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ORDER-TYPES OF MODELS OF ARITHMETIC AND A CONNECTION WITH ARITHMETIC SATURATION

(submitted by Yi Zhang)

Abstract. First, we study a question we encountered while exploring order-types of models of arithmetic. We prove that if $M \vDash \mathrm{PA}$ is resplendent and the lower cofinality of $M \setminus \mathbb{N}$ is uncountable then $(M, <)$ is expandable to a model of any consistent theory $T \supseteq \mathrm{PA}$ whose set of Gödel numbers is arithmetic. This leads to the following characterization of Scott sets closed under jump: a Scott set X is closed under jump if and only if X is the set of all sets of natural numbers definable in some recursively saturated model $M \vDash \mathrm{PA}$ with $\mathrm{lcf}(M \setminus \mathbb{N}) > \omega$. The paper concludes with a generalization of theorems of Kossak, Kotlarski and Kaye on automorphisms moving all nondefinable points: a countable model $M \vDash \mathrm{PA}$ is arithmetically saturated if and only if there is an automorphism $h\colon M \to M$ moving every nondefinable point and such that for all $x \in M$, $\mathbb{N} < x < \mathrm{Cl}\varnothing \setminus \mathbb{N}$, we have $h(x) > x$.

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1. Introduction

Peano Arithmetic (PA) is the first-order theory in the language $\mathcal{L}_{\mathrm{PA}} = \{+, \times, <, 0, 1\}$ consisting of the following axioms: associativity and commutativity of $+$ and $\times$, their neutral elements are 0 and 1 respectively, distributivity, discrete linear order axioms for $<$, adding 1 gives a successor, and the Induction Scheme:

$$\forall \overline{y}[\ \varphi(0, \overline{y}) \wedge \forall x(\varphi(x, \overline{y}) \to \varphi(x+1, \overline{y})) \to \forall x \varphi(x, \overline{y})\ ]$$

for every $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \overline{y})$.

Peano Arithmetic is an extremely powerful theory. A folklore knowledge among logicians is that all of classical analysis, number theory and combinatorics can be done within tiny subsystems of Peano Arithmetic. In pre-Gödelean era it was believed that PA comprises an axiomatization of the set of all *"truths"* about natural numbers and finite sets.

Thus, a model of Peano Arithmetic (that is, a set with operations $+$ and $\times$ defined on it so that the above axioms of PA hold) resemble the natural numbers as much as any working mathematician would hope for (all of his concrete mathematics can be conducted inside a model of PA and nobody would notice the difference). As usually for such theories, there are 2^λ non-isomorphic models of PA in every infinite cardinality λ.

The structure of models of first-order Peano Arithmetic (PA) has been extensively studied since the 1960s. Due to unclassifyability of the diverse mass of models (even in the countable case) and the elusive nature of completions of PA (especially the 'true arithmetic' $\mathrm{Th}\,\mathbb{N} = \{\varphi \mid \varphi \in \mathcal{L}_{\mathrm{PA}},\ \mathbb{N} \models \varphi\}$), models of PA are among the most difficult to deal with in the whole of model theory.

Certain classes of models were studied that could be to some extent tackled: countable models, $\varkappa$-like models (for a cardinal $\varkappa$), models coding certain sets, realizing certain types. Among the most important notions introduced is recursive saturation. A model M is recursively saturated if it realizes every type (with parameters from M) whose set of Gödel numbers is recursive. Recursively saturated models naturally occur in model theory of arithmetic. For instance, every model of PA obtained from a nonstandard model of PA by an application of the arithmetized completeness theorem is recursively saturated.

A recursively saturated model of PA is uniquely determined by its complete theory and the collection of subsets of $\mathbb{N}$ definable (coded) in the model: if $M \models \mathrm{PA}$, $N \models \mathrm{PA}$ are two recursively saturated models of the same completions of PA and code the same subsets of $\mathbb{N}$ then $M \cong N$.

Other notions were also introduced, isolating important subclasses of the class of all recursively saturated models: the most important being resplendency and arithmetic saturation. Resplendent models and arithmetically saturated models will be the main objects we study in this paper.

A model M is resplendent if for every $\overline{a} \in M$, and any statement $\varphi(\overline{a})$ containing additional relation symbols $R_1, \ldots, R_n$, if $\mathrm{Th}(M, \overline{a}) + \varphi(\overline{a})$ is consistent then there are relations $R_1, \ldots, R_n$ on M such that $(M, R_1, \ldots, R_n) \models \varphi(\overline{a})$. Resplendency implies existence of many automorphisms of a model, recursive saturation of a model and many other pleasant properties. Resplendent models are a plentiful class of very 'regular' models we can deal with.

