# 6 The Countable Ordinals

In this section, we gather some definitions and results about the countable ordinals needed to explain what $\Gamma_0$ is. This ordinal plays a central role in proof theoretic investigations of a subsystem of second-order arithmetic known as "predicative analysis", which has been studied extensively by Feferman [13] and Schütte [46]. Schütte's axiomatic presentation of the countable ordinals ([46], chapters 13, 14) is particularly convenient (and elegant), and we follow it. Most proofs are omitted. They can be found in Schütte [46].

## 6.1 A Preview of $\Gamma_0$

Proof theorists use (large) ordinals in inductive proofs establishing the consistency of certain theories. In order for these proofs to be as constructive as possible, it is crucial to describe these ordinals using systems of constructive ordinal notations. One way to obtain constructive ordinal notation systems is to build up inductively larger ordinals from smaller ones using functions on the ordinals. For example, if $\mathcal{O}$ denotes the set of countable ordinals, it is possible to define two functions $+$ and $\alpha \mapsto \omega^\alpha$ (where $\omega$ is the least infinite ordinal) generalizing addition and exponentiation on the natural numbers. Due to a result of Cantor, for every ordinal $\alpha \in \mathcal{O}$, if $\alpha > 0$, there are unique ordinals $\alpha_1 \geq \ldots \geq \alpha_n$, $n \geq 1$, such that

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}. \tag{$*$}$$

This suggests a constructive ordinal notation system. Define $\mathcal{C}$ to be the smallest set of ordinals containing 0 and closed under $+$ and $\alpha \mapsto \omega^\alpha$.

Do we have $\mathcal{C} = \mathcal{O}$? The answer is *no*. Indeed, strange things happen with infinite ordinals. For some ordinals $\alpha, \beta$ such that $0 < \alpha < \beta$, we can have $\alpha + \beta = \beta$, and even $\omega^\alpha = \alpha$!

An ordinal $\beta > 0$ such that $\alpha + \beta = \beta$ for all $\alpha < \beta$ is called an *additive principal ordinal*. It can be shown that an ordinal is an additive principal ordinal iff it is of the form $\omega^\eta$ for some $\eta$.

The general phenomenon that we are witnessing is the fact that if a function $f : \mathcal{O} \to \mathcal{O}$ satisfies a certain continuity condition, then it has *fixed points* (an ordinal $\alpha$ is a fixed point of $f$ iff $f(\alpha) = \alpha$).

The least ordinal such that $\omega^\alpha = \alpha$ (the least fixed point of $\alpha \mapsto \omega^\alpha$) is denoted by $\epsilon_0$, and $\mathcal{C}$ provides a constructive ordinal notation system for the ordinals $< \epsilon_0$. The main point here, is that for every ordinal $\alpha < \epsilon_0$, we can guarantee that $\alpha_i < \alpha$ in the decomposition $(*)$.

Unfortunately $\epsilon_0$ is too small for our purpose (which is to relate the embedding relation $\preceq$ on finite trees with the ordering on $\Gamma_0$). To go beyond $\epsilon_0$, we need functions more powerful than $\alpha \mapsto \omega^\alpha$. Such a hierarchy $(\varphi_\alpha)_{\alpha \in \mathcal{O}}$ can be defined inductively, starting from $\alpha \mapsto \omega^\alpha$.

We let $\varphi_0$ be the function $\alpha \mapsto \omega^\alpha$, and for every $\alpha > 0$, $\varphi_\alpha \colon \mathcal{O} \to \mathcal{O}$ enumerates the common fixed points of the functions $\varphi_\beta$, for all $\beta < \alpha$ (the ordinals $\eta$ such that $\varphi_\beta(\eta) = \eta$ for all $\beta < \alpha$).

Then, we have a function $\varphi \colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, defined such that $\varphi(\alpha, \beta) = \varphi_\alpha(\beta)$ for all $\alpha, \beta \in \mathcal{O}$. Note, $\varphi(1, 0) = \epsilon_0$!

The function $\varphi$ has lots of fixed points. We can have $\varphi(\alpha, \beta) = \beta$, in which case $\beta$ is called an *$\alpha$-critical ordinal*, or $\varphi(\alpha, 0) = \alpha$ (but we can't have $\varphi(\alpha, \beta) = \alpha$ for $\beta > 0$). Ordinals such that $\varphi(\alpha, 0) = \alpha$ are called *strongly critical*.

