{(k)}(x, t_1, \ldots, t_k)$ be the formula $(u)(Ev)[\varphi(x, \iota_{k+2}(u, v, t_1, \ldots, t_k)) = \mathbf{0}]$. It is obvious that every set of the form $(**)$ with a general recursive R is strongly representable in F by a formula $H_F^{(k)}(\varDelta_e, t_1, \ldots, t_k)$. It follows that there are integers r_1, r_2 such that the formulas $H_F^{(4)}(\varDelta_{r_1}, x, y, z, \varDelta_h)$ and $H_F^{(4)}(\varDelta_{r_2}, x, y, z, \varDelta_h)$ strongly represent the relations non-$\bar{C}_h$ and $\bar{C}_h^*$ of lemma 1.

Finally it is not difficult to show that conditions VI (i)-(iii) are satisfied with our choice of the formulas $H_F^{(k)}$. Thus by theorem 2 the assertion of theorem 5 is proved.

6. In this section we shall deal with the system A_ω of analysis defined in [2] and with its extensions. It will be convenient to eliminate from A_ω function variables with more than one argument. Since the pairing functions are definable in A_ω, it is clear that this simplification of A_ω is not essential. A further (essential) change is that we shall add to the axioms of A_ω the following weak form of the axiom of choice [2]:

$$(A): \quad (x)(E\beta)\varPhi(x, \beta) \equiv (E\gamma)(x, \beta)\{(z)[\beta(z) = \gamma(\varDelta_2^x(\varDelta_2 z + \varDelta_1))]\} \supset \varPhi(x, \beta).$$

---

[2] We use in A_ω the notation of [2] with the only change that the n-th numeral is denoted by $\varDelta_n$ and multiplication by juxtaposition of terms.

It is known that for every arithmetically definable relation (function) there is an elementary propositional (or numerical) formula which strongly represents the relation (function) in A_ω. We shall use the notation $\overline{\varphi}(x, y)$, $\overline{\iota}(x, y)$, $\overline{\varkappa}(x)$, $\overline{\lambda}(x)$ for elementary (numerical) formulas which represent in A_ω the following functions: $U_n(m)$ (cf. the proof of lemma 6), $J(m, n)$, $K(m)$, $L(m)$.

Let W be the set of integers e such that the relation $\leqslant_e$ defined as $U_e(J(m, n)) = 0$ is a well-ordering of N_0. The order type of $\leqslant_e$ will be denoted by $|e|$. Since the relation $\leqslant_e$ is recursive, it follows that $|e| < \omega_1$ (cf. [5], p. 412). On the other hand, every recursive relation is representable as $U_e(J(m, n)) = 0$ for a suitable e (cf. lemma 7), and hence, by a theorem of Markwald ([7], p. 142) every infinite $\xi < \omega_1$ is representable as $|e|$ for a suitable e in W. Thus we have proved

LEMMA 9. $\{|e|:\ e \,\epsilon\, W\} = \{\xi:\ \omega \leqslant \xi < \omega_1\}$.

LEMMA 10. W is weakly representable in A_ω by a formula $\Pi(x)$ of the form $(\alpha, \beta)\mathcal{C}_W(\alpha, \beta, x)$, where $\mathcal{C}_W$ is an elementary formula.

Proof. Let $Ord(\alpha)$ be the formula

$$(x)\big[\alpha\big(\overline{\iota}(x, x)\big) = 0\big] \,\&\, (x, y, z)\big\{[\alpha\big(\overline{\iota}(x, y)\big) = 0] \,\&\, [\alpha\big(\overline{\iota}(y, z)\big) = 0] \supset$$
$$[\alpha\big(\overline{\iota}(x, z)\big) = 0]\big\} \,\&\, (x, y)\big\{(x = y) \vee [\alpha\big(\overline{\iota}(x, y)\big) = 0] \vee [\alpha\big(\overline{\iota}(y, x)\big) = 0]\big\}$$
$$\&\, (x, y)\big\{[\alpha\big(\overline{\iota}(x, y)\big) = 0] \,\&\, [\alpha\big(\overline{\iota}(y, x)\big) = 0] \supset (x = y)\big\},$$

and $F(\alpha, \beta)$ the formula

$$(x)\big[\alpha\big(\overline{\iota}\big(\beta(x+1), \beta(x)\big)\big) = 0\big] \supset (Ex)[\beta(x+1) = \beta(x)].$$

Take as $\mathcal{C}_W(\alpha, \beta, x)$ the formula

$$(y, z)\big[\alpha\big(\overline{\iota}(y, z)\big) = 0 \equiv \overline{\varphi}\big(x, \overline{\iota}(y, z)\big) = 0\big] \supset [Ord(\alpha) \,\&\, F(\alpha, \beta)].$$

We immediately see that the formula $(\alpha, \beta)\mathcal{C}_W(\alpha, \beta, \Delta_e)$ is true in the principal model $\mathfrak{N}_0$ (cf. [2], p. 190) if and only if $e \,\epsilon\, W$. Hence, by theorem 3.1.E of [2]

$$e \,\epsilon\, W \equiv \,\vdash_\omega (\alpha, \beta)\, \mathcal{C}_W(\alpha, \beta, \Delta_e).$$

This proves the lemma.

Let $Im(\alpha, x, y)$, $Ims(\alpha, x, y, z)$ be the following formulas

$$(u, v)\big\{[\overline{\varphi}\big(x, \overline{\iota}(u, v)\big) = 0] \equiv \big[\overline{\varphi}\big(y, \overline{\iota}\big(\alpha(u), \alpha(v)\big)\big) = 0\big]\big\},$$

$$(u, v)\big\{[\overline{\varphi}\big(x, \overline{\iota}(u, v)\big) = 0] \equiv \big[\overline{\varphi}\big(y, \overline{\iota}\big(\alpha(u), \alpha(v)\big)\big) = 0\big]$$
$$\&\, [\overline{\varphi}\big(y, \overline{\iota}\big(\alpha(u), z\big)\big) = 0] \,\&\, \big\{[\overline{\varphi}\big(y, \overline{\iota}(v, z)\big) = 0] \supset (Ew)[\alpha(w) = v]\big\}\big\}.$$

LEMMA 11. If ψ is a function and e_1, e_2, n are integers, then ψ, e_1, e_2 satisfy $Im(\alpha, x, y)$ in $\mathfrak{N}_0$ if and only if ψ maps N_0 into itself and $p \leqslant_{e_1} q \equiv \psi(p) \leqslant_{e_2} \psi(q)$; ψ, e_1, e_2, n satisfy $Ims(\alpha, x, y, z)$ if and only if ψ maps N_0 onto a segment of $\leqslant_{e_2}$ determined by n and ψ satisfies the equivalence $p \leqslant_{e_1} q \equiv \psi(p) \leqslant_{e_2} \psi(q)$.

Lemma 11 follows directly from the definitions of formulas Im and Ims.

LEMMA 12.

(i) $\vdash_\omega \Pi(x) \,\&\, \Pi(y) \supset \{(E\gamma)[Im(\gamma, x, y)] \equiv (\gamma)(z)[\sim Ims(\gamma, y, x, z)];$

(ii) $\vdash_\omega \Pi(x) \,\&\, \Pi(y) \supset [(E\gamma)Im(\gamma, x, y) \vee (E\gamma)Im(\gamma, y, x)].$

We obtain a proof of this lemma formalizing in A_ω the proofs of the following well-known set-theoretical theorems: (1) if $\leqslant_{e_1}, \leqslant_{e_2}$ are well-orderings, then a similarity mapping of the field of $\leqslant_{e_1}$ into the field of $\leqslant_{e_2}$ exists if and only if $\leqslant_{e_2}$ is not similar to any segment of $\leqslant_{e_1}$; (2) if $\leqslant_{e_1}, \leqslant_{e_2}$ are well-orderings, then one of them is similar to a restriction of the other to a suitable subset of its field.

LEMMA 13. Relations $|e_1| \leqslant |e_2|$, $|e_1| < |e_2|$ and $|e_1| = |e_2|$ are strongly representable in A_ω relatively to the set $W \times W$.