A model $M \models \mathrm{PA}$ is arithmetically saturated if it is recursively saturated and the class of subsets of $\mathbb{N}$ definable in M is closed under jump. Thus, more subsets of $\mathbb{N}$ are definable in M than is expected from a recursively saturated model. (A recursively saturated model is only expected to code its own complete theory, see Wilmer's theorem below.) In particular, an arithmetically saturated model codes the sets $\Pi_n \,\mathrm{Th}\, \mathbb{N}$ of all true Π_n-sentences for all $n \in \mathbb{N}$. In addition, arithmetic saturation implies more homogeneity than just recursive saturation. Recursive saturation implies that the model is homogeneous (if $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$ then there is an automorphism $h\colon M \to M$ such that for all $i = 1, 2, \ldots, n$, $h(a_i) = b_i$). Arithmetic saturation gives us extra control over this automorphism. E.g., if $a_i \neq b_i$ for all $i = 1, 2, \ldots n$ then it can be ensured that this automorphism moves all nondefinable points (i.e., $h(x) \neq x$ for all $x \notin \mathrm{Cl}\varnothing$).

Section 3 starts off with an investigation of a problem concerning order-types of resplendent models of Peano arithmetic. The connection with arithmetic saturation is studied in Section 4. Section 5 uses the methods developed in Section 4 to produce a generalization of some well-known results of Kossak, Kotlarski and Kaye.

2. Definitions

If A is a linearly ordered set then $\mathrm{lcf}(A)$, the lower cofinality of A, is $\mathrm{cf}(A^*)$, where A^* is A with the order reversed.

If $B \subseteq M \models \mathrm{PA}$ then $\mathrm{Cl}_M(B)$ denotes the set of all elements of M definable in M with parameters from B, that is $a \in \mathrm{Cl}_M(B) \Leftrightarrow$ for some $\overline{b} \in B$ and some $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \overline{y})$,

$$M \models \varphi(a, \overline{b}) \wedge \exists! x \varphi(x, \overline{b}).$$

A set $Y \subseteq M$ is definable (with parameters $\overline{b} \in M$) if for some $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \overline{y})$, $Y = \{y \in M \mid M \vDash \varphi(y, \overline{b})\}$.

Definition 1. Let $M \vDash \mathrm{PA}$. We define the Standard System of M as $\mathrm{SSy}(M) = \{X \subseteq \mathbb{N} \mid$ there is a definable (with parameters) $Y \subseteq M$ such that $X = Y \cap \mathbb{N}\}$. We say that a subset A of $\mathbb{N}$ is coded in M if $A \in \mathrm{SSy}(M)$.

Definition 2. A *Scott set* X is a collection of subsets of $\mathbb{N}$ closed under $\bigcup$, $\bigcap$, $\smallsetminus$, complement, relative recursion and such that if $T \in X$ codes an infinite finitely branching tree then there is $B \in X$, $B \subseteq T$ which codes an infinite path through T.

It is known that for every $M \vDash \mathrm{PA}$, $\mathrm{SSy}(M)$ is a Scott set. The converse is known to hold for Scott sets of cardinalities ω and ω_1.

Definition 3. A model $M \vDash \mathrm{PA}$ is called *resplendent* if for every Σ^1_1-statement $\Phi(\overline{a})$, $\overline{a} \in M$, consistent with $\mathrm{Th}(M, \overline{a})$, we have $M \vDash \Phi(\overline{a})$.

Definition 4. A model $M \vDash \mathrm{PA}$ is called *recursively saturated* if every recursive type $p(x, \overline{a})$ (that is, $\{\ulcorner\varphi(x, \overline{y})\urcorner) \mid \varphi(x, \overline{y}) \in p\}$ is recursive) is realized.

It is also known (and we shall often use this fact) that a recursively saturated model M realizes all types that are coded in M, i.e. such that $\{\ulcorner\varphi(x, \overline{y})\urcorner \mid \varphi(x, \overline{a}) \in p(x, \overline{a})\} \in \mathrm{SSy}(M)$.

Fact 1. (Kleene). Let $\mathcal{L}$ be a finite language and $\{\varphi_n(\overline{x})\}_{n \in \mathbb{N}}$ be a recursive set of formulas of $\mathcal{L}$. Then there is a Σ^1_1-formula $\Phi(\overline{x})$ such that in all infinite $\mathcal{L}$-structures M, $M \vDash \forall \overline{x}\ (\Phi(\overline{x}) \leftrightarrow \bigwedge_{i \in \mathbb{N}} \varphi_n(\overline{x}))$.

It follows from Fact 1 that resplendency implies recursive saturation.

Fact 2. (Wilmers). Let X be a countable Scott set, $T \supseteq \mathrm{PA}$ be a consistent theory, $T \in X$. Then there is a countable recursively saturated $M \vDash T$ such that $\mathrm{SSy}(M) = X$.

Fact 3. ([4][7][5]). Let $M \vDash \mathrm{PA}$ be recursively saturated. Then the following are equivalent.