It can be shown that for every additive principal ordinal $\gamma = \omega^\eta$, there exist unique $\alpha, \beta$ with $\alpha \leq \gamma$ and $\beta < \gamma$, such that $\gamma = \varphi(\alpha, \beta)$. But we can't guarantee that $\alpha < \gamma$, because $\varphi(\alpha, 0) = \alpha$ when $\alpha$ is a strongly critical ordinal. This is where $\Gamma_0$ comes in!

The ordinal $\Gamma_0$ is the *least* ordinal such that $\varphi(\alpha, 0) = \alpha$ (the least strongly critical ordinal). It can be shown that for all $\alpha, \beta < \Gamma_0$, we have $\alpha + \beta < \Gamma_0$ and $\varphi(\alpha, \beta) < \Gamma_0$, and also that for every additive principal ordinal $\gamma < \Gamma_0$, $\gamma = \varphi(\alpha, \beta)$ for unique ordinals such that *both* $\alpha < \gamma$ and $\beta < \gamma$. This fact together with the Cantor normal form $(*)$ yields a constructive ordinal notation system for the ordinals $< \Gamma_0$ described in the sequel.

The reason why we were able to build the hierarchy $(\varphi_\alpha)_{\alpha \in \mathcal{O}}$ is that these functions satisfy certain conditions: they are increasing and continuous. Such functions are called *normal functions*. What is remarkable is that the function $\varphi(-, 0)$ is also a normal function, and so, it is possible to repeat the previous hierarchy construction, but this time, starting from $\varphi(-, 0)$. But there is no reason to stop there, and we can continue on and on ...!

We have what is called a *Veblen hierarchy* [53]. However, this is going way beyond the scope of these notes (transfinitely beyond!). The intrigued reader is referred to a paper by Larry Miller [34].

## 6.2 Axioms for the Countable Ordinals

Recall that a set $A$ is countable iff either $A = \emptyset$ or there is a surjective (onto) function $f \colon \mathbf{N} \to A$ with domain $\mathbf{N}$, the set of natural numbers. In particular, every finite set is countable.

Given a set $A$ and a partial order $\leq$ on $A$, we say that $A$ is *well-ordered* by $\leq$ iff every nonempty subset of $A$ has a least element.

This definition implies that a well-ordered set is totally ordered. Indeed, every subset $\{x, y\}$ of $A$ consisting of two elements has a least element, and so, either $x \leq y$ or $y \leq x$.

We say that a subset $S \subseteq A$ of $A$ is *strictly bounded* iff there is some $b \in A$ such that $x < b$ for all $x \in S$ (recall that $x < y$ iff $x \leq y$ and $x \neq y$). A subset $S$ of $A$ that is not strictly bounded is called *unbounded*. The set of countable ordinals is defined by the following axioms.

**Definition 6.1** A set $\mathcal{O}$ together with a partial order $\leq$ on $\mathcal{O}$ satisfies the axioms for the countable ordinals iff the following properties hold:

(1) $\mathcal{O}$ is well-ordered by $\leq$.

(2) Every strictly bounded subset of $\mathcal{O}$ is countable.

(3) Every countable subset of $\mathcal{O}$ is strictly bounded.

Applying axiom (3) to the empty set (which is a subset of $\mathcal{O}$), we see that $\mathcal{O}$ is nonempty. Applying axiom (1) to $\mathcal{O}$, we see that $\mathcal{O}$ has a least element denoted by 0. Repeating this argument, we see that $\mathcal{O}$ is infinite. However, $\mathcal{O}$ is *not* countable. Indeed if $\mathcal{O}$ was countable, by axiom (3), there would be some $\alpha \in \mathcal{O}$ such that $\beta < \alpha$ for all $\beta \in \mathcal{O}$, which implies $\alpha < \alpha$, a contradiction.

It is possible to show that axioms (1)-(3) define the set of countable ordinals up to isomorphism. From now on, the elements of the set $\mathcal{O}$ will be called ordinals (strictly speaking, they should be called countable ordinals).

Given a property $P(x)$ of the set of countable ordinals, the principle of *transfinite induction* is the following:

- If $P(0)$ holds, and
- for every $\alpha \in \mathcal{O}$ such that $\alpha > 0$, $\forall \beta(\beta < \alpha \supset P(\beta))$ implies $P(\alpha)$, then
- $P(\gamma)$ holds for all $\gamma \in \mathcal{O}$.

We have the following fundamental metatheorem.