Proof. By lemma 11 and theorem 3.1.E of [2]

$$|e_2| \leqslant |e_1| \equiv \{(\gamma)(z) \sim Ims(\gamma, \Delta_{e_1}, \Delta_{e_2}, z) \text{ is true in } \mathfrak{N}_0\}$$
$$\equiv \,\vdash_\omega (\gamma)(z) \sim Ims(\gamma, \Delta_{e_1}, \Delta_{e_2}, z)$$

provided that e_1, e_2 are in W. Using again lemma 11, we obtain similarly for e_1, e_2 in W

$$|e_1| > |e_2| \equiv \,\vdash_\omega (\gamma) \sim Im(\gamma, \Delta_{e_1}, \Delta_{e_2}),$$

and hence by lemmas 12 (i) and 10

$$|e_1|\text{non} \leqslant |e_2| \equiv \,\vdash_\omega \sim (\gamma)(z) \sim Ims(\gamma, \Delta_{e_2}, \Delta_{e_1}, z).$$

The lemma is thus proved for the relation $\leqslant$.

It follows immediately that it is true for the remaining two relations because if e_1, e_2 are in W, then

$$|e_1| < |e_2| \equiv \text{non}(|e_2| \leqslant |e_1|) \quad \text{and} \quad |e_1| = |e_2| \equiv (|e_1| \leqslant |e_2|) \,\&\, (|e_2| \leqslant |e_1|).$$

For later use we notice that the relations $|e_1| = |e_2|$, $|e_1| < |e_2|$ and $|e_1| \leqslant |e_2|$ are strongly represented (relatively to $W \times W$) by the formulas $(E\gamma)Im(\gamma, x, y) \,\&\, (E\gamma)Im(\gamma, y, x)$, $(\gamma) \sim Im(\gamma, y, x)$ and $(E\gamma)Im(\gamma, x, y)$. We abbreviate these formulas as $x \approx y$, $x \prec y$, and $x \preceq y$.

LEMMA 14. There is a primitive recursive function $f(e, n)$ such that if Γ is a (numerical) formula which represents f in A_ω, then

(16) $\vdash_\omega \overline{\varphi}\big(\Gamma(u, v), \overline{\iota}(x, y)\big) = 0 \equiv \Big((x = y) \vee (x < y) \,\&\, (y < v)$

$\vee (v \leqslant x) \,\&\, (y < v) \vee (v \leqslant x) \,\&\, (v \leqslant y) \,\&\, \big\{[\overline{\varkappa}_1^{(2)}(x \,\dot-\, v) < \overline{\varkappa}_1^{(2)}(y \,\dot-\, v)]$

$\vee [\overline{\varkappa}_1^{(2)}(x \,\dot-\, v) = \overline{\varkappa}_1^{(2)}(y \,\dot-\, v)] \,\&\, [\overline{\varphi}\big(u, \overline{\iota}\big(\overline{\varkappa}_2^{(2)}(x \,\dot-\, v), \overline{\varkappa}_2^{(2)}(y \,\dot-\, v)\big)\big) = 0]\big\}\Big)$

$(x \doteq y$ is here an elementary numerical formula which represents the function $m \doteq n$ in A_ω).

Proof. The relation $(m = n) \vee (m < n < p) \vee (p \leqslant m) \,\&\, (n < p)$ is recursive and hence equivalent to $U_{r_1}(J_3(m, n, p)) = 0$ for a suitable r_1. The proof of this equivalence being formalizable in A_ω, we obtain

$$\vdash_\omega \bar{\varphi}(\varDelta_{r_1}, \bar{\iota}_3(x, y, v)) = 0 \equiv (x = y) \vee (x < y) \,\&\, (y < v) \vee (v \leqslant x) \,\&\, (y < v).$$

Since axiom (Ω_{20}) of R^{**} is provable in A_ω, we obtain

$$\vdash_\omega \bar{\varphi}(\varDelta_{f_1(n)}, \bar{\iota}(x, y)) = 0 \equiv (x = y) \vee (x < y) \,\&\, (y < \varDelta_n) \vee (\varDelta_n \leqslant x) \,\&\, (y < \varDelta_n)$$

where $f_1(n) = 6 \cdot 2^{n+1}(2r_1 + 1)$. Arguing similarly we obtain primitive recursive functions $f_2(n), f_3(n), f_4(n), f_5(n)$ such that

$$\vdash_\omega \bar{\varphi}(\varDelta_{f_2(n)}, \bar{\iota}(x, y)) = 0 \equiv (\varDelta_n \leqslant x) \,\&\, (\varDelta_n \leqslant y),$$

(17)
$$\vdash_\omega \bar{\varphi}(\varDelta_{f_3(n)}, \bar{\iota}(x, y)) = 0 \equiv \bar{\varkappa}_1^{(2)}(x \doteq \varDelta_n) < \bar{\varkappa}_1^{(2)}(y \doteq \varDelta_n),$$

$$\vdash_\omega \bar{\varphi}(\varDelta_{f_4(n)}, \bar{\iota}(x, y)) = 0 \equiv \bar{\varkappa}_1^{(2)}(x \doteq \varDelta_n) = \bar{\varkappa}_1^{(2)}(y \doteq \varDelta_n),$$

$$\vdash_\omega \bar{\varphi}(\varDelta_{f_5(n)}, \bar{\iota}(x, y)) = \bar{\iota}(\bar{\varkappa}_2^{(2)}(x \doteq \varDelta_n), \bar{\varkappa}_2^{(2)}(y \doteq \varDelta_n)).$$

From the last formula and from (Ω_{17}), which is valid in A_ω we obtain

$$\vdash_\omega \bar{\varphi}(\varDelta_{f_6(e,n)}, \bar{\iota}(x, y)) = \bar{\varphi}(\varDelta_e, \bar{\iota}(\bar{\varkappa}_2^{(2)}(x \doteq \varDelta_n), \bar{\varkappa}_2^{(2)}(y \doteq \varDelta_n))),$$

where $f_6(e, n) = 12J(e, f_5(n)) + 8$. Using this formula and (17) and observing that $\vdash_\omega (x + y = 0) \equiv (x = 0) \,\&\, (y = 0)$ and that the axiom (Ω_{15}) is valid in A_ω, we obtain

$$\vdash_\omega \bar{\varphi}(\varDelta_{f_7(e,n)}, \bar{\iota}(x, y)) = 0 \equiv [\bar{\varkappa}_1^{(2)}(x \doteq \varDelta_n) = \bar{\varkappa}_1^{(2)}(y \doteq \varDelta_n)]$$
$$\&\, [\bar{\varphi}(\varDelta_e, \bar{\iota}(\bar{\varkappa}_2^{(2)}(x \doteq \varDelta_n), \bar{\varkappa}_2^{(2)}(y \doteq \varDelta_n))) = 0],$$

where $f_7(e, n) = 12J(f_6(e, n), f_4(n)) + 6$. Continuing in this way we finally obtain a primitive recursive function $f(e, n) = f_{10}(e, n)$ such that if Γ represents f in A_ω, then (16) is valid for $u = \varDelta_e$, $v = \varDelta_n$ $(e, n = 0, 1, 2, ...)$. Using rule ω we see that (16) is valid for this choice of f.

LEMMA 15. $\vdash_\omega \Pi(u) \supset \Pi(\Gamma(u, v))$.

To prove this lemma we formalize in A_ω the set-theoretical theorem saying that the formula on the right-hand side of (16) defines a well-ordering of N_0 whenever $\bar{\varphi}(u, \bar{\iota}(x, y)) = 0$ does so.

LEMMA 16. If $e \in W$, then $f(e, n) \in W$ and $|f(e, n)| = \omega |e| + n$.

Proof. If $\leqslant_e$ has the order type ξ, then the formula on the right-hand side of (16) (with u replaced by $\varDelta_e$ and v replaced by $\varDelta_n$) defines a relation of type $\omega \cdot \xi + n$.

LEMMA 17. $\vdash_\omega \Pi(u) \,\&\, \Pi(u') \,\&\, \Gamma(u, v) \approx \Gamma(u', v') \supset (u \approx u') \,\&\, (v = v')$.

The lemma is proved by formalizing in A_ω the proof of a set-theoretical theorem stating that $\omega \cdot \xi + n = \omega \cdot \xi' + n'$ implies $n = n'$ and $\xi = \xi'$.

We now take $P = W$ and define $\leqslant$ as $|f(m, n)| \leqslant |f(p, q)|$. Let Π be the formula weakly representing P constructed in lemma 10 and let M be the formula $\Gamma(x, y) \preceq \Gamma(z, t)$.

LEMMA 18. *Conditions* III *and* IV *are satisfied for the above choice of* P, $\leqslant$, Π, *and* M.