(1) M is arithmetically saturated, i.e., $\mathrm{SSy}(M)$ is closed under jump;
(2) for any $f \in M$ coding a set of pairs determining a function $f\colon \mathbb{N} \to M$, there is $c \in M \smallsetminus \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) > \mathbb{N} \Leftrightarrow f(n) > c$;
(3) for every $a \in M$, $\{\ulcorner\theta(x, y)\urcorner \mid \theta(x, y)$ is an $\mathcal{L}_{\mathrm{PA}}$-formula in two variables and $\min x\ \theta(x, a) \notin \mathrm{Cl}\varnothing\} \in \mathrm{SSy}(M)$;
(4) there is $g \in \mathrm{Aut}(M)$ such that $g(a) \neq a$ for every $a \in \mathrm{Cl}\varnothing$.

3. A question about expanding order-types of resplendent models to models of other theories

A question of H. Friedman [2] asks whether the classes of order-types of uncountable models of T are the same for all $T \supseteq \mathrm{PA}$. Having embarked on this difficult question, I realized that probably there is some chance of obtaining results in the case of *resplendent* models. For an up-to-date account of the state of Friedman's problem, see [1]. Among the results is the following theorem.

Theorem 4. If $M \vDash \mathrm{PA}$ is resplendent and $c \in M$ codes a consistent theory $T \supseteq \mathrm{PA}$ then $(M, <)$ can be expanded to a model of T.

The theorem is proved by writing down a Σ^1_1-statement expressing the existence of $N \vDash T$, $(N, <) \cong (M, <)$ and noticing that it is realized in every countable submodel of M containing c.

Using a theorem by D.Richard and J.-F.Pabion [8] which says that $M \vDash \mathrm{PA}$ is ω_1-saturated if and only if $(M, <)$ is ω_1-saturated, we obtain the following corollary.

Corollary 5. If $M \vDash \mathrm{PA}$ is resplendent and ω_1-saturated then $(M, <)$ can be expanded to a model of any consistent extension of PA.

The hunt for conditions weaker than ω_1-saturation implying expandability of $(M, <)$ to a model of $T \supset \mathrm{PA}$ led to the following theorem.

Theorem 6.
If $M \vDash \mathrm{PA}$ is resplendent and $\mathrm{lcf}(M \smallsetminus \mathbb{N}) > \omega$ then for all $n \in \omega$, $(M, <)$ is expandable to a model of $\mathrm{PA} + \Pi_n \operatorname{Th} \mathbb{N}$.

Proof. For any $n \in \omega$, let us introduce $\Sigma_n\mathrm{Def}$ = the set of all nonstandard definable points of M defined by a Σ_n-formula.

As $\mathrm{lcf}(M \smallsetminus \mathbb{N}) > \omega$, there is $a > \mathbb{N}$ such that $\Sigma_1\mathrm{Def} > a$. Define $A_1 = \{\ulcorner \forall x \varphi(x) \urcorner \mid \varphi \in \Delta_0,\ M \vDash \forall x < a\ \varphi(x)\}$. Now, $A_1 \subseteq \Pi_1 \operatorname{Th} \mathbb{N}$ because $\mathbb{N} \prec_{\Delta_0} M$. Also, $\Pi_1 \operatorname{Th} \mathbb{N} \subseteq A_1$ because if for some $\varphi \in \Delta_0$ such that $\mathbb{N} \vDash \forall x \varphi(x)$ there existed $x < a\ \neg\varphi(x)$ then $\min x \neg\varphi(x)$ would be a nonstandard Σ_1-definable point less than a. Hence, $A_1 = \Pi_1 \operatorname{Th} \mathbb{N}$. A_1 is definable, hence coded in M.

Suppose at stage n we already know that $\Pi_n \operatorname{Th} \mathbb{N} \in \mathrm{SSy}(M)$. Let $b \in M$ code $\Pi_n \operatorname{Th} \mathbb{N}$. Consider the statement

$$\Phi_n(b) \ = \ \exists \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n$$

$$\forall xy\ (x \ll_n y \longleftrightarrow x < y)\ \bigwedge$$

$$\bigwedge \text{“}(\oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n) \vDash \mathrm{PA} + \Pi_n \operatorname{Th} \mathbb{N}\text{”} \bigwedge$$
$$\bigwedge \text{“}\mathrm{SSy}(M, \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n) \subseteq \mathrm{SSy}(M)\text{”}.$$

Let us show that the last line is expressible by a Σ^1_1-sentence. Let $\varphi_m(x) = (x > m) \,\&\, \forall\, z\, \exists\, y\, \forall\, i < x\, ((z))_i = (y)_i$, where $((z))_i$ means $(z)_i$ in the language $\{\oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n\}$. The set $\{\varphi_m(x)\}_{m\in\mathbb{N}}$ is a recursive set of formulas, hence, by Kleene's Theorem, there is a Σ^1_1-sentence $\Theta(x)$ such that in any $K \vDash \mathrm{PA}$, $K \vDash \forall\, x\, (\bigwedge_{m\in\mathbb{N}} \varphi_m(x) \leftrightarrow \Theta(x))$. Then $\mathrm{SSy}(M, \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n) \subseteq \mathrm{SSy}(M)$ is implied by the Σ^1_1-sentence $\exists\, x\, \Theta(x)$. Hence, $\Phi_n(b)$ is a Σ^1_1-sentence.