**Theorem 6.2** The principle of transfinite induction is valid for $\mathcal{O}$.

*Proof*. Assume that the principle of transfinite induction does not hold. Then, $P(0)$ holds, for every $\alpha \in \mathcal{O}$ such that $\alpha > 0$, $\forall \beta(\beta < \alpha \supset P(\beta))$ implies $P(\alpha)$, but the set $W = \{\alpha \in \mathcal{O} \mid P(\alpha) = \mathbf{false}\}$ is nonempty. By axiom (1), this set has a least element $\gamma$. Clearly, $\gamma \neq 0$, and $P(\beta)$ must hold for all $\beta < \gamma$, since otherwise $\gamma$ would not be the least element of $W$. Hence, $\forall \beta < \gamma P(\beta)$ holds, and from above, this implies that $P(\gamma)$ holds, contradicting the definition of $\gamma$. $\square$

By axioms (1) and (3), for every ordinal $\alpha$, there is a smallest ordinal $\beta$ such that $\alpha < \beta$. Indeed, the set $\{\alpha\}$ is countable, hence by axiom (3) the set $\{\beta \in \mathcal{O} \mid \alpha < \beta\}$ is nonempty, and by axiom (1), it has a least element. This ordinal is denoted by $\alpha'$, and is called the *successor* of $\alpha$. We have the following properties:

$$\alpha < \alpha'$$
$$\alpha < \beta \Rightarrow \alpha' \leq \beta$$
$$\alpha < \beta' \Rightarrow \alpha \leq \beta.$$

An ordinal $\beta$ is called a *successor ordinal* iff there is some $\alpha \in \mathcal{O}$ such that $\beta = \alpha'$. A *limit ordinal* is an ordinal that is neither 0 nor a successor ordinal.

Given any countable subset $M \subseteq \mathcal{O}$, by axiom (3), the set $\{\alpha \in \mathcal{O} \mid \forall \beta \in M(\beta \leq \alpha)\}$ is nonempty, and by axiom (1), it has a least element. This ordinal denoted by $\bigsqcup M$ is the *least upper bound* of $M$, and it satisfies the following properties:

$$\alpha \in M \Rightarrow \alpha \leq \bigsqcup M$$
$$\alpha \leq \beta \text{ for all } \alpha \in M \Rightarrow \bigsqcup M \leq \beta$$
$$\beta < \bigsqcup M \Rightarrow \exists \alpha \in M \text{ such that } \beta < \alpha.$$

We have the following propositions.

**Proposition 6.3** If $M$ is a nonempty countable subset of $\mathcal{O}$ and $M$ has no maximal element, then $\bigsqcup M$ is a limit ordinal.

**Proposition 6.4** For all $\alpha, \beta \in \mathcal{O}$, if $\gamma < \beta$ for all $\gamma < \alpha$, then $\alpha \leq \beta$.

*Proof*. The proposition is clear if $\alpha = 0$. If $\alpha$ is a successor ordinal, $\alpha = \delta'$ for some $\delta$, and since $\delta < \alpha$, by the hypothesis we have $\delta < \beta$, which implies $\alpha = \delta' \leq \beta$. If $\alpha$ is a limit ordinal, we prove that $\alpha = \bigsqcup\{\gamma \in \mathcal{O} \mid \gamma < \alpha\}$, which implies that $\alpha \leq \beta$, since by the hypothesis $\beta$ is an upper bound of the set $\{\gamma \in \mathcal{O} \mid \gamma < \alpha\}$. Let $\delta = \bigsqcup\{\gamma \in \mathcal{O} \mid \gamma < \alpha\}$. First, it is clear that $\alpha$ is an upper bound of the set $\{\gamma \in \mathcal{O} \mid \gamma < \alpha\}$, and so $\delta \leq \alpha$. If $\delta < \alpha$, since $\alpha$ is a limit ordinal, we have $\delta' < \alpha$, contradicting the fact that $\delta$ is the least upper bound of the set $\{\gamma \in \mathcal{O} \mid \gamma < \alpha\}$. Hence, $\delta = \alpha$. $\square$

**Definition 6.5** The set $\mathbf{N}$ of *finite ordinals* is the smallest subset of $\mathcal{O}$ that contains 0 and is closed under the successor function.

It is not difficult to show that $\mathbf{N}$ is countable and has no maximal element. The least upper bound of $\mathbf{N}$ is denoted by $\omega$.