Proof. III is obvious. In lemmas 10, 13, and 16 we proved that Π weakly represents P and M strongly represents $\leqslant$ relatively to $P \times N_0 \times P \times N_0$. Formula IV (i) follows from lemma 17 and IV (ii) from lemmas 15 and 12 (ii). Thus it remains to prove IV (iii).

Assume that $p_0 \in W$ and that $\vdash_\omega \Phi(\varDelta_p, \varDelta_q)$ for every pair (p, q) in $W \times N_0$ such that $|f(p, q)| \leqslant |f(p_0, q_0)|$. It follows that

(18) $\vdash_\omega \Pi(\varDelta_p) \,\&\, M(\varDelta_p, \varDelta_q; \varDelta_{p_0}, \varDelta_{q_0}) \supset \Phi(\varDelta_p, \varDelta_q)$

for every such pair. If $p \in W$ and $|f(p, q)| > |f(p_0, q_0)|$, then $\vdash_\omega \sim M(\varDelta_p, \varDelta_q; \varDelta_{p_0}, \varDelta_{q_0})$ (since M strongly represents $\leqslant$ relatively to $W \times N_0 \times W \times N_0$) and hence (18) continues to hold. Finally if $p \notin W$, then the formula $M(\varDelta_p, \varDelta_q; \varDelta_{p_0}, \varDelta_{q_0})$ (i. e. $(\mathrm{E}\gamma)Im(\gamma, \Gamma(\varDelta_p, \varDelta_q), \Gamma(\varDelta_{p_0}, \varDelta_{q_0}))$) is false in the principal model $\mathfrak{R}_0$, whence it follows that its negation is true in $\mathfrak{R}_0$ and therefore provable in A_ω according to theorem 3.1.E of [2]. Thus (18) holds for arbitrary p, q, which proves (by the rule ω) that $\vdash_\omega \Pi(x) \,\&\, M(x, y, \varDelta_{p_0}, \varDelta_{q_0}) \supset \Phi(x, y)$. Lemma 18 is thus proved.

We shall now discuss condition VI. First we introduce some definitions. Let $(x)_y$ and $lh(x)$ be elementary formulas which represent in A_ω the functions $(m)_j$ and $lh(m)$ of [4], p. 230 and let $Seq(x)$ be an elementary formula which strongly represents in A_ω the relation $(j)[j \leqslant lh(n) \supset (n)_j > 0]$. The following formulas will play a fundamental role in what follows (we abbreviate the string $(a_0, ..., a_{k-1})$ of functional variables as $\boldsymbol{a}$ and the string $(x_0, ..., x_{l-1})$ of numerical variables as $\boldsymbol{x}$)

$D(\boldsymbol{a}, s, t)$:

$$Seq((s)_{\varDelta_0}) \,\&\, [lh((s)_{\varDelta_0}) = t] \,\&\, (z)[z \leqslant t \supset ((s)_{\varDelta_0})_z = a_0(z)]$$

$$\&\, Seq((s)_{\varDelta_1}) \,\&\, [lh((s)_{\varDelta_1}) = t] \,\&\, (z)[z \leqslant t \supset ((s)_{\varDelta_1})_z = a_1(z)] \,\&\, ...$$

$$\&\, Seq((s)_{\varDelta_{k-1}}) \,\&\, [lh((s)_{\varDelta_{k-1}}) = t] \,\&\, (z)[z \leqslant t \supset ((s)_{\varDelta_{k-1}})_z = a_{k-1}(z)];$$

$H^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x})$: $(\beta)(\mathrm{E}s, t)\{D(\boldsymbol{a}, \beta, s, t) \,\&\, [\bar{\varphi}(y, \bar{\iota}_{l+1}(s, \boldsymbol{x})) = 0]\}$,

$\bar{H}^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x})$: $(\mathrm{E}\beta)(s, t)\{D(\boldsymbol{a}, \beta, s, t) \supset [\bar{\varphi}(y, \bar{\iota}_{l+1}(s, \boldsymbol{x})) = 0]\}$,

$I_j^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x})$: $(\iota z)H^{(k,l+1)}(y, \boldsymbol{a}, x_0, ..., x_{j-1}, z, x_j, ..., x_{l-1})$,

$\bar{I}_j^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x})$: $(\iota z)\bar{H}^{(k,l+1)}(y, \boldsymbol{a}, x_0, ..., x_{j-1}, z, x_j, ..., x_{l-1})$.

The formula D can be read: s is a diagram of functions $\boldsymbol{a}$ in the interval $(0, t)$; formulas $H^{(k,l)}$, $\bar{H}^{(k,l)}$ are formalizations of the relation

$$(\psi)(\mathrm{E}p)\left[U_n\big(J_{l+1}(2^{\bar{\varphi}_0(p)} \cdot 3^{\bar{\varphi}_1(p)} \ldots p_{k-1}^{\bar{\varphi}_{k-1}(p)} p_k^{\bar{\varphi}(p)}, m_0, \ldots, m_{l-1})\big) = 0\right]$$

and of a similar relation with the order of quantifiers reversed.

In the following lemmas 19-27 we show that theorems 3.2-3.5 of Kleene [6] have formal counterparts provable in A_ω.

LEMMA 19. *There is a primitive recursive function $g_0(m)$ such that if ν is an elementary numerical formula representing g_0 in A_ω, then*

$$\vdash_\omega H^{(k,l)}\big(\nu(y), \boldsymbol{a}, \boldsymbol{x}\big) \equiv {\sim} \bar{H}^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}),$$
$$\vdash_\omega \bar{H}^{(k,l)}\big(\nu(y), \boldsymbol{a}, \boldsymbol{x}\big) \equiv {\sim} H^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}).$$

Proof. Since 0^n is a primitive recursive function, there is an r such that $U_r(n) = 0^n$. The recursion equations for $\varphi(\Delta_r, \boldsymbol{x})$ being deducible from the axioms (Ω_1)-(Ω_{24}) and these axioms being provable in A_ω, we infer that

$$\vdash_\omega \bar{\varphi}(\Delta_r, \Delta_0) = \Delta_1, \qquad \vdash_\omega x > \Delta_0 \supset \bar{\varphi}(\Delta_r, x) = \Delta_0.$$

Now put $g_0(n) = 12J(n, r) + 8$ and let ν be an elementary numerical formula representing g_0 in A_ω. It follows that

$$\vdash_\omega \bar{\varphi}\big(\nu(y), z\big) = 0 \equiv \bar{\varphi}(y, z) \neq 0,$$

whence immediately follow both formulas of lemma 19.

LEMMA 20. *There are primitive recursive functions $g_k(m, n)$, $\bar{g}_k(m, n)$ such that if π_k, $\bar{\pi}_k$ are elementary numerical formulas representing g_k and $\bar{g}_k$ in A_ω, then*

$$\vdash_\omega H^{(k,l)}\big(\pi_k(y_1, y_2), \boldsymbol{a}, \boldsymbol{x}\big) \equiv [H^{(k,l)}(y_1, \boldsymbol{a}, \boldsymbol{x}) \,\&\, H^{(k,l)}(y_2, \boldsymbol{a}, \boldsymbol{x})],$$
$$\vdash_\omega \bar{H}^{(k,l)}\big(\bar{\pi}_k(y_1, y_2), \boldsymbol{a}, \boldsymbol{x}\big) \equiv [\bar{H}^{(k,l)}(y_1, \boldsymbol{a}, \boldsymbol{x}) \lor \bar{H}^{(k,l)}(y_2, \boldsymbol{a}, \boldsymbol{x})].$$

In view of lemma 19 it will be sufficient to prove only the second part of the lemma. The formula in square brackets on the right-hand side is equivalent to

$$(19) \quad (\mathrm{E}\beta)(s', s'', t', t'')\{D(\boldsymbol{a}, \beta, s', t')$$
$$\& \; D(\boldsymbol{a}, \beta, s'', t'') \supset [\bar{\varphi}\big(y_1, \bar{\iota}_{l+1}(s', \boldsymbol{x})\big) \cdot \bar{\varphi}\big(y_2, \bar{\iota}_{l+1}(s'', \boldsymbol{x})\big) = 0]\}.$$

Consider the primitive recursive functions

$$b(m, q) = \prod_{i<q} p_i^{(m)_i},$$

$$f_1(m, q, k) = \prod_{i<k} p_i^{b((m)_i, K(q))}, \qquad f_2(m, q, k) = \prod_{i<k} p_i^{b((m)_i, L(q))}.$$