$\Phi_n(b)$ is consistent because, by Wilmers' Theorem, as $(\mathrm{PA}+\Pi_n \operatorname{Th}\mathbb{N}) \in \mathrm{SSy}(M)$, there is a countable model

$$N \vDash \mathrm{PA} + \Pi_n \operatorname{Th}\mathbb{N}, \quad \mathrm{SSy}(N) = \mathrm{SSy}(\mathrm{Cl}_M(b)).$$

Hence, by resplendency, $\Phi_n(b)$ is already realized in M.

Denote the model $(M, \oplus_n, \otimes_n, \ll_n, \mathbb{O}_n, \mathbb{S}_n)$ by M_n. By construction, $M_n \vDash \mathrm{PA} + \Pi_n \operatorname{Th}\mathbb{N}$, $(M_n, <) \cong (M, <)$, $\mathrm{SSy}(M_n) \subseteq \mathrm{SSy}(M)$.

Let $(\Sigma_n \mathrm{Def})_{M_n} > a > \mathbb{N}$. Consider

$$A_{n+1} = \{\ulcorner \forall x \varphi(x) \urcorner \mid \varphi \in \Sigma_n,\ M_n \vDash \forall x < a\ \varphi(x)\}.$$

$A_{n+1} \subseteq \Pi_{n+1} \operatorname{Th}\mathbb{N}$ because if $M_n \vDash \forall x < a\ \varphi(x)$ but $\mathbb{N} \vDash \exists x\ \neg\varphi(x)$ then for some $k \in \mathbb{N}$, $\mathbb{N} \vDash \neg\varphi(k)$, which is a Π_n-statement. Hence, as $M_n \vDash \Pi_n \operatorname{Th}\mathbb{N}$, $M_n \vDash \neg\varphi(k)$, contradiction.

$\Pi_{n+1} \operatorname{Th}\mathbb{N} \subseteq A_{n+1}$. Let $\mathbb{N} \vDash \forall x \varphi(x)$, where $\varphi(x) \in \Sigma_n$. If $M_n \vDash \exists x < a\ \neg\varphi(x)$ then $c =: \min x \neg\varphi(x)$ is a Σ_n-definable point less than a. If $c \in \mathbb{N}$ then $M_n \vDash \neg\varphi(c)$, which is a Π_n-statement not belonging to $\Pi_n \operatorname{Th}\mathbb{N}$. Contradiction with $M_n \vDash \Pi_n \operatorname{Th}\mathbb{N}$. Hence $\mathbb{N} < c < a$ contradicting the assumption that $(\Sigma_n \mathrm{Def}) > a$.

Therefore $\Pi_{n+1} \operatorname{Th}\mathbb{N} = A_{n+1}$, which is coded in M_n. As $\mathrm{SSy}(M_n) \subseteq \mathrm{SSy}(M)$, $\Pi_{n+1} \operatorname{Th}\mathbb{N}$ is coded also in M.

Now, by Theorem 4, $(M, <)$ is expandable to a model of $\mathrm{PA} + \Pi_n \operatorname{Th}\mathbb{N}$ for every $n \in \omega$. □

Now, let us study a corollary. A theory T is called arithmetic if it has an axiomatization S such that $S = \{n \in \mathbb{N} \mid \mathbb{N} \vDash \theta(n)\}$ for some formula $\theta(x) \in \mathcal{L}_{\mathrm{PA}}$. Recursive extensions of PA are examples of arithmetic theories. Also, there are complete arithmetic theories by the arithmetized completeness theorem.

Corollary 7.
For any consistent arithmetic theory $T \supseteq \mathrm{PA}$, if $M \vDash \mathrm{PA}$ is resplendent and $\mathrm{lcf}(M \setminus \mathbb{N}) > \omega$ then there is $N \vDash T$ such that $(N, <) \cong (M, <)$.

Proof.
Since T is arithmetic, T is recursive in the set $\Pi_n \operatorname{Th} \mathbb{N}$ for some n. Hence T is coded in M. Hence, by Theorem 4, $(M, <)$ is expandable to a model of T. $\square$

However there is a proof that $\Pi_n \operatorname{Th} \mathbb{N}$ is coded in M different from the one above. Indeed, resplendency implies recursive saturation and for any $f \colon \mathbb{N} \to M$ there is $a \in M$ such that $\forall n \in \mathbb{N}(f(n) > \mathbb{N} \Rightarrow f(n) > a)$ because $\operatorname{lcf}(M \smallsetminus \mathbb{N}) > \omega$. Hence, M is arithmetically saturated, thus, applying the machinery of arithmetic saturation (Fact 3), we can conclude that $\operatorname{SSy}(M)$ is closed under jump, hence contains $\Pi_n \operatorname{Th} \mathbb{N}$ for all $n \in \omega$.

In the next section we shall investigate whether recursive saturation and uncountable lower cofinality give us more information about which sets are coded in M than just arithmetic saturation. The answer will be "No".

We can also reformulate this question as follows. If M is recursively saturated and $\operatorname{lcf}(M \smallsetminus \mathbb{N}) > \omega$ then $\operatorname{SSy}(M)$ is closed under jump. Does every countable Scott set closed under jump occur in this way? The answer will be "Yes".

4. Do recursive saturation and uncountable lower cofinality say more about coding than arithmetic saturation?

Lemma 8. Let $M \models \mathrm{PA}$ be recursively saturated. Then M is arithmetically saturated if and only if for all $a \in M$, $\operatorname{Cl}(a) \smallsetminus \mathbb{N}$ is bounded below in $M \smallsetminus \mathbb{N}$.

Proof. Suppose, for all $a \in M$, $\operatorname{Cl}(a) \smallsetminus \mathbb{N}$ is bounded below. Let $f \in M$ code a function whose domain contains $\mathbb{N}$. For every $n \in \mathbb{N}$, $f(n) \in \operatorname{Cl}(f)$. If $b \in M \smallsetminus \mathbb{N}$ is such that $\operatorname{Cl}(f) > b$ then for all $n \in \mathbb{N}$, $(f(n) > \mathbb{N} \Leftrightarrow f(n) > b)$.

Let M be arithmetically saturated, $c > \mathbb{N}$. The type which says: $F \in M$ codes a function $F \colon [0, c] \longrightarrow M$ with $F(\ulcorner\theta\urcorner) = t_\theta(a)$ (where θ ranges over all formulas of L_{PA} with two variables and t_θ is the Skolem term defined by θ) is recursive, hence realized. But if $\operatorname{Cl}(a) \smallsetminus \mathbb{N}$ is unbounded below then $\{F(\ulcorner\theta\urcorner)\} \cap (M \smallsetminus \mathbb{N})$ is not separated from $\mathbb{N}$ contradicting arithmetic saturation. $\square$

Let $E = \{x \in M \mid \text{there are no nonstandard definable points below } x\}$. If $a \in M \smallsetminus \operatorname{Cl}\varnothing$, $E_a = \{x \in M \mid \text{ for all } c \in \operatorname{Cl}\varnothing, c < x \leftrightarrow c < a \}$.

By Lemma 8, $E \neq \varnothing$ and for any a such that $\mathbb{N} < a < \mathrm{Cl}\varnothing \setminus \mathbb{N}$, $E_a = E$. The following lemma establishes some homogeneity properties of E_a which will be important in the rest of this section.

Lemma 9. Let M be recursively saturated, $a \in M \setminus \mathrm{Cl}\varnothing$.

(1) If $p(x, \overline{b})$ is realized by $c \in E_a$, $c > \mathrm{Cl}(\overline{b}) \cap E_a$ then for all $x \in E_a$ there is $y > x$ such that $p(y, \overline{b})$.
(2) If $p(x, \overline{b})$ is realized by $c \in E_a$, $c < \mathrm{Cl}(\overline{b}) \cap E_a$ then for all $x \in E_a$ there is $y < x$ such that $p(y, \overline{b})$.

Proof. 1. Let $A_{upper} = \{x \in M \mid \exists\ y \in \mathrm{Cl}\varnothing,\ a < y < x\}$. For an arbitrary $e \in E_a$, let us find $d > e$ such that $p(d, \overline{b})$. Let us show that for all $\theta(x, \overline{b}) \in p(x, \overline{b})$, $M \vDash \theta(y, \overline{b})$ for unboundedly-many $y \in E_a$. Consider the two cases. If $A = \{x \in A_{upper} \mid M \vDash \theta(x, \overline{b})\}$ is unbounded below then $M \vDash \theta(y, \overline{b})$ for unboundedly-many $y \in E_a$ by overspill. Otherwise, let $k \in \mathrm{Cl}\varnothing$, $a < k < A$. Define $g = \max x < k\ \theta(x, \overline{b})$. We observe that $g \in \mathrm{Cl}(\overline{b})$, while $c \leq g < A_{upper}$, which is a contradiction.

Thus for any $e \in E$, $p(x, \overline{b}) \cup \{x > e\}$ is finitely satisfied. By recursive saturation, $p(x, \overline{b}) \cup \{x > e\}$ is coded, hence realized.
2. Analogous proof. □

Lemma 10. Let $M \vDash \mathrm{PA}$ be a countable arithmetically saturated model, $\mathbb{N} < e < \mathrm{Cl}\varnothing \setminus \mathbb{N}$. Then there is an elementary embedding $h : M \to M$ such that for all $x > \mathbb{N}$, $h(x) > e$.