It is then easy to prove that

$$\vdash_\omega D(\boldsymbol{a}, \beta, \Delta_m, \Delta_n) \supset D(\boldsymbol{a}, \beta, \Delta_{f_1(m,n,k+1)}, \Delta_{K(n)}),$$
$$\vdash_\omega D(\boldsymbol{a}, \beta, \Delta_m, \Delta_n) \supset D(\boldsymbol{a}, \beta, \Delta_{f_2(m,n,k+1)}, \Delta_{L(n)}).$$

Let r_i $(i = 1, 2)$ be integers such that $\bar{\varphi}\big(\Delta_{r_i}, \bar{\iota}_3(x, y, z)\big)$ represents f_i in A_ω and let $\varrho_i(k) = 6 \cdot 2^{k+2}(2r_i + 1)$. From axiom (Ω_{20}) it follows that $\bar{\varphi}\big(\Delta_{\varrho_i(k)}, \bar{\iota}(x, y)\big)$ represents $f_i(m, n, k+1)$ considered as the function of m, n alone. Using rule ω we infer that

$$\vdash_\omega D(\boldsymbol{a}, \beta, s, t) \supset D\big(\boldsymbol{a}, \beta, \bar{\varphi}(\Delta_{\varrho_1(k)}, \bar{\iota}(s, t)), \bar{\varkappa}(t)\big) = 0,$$
$$\vdash_\omega D(\boldsymbol{a}, \beta, s, t) \supset D\big(\boldsymbol{a}, \beta, \bar{\varphi}(\Delta_{\varrho_2(k)}, \bar{\iota}(s, t)), \bar{\lambda}(t)\big) = 0.$$

Obviously in both formulas we can replace t by $lh(s)$.

From these formulas we infer that (19) implies (in A_ω) the following formula

$$(\mathrm{E}\beta)(s, t) D(\boldsymbol{a}, \beta, s, t) \supset \bar{\varphi}\Big(y_1, \bar{\iota}_{l+1}\big(\bar{\varphi}(\Delta_{\varrho_1(k)}, \bar{\iota}(s, lh(s))), \boldsymbol{x}\big)\Big)$$
$$\cdot \bar{\varphi}\Big(y_2, \bar{\iota}_{l+1}\big(\bar{\varphi}(\Delta_{\varrho_2(k)}, \bar{\iota}(s, lh(s))), \boldsymbol{x}\big)\Big) = 0.$$

Conversely this formula implies (19), as we easily see using the theorem $\vdash_\omega (\mathrm{E}!s) D(\boldsymbol{a}, \beta, s, t)$.

We can simplify the formula obtained above by observing that $\vdash_\omega \bar{\iota}\big(s, lh(s)\big) = \bar{\varphi}(\Delta_q, s)$ for a suitable q and hence $\vdash_\omega \bar{\varphi}\big(\Delta_{\varrho_i(k)}, \bar{\iota}(s, lh(s))\big) = \bar{\varphi}(\Delta_{\bar{\varrho}_i(k)}, s)$ with $\bar{\varrho}_i(k) = 12J(\varrho_i(k), q) + 8$ (see axiom (Ω_{17})). Thus (19) is equivalent to

$$(\mathrm{E}\beta)(s, t)\Big[D(\boldsymbol{a}, \beta, s, t) \supset \bar{\varphi}\big(y_1, \bar{\iota}_{l+1}(\bar{\varphi}(\Delta_{\bar{\varrho}_1(k)}, s), \boldsymbol{x})\big)$$
$$\cdot \bar{\varphi}\big(y_2, \bar{\iota}_{l+1}(\bar{\varphi}(\Delta_{\bar{\varrho}_2(k)}, s), \boldsymbol{x})\big) = 0\Big].$$

Finally, we notice that in view of axioms (Ω_{16})-(Ω_{18})

$$\vdash_\omega \bar{\varphi}\big(y_1, \bar{\iota}_{l+1}(\bar{\varphi}(\Delta_{\bar{\varrho}_1(k)}, s), \boldsymbol{x})\big) \cdot \bar{\varphi}\big(y_2, \bar{\iota}_{l+1}(\Delta_{\bar{\varrho}_2(k)}, s), \boldsymbol{x})\big) = 0$$
$$\equiv \bar{\varphi}\big(\pi_k(y_1, y_2), \bar{\iota}_{l+1}(s, \boldsymbol{x})\big) = 0,$$

where $\pi_k(y_1, y_2)$ represents the function

$$12J\big(12J(m_1, \bar{\varrho}_1(k)) + 8, \; 12J(m_2, \bar{\varrho}_2(k)) + 8\big) + 7.$$

LEMMA 21. *For every l there are primitive recursive functions $g_1(m, n)$, $\bar{g}_1(m, n)$ such that if σ, $\bar{\sigma}$ are elementary numerical formulas representing g_1 and $\bar{g}_1$, then*

$$\vdash_\omega H^{(k,l)}\big(\sigma(y_1, y_2), \boldsymbol{a}, \boldsymbol{x}\big) \equiv [H^{(k,l)}(y_1, \boldsymbol{a}, \boldsymbol{x}) \lor H^{(k,l)}(y_2, \boldsymbol{a}, \boldsymbol{x})],$$

Proof. Again we shall prove only the second equivalence. Let $S(\gamma, \beta_1, \beta_2)$ be the formula $(z)\big[\gamma(z) = \bar{\imath}\big(\beta_1(z), \beta_2(z)\big)\big]$ and $S^*(s, s', s'')$ the formula $[lh(s) = lh(s') = lh(s'')] \,\&\, (z)\big\{z \leqslant lh(s) \supset [(s)_z = \bar{\imath}\big((s')_z, (s'')_z\big)]\big\}$. It is obvious that

$$\vdash_\omega (\beta_1, \beta_2)(\mathbf{E}!\gamma)\, S(\gamma, \beta_1, \beta_2)\,, \qquad \vdash_\omega (\gamma)(\mathbf{E}!\,\beta_1, \beta_2)\, S(\gamma, \beta_1, \beta_2)$$

and

$$\vdash_\omega D(\gamma, s, t) \,\&\, D(\beta_1, s', t) \,\&\, D(\beta_2, s'', t) \,\&\, S(\gamma, \beta_1, \beta_2) \supset S^*(s, s', s'')\,.$$

If m, m', m'' satisfy S^* in $\mathfrak{N}_0$, then $m' = 2^{K((m)_0)} \cdot 3^{K((m)_1)} \dots p_{lh(m)}^{K((m)_{lh(m)})}$ $= f'(m),\ \ m'' = 2^{L((m)_0)} \cdot 3^{L((m)_1)} \dots p_{lh(m)}^{L((m)_{lh(m)})} = f''(m)$. The function f', f'' are primitive recursive; let r_1, r_2 be integers such that $\bar{\varphi}(\varDelta_{r_1}, x)$ represents f' in A_ω and $\bar{\varphi}(\varDelta_{r_2}, x)$ represents f'' in A_ω. We then have $\vdash_\omega S^*\big(s, \bar{\varphi}(\varDelta_{r_1}, s), \bar{\varphi}(\varDelta_{r_2}, s)\big)$ and $\vdash_\omega(s)(\mathbf{E}!\,s', s'')\,S^*(s, s', s'')$.