Proof.
A forth-argument. Let us enumerate M as $\{a_1, a_2, \ldots, a_i, \ldots\}_{i<\omega}$ and build inductively a sequence $\{b_1, b_2, \ldots, b_i, \ldots\}_{i<\omega}$ with $\mathrm{tp}(b_1, \ldots, b_i) = \mathrm{tp}(a_1, \ldots, a_i)$ and $b_i > e \iff b_i > \mathbb{N}$ for all i and define $h(a_i) = b_i$.

Suppose at stage i we already have $\mathrm{tp}(a_1, \ldots, a_i) = \mathrm{tp}(b_1, \ldots b_i)$, $e < \mathrm{Cl}(b_1, \ldots, b_i) \setminus \mathbb{N}$. By Lemma 8, $\mathrm{Cl}(a_1, \ldots, a_{i+1}) \setminus \mathbb{N}$ is bounded below. Let $c < \mathrm{Cl}(a_1, \ldots, a_{i+1}) \setminus \mathbb{N}$. By Lemma 9 (2),

$$p(x, a_1, \ldots, a_i, c) = \{\theta(a_1, \ldots, a_i, x) \mid M \vDash \theta(b_1, \ldots, b_i, e)\} \cup \{x < c\}$$

is satisfied, say, by $e^* \in E$.

As $\mathrm{tp}(a_1, \ldots, a_i, e^*) = \mathrm{tp}(b_1, \ldots, b_i, e)$, by recursive saturation, there is an elementary embedding (actually, an automorphism) $h \colon M \to M$ such that $h(a_1) = b_1, \ldots, h(a_i) = b_i, h(e^*) = e$. Put $b_{i+1} = h(a_{i+1})$. By construction, $e < \mathrm{Cl}(b_1, \ldots, b_{i+1})$. □

Hence, M has an elementary extension $N \succ M$, $N \cong M$ and there is $a \in N \setminus M$ such that $\mathbb{N} < a < M$. Since a union of an elementary chain of recursively saturated models is recursively saturated, we can repeat this extension ω_1 times to obtain the following Theorem, which was promised earlier.

Theorem 11.
Let X be a countable Scott Set. Then X is closed under jump if and only if there is a recursively saturated $M \models \mathrm{PA}$, $\mathrm{lcf}(M \setminus \mathbb{N}) > \omega$, $\mathrm{SSy}(M) = X$.

The countability assumption cannot be dropped yet because for Scott sets X with $\operatorname{card} X > \omega_1$, the existence of a model $M \models \mathrm{PA}$ such that $\mathrm{SSy}(M) = X$ is still an open problem.

5. Automorphisms moving all nondefinable points

Now, as we are discussing arithmetic saturation, let us turn to automorphism groups where arithmetic saturation has profound consequences. We shall employ lemmas and methods of the previous section.

Fact 12. (Kaye, Kossak, Kotlarski [3]) If $M \models \mathrm{PA}$ is countable and arithmetically saturated then M has an automorphism which moves every nondefinable point.

Fact 13. (Kossak [6]) If $M \models \mathrm{Th}\,\mathbb{N}$ is countable and arithmetically saturated then there exists $h \in \mathrm{Aut}(M)$ such that for all $x > \mathbb{N}$, $h(x) > x$, i.e. h moves every nonstandard point upwards.

Fact 14. (Kossak, Schmerl [7]) If M is countable and arithmetically saturated then there is an automorphism f of M such that for every $x > \mathrm{Cl}\varnothing$, $f(x) < x$.

Notice that Fact 14 generalizes Fact 13. We are going to prove a theorem generalizing Fact 13 in a different direction and at the same time fusing it somehow with Fact 12.

Recall the notation $E = \{x \in M \mid \text{there are no nonstandard elements of } \mathrm{Cl}\varnothing \text{ below x}\}$. In general, if $\mathrm{Th}(M) \neq \mathrm{Th}\,\mathbb{N}$, there exists no $h \in \mathrm{Aut}(M)$ such that for all $x \notin \mathrm{Cl}\varnothing$, $h(x) > x$.

Proof. Let $a < b < e$, $e \in \mathrm{Cl}\varnothing \setminus \mathbb{N}$, $h(a) = b$. Then $e - a > e - b$, hence $h(e - a) > h(e - b)$, hence $e - b > h(e - b)$. □

But what we can expect is the following Theorem.

Theorem 15.
If $M \models \mathrm{PA}$ is countable and arithmetically saturated then there is $h \in \mathrm{Aut}(M)$ such that for all $x \in E$, $h(x) > x$ and h moves every nondefinable point.

Since in the case of $\mathrm{Th}(M) = \mathrm{Th}\,\mathbb{N}$, we have $E = M \smallsetminus \mathbb{N}$, this Theorem generalizes Fact 13. The proof uses Kossak's method and the following two lemmas.

Lemma 16. (Kaye, Kotlarski) If M is arithmetically saturated, $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$ and for any Skolem term t,

$$t(\overline{a}) = t(\overline{b}) \Rightarrow t(\overline{a}) \in \mathrm{Cl}\varnothing$$

then for any $c \in M$ there is d such that $\mathrm{tp}(\overline{a}, c) = \mathrm{tp}(\overline{b}, d)$ and for any Skolem term t,

$$t(\overline{a}, c) = t(\overline{b}, d) \Rightarrow t(\overline{a}, c) \in \mathrm{Cl}\varnothing.$$

Notice that Fact 12 follows from Lemma 16 by a back-and-forth argument.

Lemma 17. Let $M \vDash \mathrm{PA}$ be recursively saturated.

(1) If $c < \mathrm{Cl}(\overline{a}) \smallsetminus \mathbb{N}$ then for any b there is b' such that $\mathrm{tp}(\overline{a}, b) = \mathrm{tp}(\overline{a}, b')$,
$c < \mathrm{Cl}(\overline{a}, b') \smallsetminus \mathbb{N}$.

(2) If $c > \mathrm{Cl}(\overline{a}) \cap E$ then for any b there is b' such that $\mathrm{tp}(\overline{a}, b) = \mathrm{tp}(\overline{a}, b')$,
$c > \mathrm{Cl}(\overline{a}, b') \cap E$.

Proof. 1) By Lemma 9, (1), there is $d < \mathrm{Cl}(\overline{a}, b)$ such that $\mathrm{tp}(\overline{a}, c) = \mathrm{tp}(\overline{a}, d)$. By recursive saturation, there is $h : M \to M,\ h(\overline{a}) = \overline{a}, h(d) = c$. Denote $h(b)$ by b'. As h is elementary, $\mathrm{tp}(\overline{a}, b) = \mathrm{tp}(\overline{a}, b')$. As $d < \mathrm{Cl}(\overline{a}, b)$, $c < \mathrm{Cl}(\overline{a}, b')$.
2) Similar proof. □

Proof. We shall construct a string of points $\{d_i\}_{i \in \mathbb{Z}}$ unbounded above and below in E such that our future automorphism h takes d_i to d_{i+1} which will guarantee that each point of E moves upwards: if $a \in (d_i, d_{i+1})$ then $h(a) \in (hd_i, hd_{i+1}) = (d_{i+1}, d_{i+2})$. Also, it obviously follows that there will be no h-fixed initial segment in E other than $\sup E$ and $\mathbb{N}$.

By Lemma 16 there are $c_0, c_1 \in E$ such that $\mathrm{tp}(c_0) = \mathrm{tp}(c_1)$ and $(t(c_0) = t(c_1) \Rightarrow t(c_0) \in \mathrm{Cl}\varnothing)$, hence, considering the type $\{\varphi(x) \mid M \vDash \varphi(c_1)\} \cup \{t(x) \neq t(c_1) \mid t(c_1) \notin \mathrm{Cl}\varnothing\}$, we deduce, using Lemma 9, that there are $d_0, d_1 \in E$ such that

$$\mathrm{tp}(d_0) = \mathrm{tp}(d_1),$$

$$t(d_0) = t(d_1) \Rightarrow t(d_0) \in \mathrm{Cl}\varnothing,$$

$$\mathrm{Cl}(d_0) \cap E < d_1,$$

$$\mathrm{Cl}(d_1) \smallsetminus \mathbb{N} > d_0.$$

Let $\{s_i\}_{i\in\omega}$ be an enumeration of the whole of $M \smallsetminus \mathrm{Cl}\varnothing$. By stage n we shall already have:

$$\overline{a} = a_0, \dots, a_{2n-1},$$

$$\overline{b} = b_0, \dots, b_{2n-1},$$

$$\overline{d} = d_{-n}, d_{-n+1}, \dots, d_n, d_{n+1}$$

satisfying the following conditions:

$$\mathrm{tp}(\overline{a}, d_{-n}, \dots, d_n) = \mathrm{tp}(\overline{b}, d_{-n+1}, \dots, d_{n+1}),$$

$$d_{-n} < \mathrm{Cl}(\overline{b}, d_{-n+1}, \dots, d_{n+1}) \cap E,$$

$$d_{n+1} > \mathrm{Cl}(\overline{a}, d_{-n}, \dots, d_n) \cap E,$$

$$t(\overline{a}, d_{-n}, \dots, d_n) = t(\overline{b}, d_{-n+1}, \dots, d_{n+1}) \;\Rightarrow\; t(\overline{a}, d_{-n}, \dots, d_n) \in \mathrm{Cl}\varnothing.$$

(At stage $n = 0$, $\overline{a}$ and $\overline{b}$ are empty.)