It is now easily seen that the right-hand side of the second formula in the lemma is equivalent to

$$(20) \qquad (\mathbf{E}\gamma)(s, t)\Big\{D(\boldsymbol{a}, \gamma, s, t) \supset \big[\bar{\varphi}\big(y_1, \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_{r_1}, s), \boldsymbol{x}\big)\big) \\ + \bar{\varphi}\big(y_2, \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_{r_2}, s), \boldsymbol{x}\big)\big) = 0\big]\Big\}\,.$$

We reduce this formula to the desired form as follows: since the function $J_{l+1}\big(f'(K_1^{(l+1)}(p)), K_2^{(l+1)}(p), \dots, K_{l+1}^{(l+1)}(p)\big)$ is primitive recursive, there is an integer r' such that

$$\vdash_\omega \bar{\varphi}(\varDelta_{r'}, z) = \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_{r_1}, \bar{\varkappa}_1^{(l+1)}(z)), \bar{\varkappa}_2^{(l+1)}(z), \dots, \bar{\varkappa}_{l+1}^{(l+1)}(z)\big)\,.$$

This gives

$$\vdash_\omega \bar{\varphi}\big(y_1, \bar{\varphi}(\varDelta_{r'}, \bar{\imath}_{l+1}(s, \boldsymbol{x}))\big) = \bar{\varphi}\big(y_1, \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_{r_1}, s), \boldsymbol{x}\big)\big)\,.$$

Similar equation is provable with r_1 replaced by r_2 and r' by r''. Thus (20) is equivalent to $\bar{H}^{(k,l)}\big(\bar{\sigma}(y_1, y_2), \boldsymbol{a}, \boldsymbol{x}\big)$ with

$$\bar{\sigma}(y_1, y_2) = \varDelta_{12}\bar{\imath}\big(\varDelta_{12}\bar{\imath}(y_1, \varDelta_{r'}) + \varDelta_8, \varDelta_{12}\bar{\imath}(y_2, \varDelta_{r''}) + \varDelta_8\big) + \varDelta_6\,.$$

Hence $\bar{g}_1(m, n) = 12J\big(12J(m, r') + 8, 12J(n, r'') + 8\big) + 6\,.$

Lemma 22. *There are primitive recursive functions $g_2, \bar{g}_2$ such that if $l \geqslant 1$ and $\tau, \bar{\tau}$ are numerical formulas representing g_2 and $\bar{g}_2$, then*

$$\vdash_\omega H^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}) \equiv H^{(k,l-1)}\big(\tau(y, x_{l-1}), \boldsymbol{a}, x_0, \dots, x_{l-2}\big)\,,$$

$$\vdash_\omega \bar{H}^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}) \equiv \bar{H}^{(k,l-1)}\big(\bar{\tau}(y, x_{l-1}), \boldsymbol{a}, x_0, \dots, x_{l-2}\big)\,.$$

Proof. By axiom (Ω_{20}) we obtain

$$\vdash_\omega H^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}) \equiv H^{(k,l-1)}\big(\varDelta_6\, \varDelta_2^{x_{l-1}+1}(\varDelta_2 y + \varDelta_1), \boldsymbol{a}, x_0, \dots, x_{l-2}\big)\,.$$

Lemma 23. *For every l there are primitive recursive functions $g_3, \bar{g}_3$ such that if $v, \bar{v}$ are elementary terms representing them in A_ω, then for $j = 0, 1, \dots, l-1$*

$$\vdash_\omega H^{(k,l-1)}\big(v(y, \varDelta_j), \boldsymbol{a}, x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{l-1}\big) \equiv (\mathbf{E}x_j)\, H^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x})\,,$$

$$\vdash_\omega \bar{H}^{(k,l-1)}\big(\bar{v}(y, \varDelta_j), \boldsymbol{a}, x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{l-1}\big) \equiv (x_j)\, \bar{H}^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x})\,.$$

Proof. The right-hand side of the second formula is equivalent to

$$(x_j)(\mathbf{E}\beta)(s, t)\big[D(\boldsymbol{a}, \beta, s, t) \supset \bar{\varphi}\big(y, \bar{\imath}_{l+1}(s, \boldsymbol{x})\big) = 0\big]\,.$$

Using axiom (A) of A_ω (cf. p. 217) we transform this formula to

$$(21) \qquad (\mathbf{E}\gamma)(\beta, x_j, s, t)\big\{(z)\big[\beta(z) = \gamma\big(\varDelta_2^{x_j}(\varDelta_2 z + \varDelta_1)\big)\big] \\ \&\, D(\boldsymbol{a}, \beta, s, t) \supset \bar{\varphi}\big(y, \bar{\imath}_{l+1}(s, \boldsymbol{x})\big) = 0\big]\big\}\,.$$

Put $q(m, n) = [lh(m)/2^n - \tfrac{1}{2}]$, $h(n, m) = \prod\limits_{0 \leqslant j \leqslant q(m,n)} p^{(m)_k}_{2^n(2j+1)}$ and let r be an integer such that $\bar{\varphi}\big(\varDelta_r, \bar{\imath}(x, y)\big)$ represents in A_ω the function $\prod\limits_{0 \leqslant j < k} p_j^{b((m)_j, q(m,n))} \cdot p_k^{h(m,n)}$. We shall show (informally) that (21) is equivalent to

$$(22) \qquad (\mathbf{E}\gamma)(s, t)\Big\{D(\boldsymbol{a}, \gamma, s, t) \\ \supset (x_j)\big[x_j \leqslant t \supset \bar{\varphi}\big(y, \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_r, \bar{\imath}(x_j, s))\big), \boldsymbol{x}\big) = 0\big]\Big\}\,.$$

Assume (21) and choose s, t, x_j so that $D(\boldsymbol{a}, \gamma, s, t)$ and $x_j \leqslant t$. If $\beta(z) = \gamma\big(\varDelta_2^{x_j}(\varDelta_2 z + 1)\big)$, then $\bar{\varphi}\big(\varDelta_r, \bar{\imath}(x_j, s)\big)$ satisfies the condition $D\big(\boldsymbol{a}, \beta, \bar{\varphi}(\varDelta_r, \bar{\imath}(x_j, s)), t'\big)$ for a suitable t', and hence, by (21), we obtain

$$\bar{\varphi}\big(y, \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_r, \bar{\imath}(x_j, s)), \boldsymbol{x}\big)\big) = 0\,.$$

The converse implication is proved similarly. It remains to reduce (22) to the form indicated in the lemma. By the technique already used we find a primitive recursive function g_4 such that if ξ is an elementary formula representing g_4, then

$$\vdash_\omega \big[\bar{\varphi}\big(y, \bar{\imath}_{l+1}\big(\bar{\varphi}(\varDelta_r, \bar{\imath}(x_j, s)), \boldsymbol{x}\big) = 0\big] \\ \equiv \big[\bar{\varphi}\big(\xi(y, \varDelta_j), \bar{\imath}(\bar{\imath}_1(s, x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{l-1}), x_j)\big) = 0\big]\,.$$

It follows by axiom (Ω_{22}) that

$$\vdash_\omega \bar\varphi\Big(\Delta_{12}\xi(y,\Delta_j)+\Delta_{11},\, \bar\imath\big(lh(s),\bar\imath_q(s,x_0,...,x_{j-1},x_{j+1},...,x_{l-1})\big)\Big)=0$$
$$\equiv (x_j)\Big[x_j\leqslant lh(s)\supset \bar\varphi\big(\xi(y,\Delta_j),\,\bar\imath(\bar\imath_q(s,x_0,...,x_{j-1},x_{j+1},...,x_{l-1}),x_j)\big)=0\Big].$$

The left-hand side of this equivalence can obviously be represented in the form $\bar\varphi\big(\bar v(y,\Delta_j),\,\bar\imath_q(s,x_0,...,x_{j-1},x_{j+1},...,x_{l-1})\big)=0$ where $\bar v$ is an elementary term representing a primitive recursive function. Since $\vdash_\omega D(a,\gamma,s,t)\supset t=lh(s)$, we see that (22) is equivalent to

$$\bar H^{(k,l-1)}\big(\bar v(y,\Delta_j),\,a,x_0,...,x_{j-1},x_{j+1},...,x_{l-1}\big).$$

LEMMA 24. *There are primitive recursive functions $g_5,\bar g_5$ such that if $\eta,\bar\eta$ are elementary terms representing them in A_ω, then for $j=0,1,...,l-1$*

$$\vdash_\omega H^{(k,l)}(y,a,x)\equiv H^{(k,l+1)}\big(\eta(y,\Delta_j,\Delta_l),\,a,x_0,...,x_{j-1},z,x_j,...,x_{l-1}\big),$$
$$\vdash_\omega \bar H^{(k,l)}(y,a,x)\equiv \bar H^{(k,l+1)}\big(\bar\eta(y,\Delta_j,\Delta_l),\,a,x_0,...,x_{j-1},z,x_j,...,x_{l-1}\big).$$

The proof is similar to that of previous lemmas.