Back

Let $b_{2n} = s_n$. Let $e < \mathrm{Cl}(b_{2n}, \overline{b}, d_{-n}, \dots, d_{n+1})$. By Lemma 16 (applied to the tuples $(\overline{b}, d_{-n+1}, \dots, d_{n+1})$ and $(\overline{a}, d_{-n}, \dots, d_n)$ and the new point d_{-n}), the set of formulas

$$p(x) = \{\varphi(\overline{a}, x, d_{-n}, \dots, d_n) \mid M \vDash \varphi(\overline{b}, d_{-n}, d_{-n+1}, \dots, d_{n+1})\} \cup$$
$$\cup \{t(\overline{a}, x, d_{-n}, \dots, d_n) \neq t(\overline{b}, d_{-n}, d_{-n+1}, \dots, d_{n+1}) \mid$$
$$\mid \; t(\overline{b}, d_{-n}, d_{-n+1}, \dots, d_{n+1}) \notin \mathrm{Cl}\varnothing\}$$

is realized, hence, by Lemma 9 (2), is realized by a point less than e, hence, by Lemma 17 (2), is realized by a point $d_{-n-1} < e$ such that

$$\mathrm{Cl}(\overline{a}, d_{-n-1}, \dots, d_n) \cap E < d_{n+1}.$$

$$\text{Let } q(x) = \{\varphi(\overline{a}, d_{-n-1}, \dots, d_n, x) \mid M \vDash \varphi(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}\} \cup$$
$$\cup \{t(\overline{a}, d_{-n-1}, \dots, d_n, x) \neq t(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}) \mid$$
$$\mid t(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}) \notin \mathrm{Cl}\varnothing\}.$$

By Lemma 16, $q(x)$ is realized, hence, by Lemma 17 (2), is realized by some point a_{2n} such that

$$\mathrm{Cl}(\overline{a}, d_{-n-1}, \dots, d_n, a_{2n}) \cap E < d_{n+1}.$$

By construction,

$$t(\overline{a}, d_{-n-1}, \dots, d_n, a_{2n}) = t(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}) \Rightarrow$$
$$\Rightarrow t(\overline{b}, d_{-n}, \dots, d_{n+1}, b_{2n}) \in \mathrm{Cl}\varnothing,$$

i.e., every nondefinable point of $\mathrm{Cl}(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n})$ moves. Let us show that if $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in E$ then

$$t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) < t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}).$$

If $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) > d_{n+1}$ then $t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) \in (d_n, d_{n+1})$ because $\mathrm{Cl}(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) \cap E < d_{n+1}$. If $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in (d_i, d_{i+1}),\ i = -n, \ldots, n$, then $t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) \in (d_{i-1}, d_i)$.
If $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) < d_{-n}$ then, by construction of d_{-n-1}, $t(\overline{b}, d_{-n}, \ldots, d_{n+1}, b_{2n}) \in (d_{-n-1}, d_{-n})$, hence $t(\overline{a}, d_{-n-1}, \ldots, d_n, a_{2n}) < d_{-n-1}$.

Forth

Let $a_{2n+1} = s_n$. Using Lemmas 16, 9 (1), 17 (1), we choose d_{n+2} such that

$$\mathrm{tp}(\overline{b}, b_{2n}, d_{-n}, \ldots, d_{n+2}) = \mathrm{tp}(\overline{a}, a_{2n}, d_{-n-1}, \ldots, d_{n+1}),$$
$$t(\overline{b}, b_{2n}, d_{-n}, \ldots, d_{n+2}) = t(\overline{a}, a_{2n}, d_{-n-1}, \ldots, d_{n+1}) \Rightarrow$$
$$\Rightarrow t(\overline{a}, a_{2n}, d_{-n-1}, \ldots, d_{n+1} \in \mathrm{Cl}\varnothing,$$
$$d_{n+2} > \mathrm{Cl}(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}) \cap E,$$
$$d_{-n-1} < \mathrm{Cl}(\overline{b}, b_{2n}, d_{-n}, \ldots, d_{n+2}) \cap E.$$

Now, using Lemmas 16 and 17 (1), we choose b_{2n+1} such that

$$\mathrm{tp}(\overline{b}, b_{2n}, b_{2n+1}, d_{-n}, \ldots, d_{n+2}) = \mathrm{tp}(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}),$$
$$t(\overline{b}, b_{2n}, b_{2n+1}, d_{-n}, \ldots, d_{n+2}) = t(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}) \Rightarrow$$
$$\Rightarrow t(\overline{a}, a_{2n}, a_{2n+1}, d_{-n-1}, \ldots, d_{n+1}) \in \mathrm{Cl}\varnothing,$$
$$\text{and } \ d_{-n-1} < \mathrm{Cl}(\overline{b}, b_{2n}, b_{2n+1}, d_{-n}, \ldots, d_{n+2}).$$

Having obtained the points a_i, b_i for all $i \in \omega$, we observe that $h\colon M \to M$ defined as $h(a_i) = b_i$ for all $i \in \omega$ is an elementary isomorphism possessing the required properties. □

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