LEMMA 25. *For every l there are primitive recursive functions $g_6,\bar g_6$ such that if $\zeta,\bar\zeta$ are elementary terms representing them in A_ω, then for $j=0,1,...,l-1$*

$$\vdash_\omega H^{(k,l-1)}\big(\zeta(y,\Delta_j),\,a,x_0,...,x_{j-1},x_{j+1},...,x_{l-1}\big)\equiv (x_j)H^{(k,l)}(y,a,x),$$
$$\vdash_\omega \bar H^{(k,l-1)}\big(\bar\zeta(y,\Delta_j),\,a,x_0,...,x_{j-1},x_{j+1},...,x_{l-1}\big)\equiv (Ex_j)\bar H^{(k,l)}(y,a,x).$$

Proof. Let us again consider only the second formula. The right-hand side of it is equivalent to

$$(E\beta)(s,t)\Big[D(a,\beta,s,t)\supset \bar\varphi\big(y,\bar\imath_{l+1}(s,x_0,...,x_{j-1},\beta(\Delta_0),x_{j+1},...,x_{l-1})\big)=0\Big].$$

We can replace $\beta(\Delta_0)$ by $\big((s)_{\Delta_k}\big)_{\Delta_0}$ and then use the technique of the preceding lemmas to reduce the right-hand side to the desired form.

LEMMA 26. *Let Γ be an elementary numerical formula and C an elementary propositional formula and let the variables (free and bound) occurring in them be some of the variables $a_0,...,a_{k-1},x_0,...,x_{l-1}$. Then there exist integers $j,f,\bar f,e,\bar e$ such that*

$$\vdash_\omega \Gamma=I_j^{(k,l)}(\Delta_f,a,x),\qquad \vdash_\omega C\equiv H^{(k,l)}(\Delta_e,a,x),$$
$$\vdash_\omega \Gamma=\bar I_j^{(k,l)}(\Delta_{\bar f},a,x),\qquad \vdash_\omega C\equiv \bar H^{(k,l)}(\Delta_{\bar e},a,x).$$

Proof. From lemmas 23 and 25 it is obvious that if lemma 26 holds for the formula C, then it does so for the formulas $(x_j)C$ and $(Ex_j)C$.

Lemmas 19, 20 and 21 show that if lemma 26 holds for formulas C_1,C_2, then it does so for the formulas $\sim C_1$, $C_1\vee C_2$ and $C_1\,\&\,C_2$.

Let us assume that the lemma holds for numerical formulas Γ_1,Γ_2 and that $f_i,\bar f_i$ are the corresponding integers, $i=1,2$. For convenience of notation replace l by $l+1$ and assume $j=l+1$. Then

$$\vdash_\omega z=\Gamma_i\equiv H^{(k,l+1)}(\Delta_{f_i},a,x,z)\,\&\,(t)\big[H^{(k,l+1)}\big(\nu(\Delta_{\bar f_i}),a,x,t\big)\vee z=t\big]$$
$$\vee (t)H^{(k,l+1)}\big(\nu(\Delta_{\bar f_i}),a,x,t\big)\,\&\,(z=\Delta_0)$$

and we easily see, using lemmas 19, 20, 25, that there are integers $f_i',\bar f_i'$ such that

$$(*)\qquad
\begin{aligned}
&\vdash_\omega z_i=\Gamma_i\equiv H^{(k,l+1)}(\Delta_{f_i'},a,x,z_i),\\
&\vdash_\omega z_i=\Gamma_i\equiv \bar H^{(k,l+1)}(\Delta_{\bar f_i'},a,x,z_i).
\end{aligned}$$

Since

$$\vdash_\omega \Gamma_1=\Gamma_2\equiv (Ez_1,z_2)\big[H^{(k,l+1)}(\Delta_{f_1'},a,x,z_1)$$
$$\&\, H^{(k,l+1)}(\Delta_{f_2'},a,x,z_2)\,\&\,(z_1=z_2)\big]$$

and since the formula $z_1=z_2$ can be brought to the form

$$H^{(k,l+2)}(\Delta_r,a,x,z_1,z_2),$$

we easily infer, using lemmas 20, 23, and 24, that there is an integer e such that $\vdash_\omega \Gamma_1=\Gamma_2\equiv H^{(k,l)}(\Delta_e,a,x)$. Similarly we find an integer $\bar e$, thus showing that lemma 26 holds for the formula $\Gamma_1=\Gamma_2$.

Replacing in the above proof the formula $z_1=z_2$ by $z=z_1+z_2$ or $z=z_1\cdot z_2$, we show similarly that lemma 26 holds for the numerical formulas $\Gamma_1+\Gamma_2$ and $\Gamma_1\times\Gamma_2$.

Since x_j can be represented as $I_j^{(k,l)}(\Delta_f,a,x)$ and in a similar form with $\bar I_j^{(k,l)}$ instead of $I_j^{(k,l)}$, we see that lemma 26 is true for the numerical formulas x_j, $j=0,1,...,l-1$. It is also obvious that if lemma 26 holds for the propositional formula C, it does so for the formula $(\iota x_j)C$. Hence it remains to prove lemma 26 for the formula $a_j(\Gamma_1)$ under the assumption that it holds for the (numerical) formula Γ_1.

Assume for simplicity that $\vdash_\omega \Gamma_1=(\iota z)H^{(k,l+1)}(\Delta_{f_1},a,x,z)$ and $\vdash_\omega \Gamma_1=(\iota z)\bar H^{(k,l+1)}(\Delta_{\bar f_1},a,x,z)$. Then by $(*)$

$$\vdash_\omega a_j(\Gamma_1)=(\iota v)(Ez)(\beta)(Es,t)\Big\{D(a,\beta,s,t)\,\&\,\big(lh(s)\geqslant z\big)$$
$$\&\,\big[v=\big((s)_{\Delta_j}\big)_z\big]\,\&\,\big[\bar\varphi\big(\Delta_{f_1'},\bar\imath_{l+2}(s,x,z)\big)=0\big]\Big\},$$

$$\vdash_\omega a_j(\Gamma_1)=(\iota v)(Ez)(E\beta)(s,t)\Big(D(a,\beta,s,t)\supset \big\{\big(lh(s)<z\big)$$
$$\vee \big[v=\big((s)_{\Delta_j}\big)_z\big]\,\&\,\big[\bar\varphi\big(\Delta_{\bar f_1'},\bar\imath_{l+2}(s,x,z)\big)=0\big]\big\}\Big).$$

It is obvious that we can determine integers $f_1^*, \bar{f}_1^*$ such that

$$\vdash_\omega \big(lh(s) \geqslant z\big) \,\&\, \big[v = \big((s)_{\varDelta_j}\big)_z\big] \,\&\, \big[\overline{\varphi}\big(\varDelta_{f_1'}, \bar{\iota}_{l+2}(s, \boldsymbol{x}, z)\big) = 0\big]$$
$$\equiv \overline{\varphi}\big(\varDelta_{f_1^*}, \bar{\iota}_{l+3}(s, \boldsymbol{x}, z, v)\big) = 0 \; ,$$

$$\vdash_\omega \big(lh(s) < z\big) \vee \big[v = \big((s)_{\varDelta_j}\big)_z\big] \,\&\, \big[\overline{\varphi}\big(\varDelta_{\bar{f}_1'}, \bar{\iota}_{l+2}(s, \boldsymbol{x}, z)\big) = 0\big]$$
$$\equiv \overline{\varphi}\big(\varDelta_{\bar{f}_1^*}, \bar{\iota}_{l+3}(s, \boldsymbol{x}, z, v)\big) = 0 \; .$$

We thus obtain

$$\vdash_\omega a_j(\varGamma_1) = (w)(\mathbf{E}z)\, H^{(k,l+2)}(\varDelta_{f_1^*}, \boldsymbol{a}, \boldsymbol{x}, z, v) \; ,$$
$$\vdash_\omega a_j(\varGamma_1) = (w)(\mathbf{E}z)\, \bar{H}^{(k,l+2)}(\varDelta_{\bar{f}_1^*}, \boldsymbol{a}, \boldsymbol{x}, z, v) \; ,$$

whence we obtain the desired result using lemmas 23 and 25.

LEMMA 27. *For every k, l there are primitive recursive functions $g_7, \bar{g}_7$ such that if $\vartheta, \overline{\vartheta}$ are elementary numerical formulas representing them in A_ω and $\boldsymbol{a}' = (a_1, \ldots, a_{k-1})$, then*

$$\vdash_\omega H^{(k-1,l)}\big(\vartheta(y, \varDelta_k), \boldsymbol{a}', \boldsymbol{x}\big) \equiv (a_0)\, H^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}) \; ,$$
$$\vdash_\omega \bar{H}^{(k-1,l)}\big(\overline{\vartheta}(y, \varDelta_k), \boldsymbol{a}', \boldsymbol{x}\big) \equiv (\mathbf{E}a_0)\, \bar{H}^{(k,l)}(y, \boldsymbol{a}, \boldsymbol{x}) \; .$$

Proof. We shall prove the first formula. The right-hand side of this formula is equivalent to

$$(23) \qquad (\gamma)(\mathbf{E}a_0, \beta)(\mathbf{E}s, t)\big[S(\gamma, a_0, \beta) \,\&\, D(\boldsymbol{a}, \beta, s, t) \,\&\, \overline{\varphi}\big(y, \bar{\iota}_{l+1}(s, \boldsymbol{x})\big) = 0\big] \; ,$$

where S is the formula used in the proof of lemma 21. Let $h(m, k)$ be the primitive recursive function

$$h(m, k) = 2^{f'((m)_k)} \cdot 3^{(m)_1} \ldots p_{k-1}^{(m)_{k-1}} p_k^{f''((m)_k)}$$

where f' and f'' have the same meaning as in the proof of lemma 21 and let r be an integer such that $\overline{\varphi}\big(\varDelta_r, \bar{\iota}(x, y)\big)$ represents $h(m, k)$ in A_ω. Then

$$\vdash_\omega S(\gamma, a_0, \beta) \supset \Big[D(\boldsymbol{a}', \gamma, s, t) \equiv D\big(\boldsymbol{a}, \beta, \overline{\varphi}(\varDelta_r, \bar{\iota}(s, \varDelta_k)), t\big)\Big]$$

and we infer that (23) is equivalent to

$$(\gamma)(\mathbf{E}s, t)\Big[D(\boldsymbol{a}', \gamma, s, t) \,\&\, \overline{\varphi}\Big(y, \bar{\iota}_{l+1}\big(\overline{\varphi}(\varDelta_r, \bar{\iota}(s, \varDelta_k)), \boldsymbol{x}\big)\Big) = 0\Big] \; .$$

This formula is reducible to the form required in the lemma in the way used several times in the preceding proofs.

LEMMA 28. *Formulas $H^{(0,l)}(y, \boldsymbol{x})$ satisfy condition VI.*

Proof. (i) follows from lemma 21. To prove (iii) we notice that $\vdash_\omega \overline{\varphi}\big(\varDelta_j, \bar{\iota}_l(t_0, \ldots, t_{l-3}, \varDelta_m, \varDelta_n)\big) = \overline{\varphi}\big(\varDelta_{\zeta(j,m,n)}, \bar{\iota}_{l-2}(t_0, \ldots, t_{l-3})\big)$, where ζ is prim-

itive recursive (cf. axiom (Ω_{20})), and hence $\vdash_\omega H^{(0,l)}(\varDelta_j, x_0, \ldots, x_{l-3}, \varDelta_m, \varDelta_n)$ $\equiv H^{(0,l-2)}(\varDelta_{\zeta(j,m,n)}, x_0, \ldots, x_{l-3})$.

Formula (ii) has in our case the form

$$(24) \qquad \Pi(t_1) \supset \big\{H^{(0,4)}(\varDelta_{\tau(j)}, t_1, \ldots, t_4)$$
$$\equiv (\mathbf{E}u, v)\big[\Pi(u) \,\&\, \big(\Gamma(u, v) \preceq \Gamma(t_1, t_2)\big) \,\&\, \sim\!\big(\Gamma(t_1, t_2) \preceq \Gamma(u, v)\big)$$
$$\&\, H^{(0,4)}(\varDelta_j, u, v, t_3, t_4)\big]\big\} \; .$$

Since

$$\vdash_\omega \Pi(t_1) \,\&\, \Pi(u) \supset \big\{\big[\big(\Gamma(u, v) \preceq \Gamma(t_1, t_2)\big) \,\&\, \sim\!\big(\Gamma(t_1, t_2) \preceq \Gamma(u, v)\big)\big]$$
$$\equiv (\gamma)\big[\sim\! Im\big(\gamma, \Gamma(t_1, t_2), \Gamma(u, v)\big)\big]\big\}$$

(cf. lemma 15 and the remark following lemma 13), we can give to (24) the form (cf. lemma 10)

$$\Pi(t_1) \supset \big\{H^{(0,4)}(\varDelta_{\tau(j)}, t_1, \ldots, t_4) \equiv (\mathbf{E}u, v)(\alpha, \beta, \gamma)$$
$$\big\{\mathcal{C}_W(\alpha, \beta, u) \,\&\, \big[\sim\! Im\big(\gamma, \Gamma(t_1, t_2), \Gamma(u, v)\big)\big] \,\&\, H^{(0,4)}(\varDelta_j, u, v, t_3, t_4)\big\}\big\} \; .$$

It follows from lemmas 26, 27, 23, and 20 that there is a primitive recursive τ for which this formula is provable in A_ω. Lemma 28 is thus proved.

It remains still to give examples of families $\{A_j\}$ of extensions of A_ω, which satisfy the assumptions of theorem 2.

Let B_j be a recursive family of sets of closed formulas. Hence there is a recursive relation R such that

$$\Phi \in B_j \equiv R(j, \ulcorner\Phi\urcorner) \; .$$

We denote by r an integer such that the formula $\overline{\varphi}\big(\varDelta_r, \bar{\iota}(x, y)\big) = 0$ strongly represents R in A_ω. Let us further assume that the sets $A_j = Cn_\omega(B_j)$ are consistent. We shall show that the assumptions of theorem 2 are satisfied for the family $\{A_j\}$.

Let $S_{\xi,j}$ be defined by transfinite induction on ξ as follows:

$$S_{0,j} = B_j, \qquad S_{\lambda,j} = \bigcup_{\xi < \lambda} S_{\xi,j} \quad \text{for limit ordinals } \lambda \; ,$$

$$S_{\xi+1} = \big\{\Phi\colon (\Phi \text{ is a closed formula}) \,\&\, (\mathbf{E}\Psi)\big[(\Psi \text{ is a formula with exactly}$$
$$\text{one free variable } x) \,\&\, (n)\big(\Psi(\varDelta_n) \in S_{\xi,j}\big) \,\&\, \vdash\!\big((x)[\Psi(x) \supset \Phi]\big)\big]\big\} \; .$$

Thus $S_{\xi,j}$ is the set of (closed) formulas which can be derived from B_j by ξ applications of the rule ω.

Spector [10] proved that $A_j = Cn_\omega(B_j) = S_{\omega_1,j}$.

Let $Flm(x)$, $Th(x)$ be elementary formulas which (strongly) represent in A_ω the set of formulas with one free variable x_1 and the set of Gödel numbers of closed formulas Φ such that $\vdash \Phi$.

Further, let $sb(x, y)$, $gen(x)$, $x\, imp\, y$ be elementary numerical formulas which represent in A_ω the primitive recursive functions $Sb(1, l, \ulcorner\Phi\urcorner)$, $NegEx(1, 1, Neg(\ulcorner\Phi\urcorner)$, $\ulcorner\Phi\urcorner Imp \ulcorner\Psi\urcorner$ (cf. II, p. 206).

We consider the following formulas:

$$Z_0(u, x): \quad (y)\big[\overline{\varphi}\big(u, \bar{\iota}(x, y)\big) = 0\big] \quad (x \text{ is the minimal element of } \leqslant_u),$$

$$Z_s(u, x, x'): \quad (y)\big\{\overline{\varphi}\big(u, \bar{\iota}(y, x)\big) = 0 \supset \big[\overline{\varphi}\big(u, \bar{\iota}(x, y)\big) = 0 \lor \overline{\varphi}\big(u, \bar{\iota}(y, x')\big) = 0\big]$$

$$(x \text{ is the successor of } x' \text{ in the ordering } \leqslant_u),$$

$$Z_l(u, x): \quad \sim Z_0(u, x) \,\&\, (y)\big[\big[\overline{\varphi}\big(u, \bar{\iota}(y, x)\big) = 0\big] \,\&\, (y \neq x)$$

$$\supset (Ey')\big\{y \neq y' \,\&\, x \neq y' \,\&\, \big[\overline{\varphi}\big(u, \bar{\iota}(y, y')\big) = 0\big] \,\&\, \big[\overline{\varphi}\big(u, \bar{\iota}(y', x)\big) = 0\big]\big\}\big]$$

$$(x \text{ is a limit element of the ordering } \leqslant_u).$$

Let $Z(a, u, v)$ be the formula

$$(x)\big\{Z_0(u, x) \supset (y)\big[a\big(\bar{\iota}(x, y)\big) = 0 \equiv \overline{\varphi}\big(\Delta_r, \bar{\iota}(v, y)\big) = 0\big]\big\}$$

$$\&\, (x, x')\big(Z_s(u, x, x') \supset \big\{(y)\big[\bar{a}\big(\bar{\iota}(x, y)\big) = 0\big] \equiv (Et)(z)\big[Flm(t)$$

$$\&\, Th\big(sb(z, t)\big) \,\&\, Th\big(gen(t\, imp\, y)\big)\big]\big\}\big)$$

$$(x)\big(Z_l(u, x) \supset (y)\big\{\big[a\big(\bar{\iota}(x, y)\big) = 0\big] \equiv (Ex')\big[\overline{\varphi}\big(u, \bar{\iota}(x', x)\big) = 0$$

$$\&\, (x' \neq x) \,\&\, a\big(\bar{\iota}(x', y)\big) = 0\big]\big\}\big).$$

The following lemma explains the meaning of this formula:

LEMMA 29. *Let p be in W and let ψ, p, j satisfy Z in $\mathfrak{N}_0$. Then $\big[\psi\big(J(n, k)\big) = 0\big] \equiv [k$ is the Gödel number of a formula Φ in $S_{\xi_n, j}$ where ξ_n is the order type of a segment of N_0 determined by n in the well-ordering $\leqslant_p$].*

Proof by induction on ξ_n presents no difficulties.

LEMMA 30. $\vdash_\omega \Pi(u) \supset (E!a) Z(a, u, v)$.

Proof: by the formalization in A_ω of the usual existence and uniqueness proofs of functions defined by transfinite induction.

LEMMA 31. $\vdash_\omega \Pi(u) \,\&\, \Pi(u') \,\&\, [\Gamma(u, v) \approx \Gamma(u', v')] \,\&\, Z(a, u, v)$ $\&\, Z(a', u', v') \supset \big\{(Ex)\big[a\big(\bar{\iota}(x, y)\big) = 0\big] \equiv (Ex')\big[a'\big(\bar{\iota}(x', y)\big) = 0\big]\big\}.$

Proof (informal). The antecedent implies (cf. lemma 17) that $v = v'$ and $u \approx u'$. Let β be a similarity mapping of N_0 onto itself carrying the relation $\overline{\varphi}\big(u, \bar{\iota}(x_1, x_2)\big) = 0$ over into $\overline{\varphi}\big(u', \bar{\iota}(x_1', x_2')\big) = 0$. We then prove by transfinite induction on the order type of the segment of the first relation determined by x that $a\big(\bar{\iota}(x, y)\big) = 0 \equiv a\big(\bar{\iota}(\beta(x), y)\big) = 0$.

LEMMA 32. *The formulas*

$$\Gamma(u, v, w): \quad (a)\big\{Z(a, u, v) \supset (Ex)\big[a\big(\bar{\iota}(x, w)\big) = 0\big]\big\},$$

$$\Gamma^*(u, v, w): \quad (a)\big\{Z(a, u, v) \supset (Ex)\big[a\big(\bar{\bar{\iota}}(x, \nu(w))\big) = 0\big]$$

(where $\nu(w)$ is an elementary numerical formula which represents the function Neg in A_ω) *strongly represent in A_ω relatively to $W \times N_0^2$ the relations*

$$C(p, j, n): \quad (E\Phi)\big[(n = \ulcorner\Phi\urcorner) \,\&\, (\Phi \in \bigcup_{\xi < |p|} S_{\xi, j})\big],$$

$$C^*(p, j, n): \quad (E\Phi)\big[(n = \ulcorner \sim \Phi\urcorner) \,\&\, (\Phi \in \bigcup_{\xi < |p|} S_{\xi, j})\big].$$

Proof. Let $p \in W$, whence $\vdash \Pi(\Delta_p)$ (cf. lemma 10). Assume that $C(p, j, n)$, whence that $n = \ulcorner\Phi\urcorner$ and $\Phi \in S_{\xi, j}$ where $\xi < |p|$. We shall show that $\Gamma(\Delta_p, \Delta_j, \Delta_n)$ is true in $\mathfrak{N}_0$. Indeed, if ψ together with p, j satisfies Z in $\mathfrak{N}_0$, then there is an integer q such that (with the notation of lemma 29) $\xi = \xi_q$ and hence $\psi\big(J(q, n)\big) = 0$, i.e. ψ and n satisfy in $\mathfrak{N}_0$ the formula $(Ex)\big[a\big(\bar{\iota}(x, w)\big) = 0\big]$. It follows that $\vdash_\omega \Gamma(\Delta_p, \Delta_j, \Delta_n)$.

Now assume that non-$C(p, j, n)$. According to lemma 30

$$\vdash_\omega \sim \Gamma(\Delta_p, \Delta_j, \Delta_n) \equiv (a)\big\{Z(a, \Delta_p, \Delta_j) \supset (x)\big[a\big(\bar{\iota}(x, \Delta_n)\big) \neq 0\big]\big\}.$$

Thus the implication non-$C(p, j, n) \supset \vdash_\omega \sim \Gamma(\Delta_p, \Delta_j, \Delta_n)$ will be proved if we succeed in showing that $\Gamma(\Delta_p, \Delta_j, \Delta_n)$ is false in $\mathfrak{N}_0$. However, this is obvious because by lemma 30 there is exactly one function ψ which satisfies in $\mathfrak{N}_0$ the formula $Z(a, \Delta_p, \Delta_j)$ and for this function (by lemma 29) we have $\psi\big(J(q, n)\big) \neq 0$ for every q.

Proof of the second part of the lemma is similar.

THEOREM 6. *If $\{B_j\}$ is a recursive family of closed formulas and if the sets $A_j = Cn_\omega(B_j)$ are consistent, then there is a formula free for the family $\{A_j\}$.*

Proof. Starting with formulas Γ, Γ^* of lemma 32 we construct formulas $\overline{\Gamma}_h, \overline{\Gamma}_h^*$ as in lemma 1. It has been proved in lemma 1 that these formulas strongly represent relations $\overline{C}_h, \overline{C}_h^*$ relatively to $W \times N_0^2$ and satisfy condition (g), p. 207. From lemmas 26, 27, 23, 24, and 20 it follows that there are integers r_1, r_2 such that

$$\vdash_\omega H^{(0, 4)}(\Delta_{r_1}, x, y, z, \Delta_h) \equiv \overline{\Gamma}_h(x, y, z),$$

$$\vdash_\omega H^{(0, 4)}(\Delta_{r_2}, x, y, z, \Delta_h) \equiv \overline{\Gamma}_h^*(x, y, z)$$

and hence that all assumptions of theorem 2 are satisfied.

It is rather remarkable that theorem 1 fails for the system A_β of analysis discussed in [8]. Indeed, it has been proved in [8] that there is a finite complete extension of A_β. It is extremely unikely that there be a formula free for A_β; this question, however, is open and seems to be rather difficult.

References

[1] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik 38 (1931), pp. 173-198.

[2] A. Grzegorczyk, A. Mostowski, Cz. Ryll-Nardzewski, *The classical and the ω-complete arithmetic*, The Journal of Symbolic Logic 23 (1958), pp. 188-206.

[3] S. Feferman and G. Kreisel, *Faithful interpretations*, Notices of the American Mathematical Society 6 (1959), p. 516.

[4] S. C. Kleene, *Introduction to meta-mathematics*, North Holland Publishing Company, Amsterdam 1952.

[5] — *On the forms of the predicates in the theory of constructive ordinals (second paper)*, American Journal of Mathematics 77 (1955), pp. 405-428.

[6] — *Arithmetical predicates and function quantifiers*, Transactions of the American Mathematical Society 79 (1955), pp. 312-340.

[7] W. Markwald, *Zur Theorie der konstruktiven Wohlordnungen*, Mathematische Annalen 127 (1954), pp. 135-149.

[8] A. Mostowski, *A system of analysis based on an infinitary rule of proof*, Proceedings of the International Symposium on Foundations of Mathematics: *Infinitistic Methods*, Warszawa 1961.

[9] J. B. Rosser, *Extensions of some theorems of Gödel and Church*, The Journal of Symbolic Logic 1 (1936), pp. 87-91.

[10] C. Spector, *Inductively defined sets of natural numbers*, Proceedings of the International Symposium on Foundations of Mathematics: *Infinitistic Methods*. Warszawa 1961.

[11] A. Tarski, A. Mostowski, R. M. Robinson, *Undecidable Theories*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1953.

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