**Proposition 6.6** The ordinal $\omega$ is the least limit ordinal. For every $\alpha \in \mathcal{O}$, $\alpha < \omega$ iff $\alpha \in \mathbf{N}$.

It is easy to see that limit ordinals satisfy the following property: For every limit ordinal $\beta$

$$\alpha < \beta \Rightarrow \alpha' < \beta.$$

## 6.3 Ordering Functions

Given any ordinal $\alpha \in \mathcal{O}$, let $\mathcal{O}(\alpha)$ be the set $\{\beta \in \mathcal{O} \mid \beta < \alpha\}$. Clearly, $\mathcal{O}(0) = \emptyset$, $\mathcal{O}(\omega) = \mathbf{N}$, and by axiom (2), each $\mathcal{O}(\alpha)$ is countable.

**Definition 6.7** A subset $A \subseteq \mathcal{O}$ is an *$\mathcal{O}$-segment* iff for all $\alpha, \beta \in \mathcal{O}$, if $\beta \in A$ and $\alpha < \beta$, then $\alpha \in A$.

The set $\mathcal{O}$ itself is an $\mathcal{O}$-segment, and an $\mathcal{O}$-segment which is a proper subset of $\mathcal{O}$ is called a *proper $\mathcal{O}$-segment*. It is easy to show that $A$ is a proper $\mathcal{O}$-segment iff $A = \mathcal{O}(\alpha)$ for some $\alpha \in \mathcal{O}$.

We now come to the crucial concept of an ordering function.

**Definition 6.8** Given a subset $B \subseteq \mathcal{O}$, a function $f : A \to B$ is an *ordering function* for $B$ iff:

(1) The domain of $f$ is an $\mathcal{O}$-segment.

(2) The function $f$ is *strictly monotonic (or increasing)*, that is, for all $\alpha, \beta \in \mathcal{O}$, if $\alpha < \beta$, then $f(\alpha) < f(\beta)$.

(3) The range of $f$ is $B$.

Intuitively speaking, an ordering function $f$ of a set $B$ *enumerates* the elements of the set $B$ in increasing order. Observe that an ordering function $f$ is bijective, since by (3), $f(A) = B$, and by (2), $f$ is injective. Note that the ordering function for the empty set is the empty function. The following fundamental propositions are shown by transfinite induction.

**Proposition 6.9** If $f : A \to B$ is an ordering function, then $\alpha \leq f(\alpha)$ for all $\alpha \in A$

*Proof*. Clearly, $0 \leq f(0)$. Given any ordinal $\alpha > 0$, for every $\beta < \alpha$, by the induction hypothesis, $\beta \leq f(\beta)$. Since $f$ is strictly monotonic, $f(\beta) < f(\alpha)$. Hence, $\beta < f(\alpha)$ for all $\beta < \alpha$, and by proposition 6.4, this implies that $\alpha \leq f(\alpha)$. $\square$

**Proposition 6.10** Every subset $B \subseteq \mathcal{O}$ has at most one ordering function $f : A \to B$.

*Proof.* Let $f_i : A_i \to B$, $i = 1, 2$, be two ordering functions for $B$. We show by transfinite induction that, if $\alpha \in A_1$, then $\alpha \in A_2$ and $f_1(\alpha) = f_2(\alpha)$. If $B = \emptyset$, then clearly $f_1 = f_2 : \emptyset \to \emptyset$. Otherwise, since $A_1$ and $A_2$ are $\mathcal{O}$-segments, $0 \in A_1$ and $0 \in A_2$. Since $f_2$ is surjective, there is some $\alpha \in A_2$ such that $f_2(\alpha) = f_1(0)$. By (strict) monotonicity of $f_2$, we have $f_2(0) \leq f_1(0)$. Similarly, since $f_1$ is surjective, there is some $\beta \in A_1$ such that $f_1(\beta) = f_2(0)$, and by (strict) monotonicity of $f_1$, we have $f_1(0) \leq f_2(0)$. Hence $f_1(0) = f_2(0)$. Now, assume $\alpha > 0$. Since $f_2$ is surjective, there is some $\beta \in A_2$ such that $f_2(\beta) = f_1(\alpha)$. If $\beta < \alpha$, since $A_1$ is an $\mathcal{O}$-segment, $\beta \in A_1$, and by the induction hypothesis, $\beta \in A_2$ and $f_1(\beta) = f_2(\beta)$. By strict monotonicity, $f_2(\beta) = f_1(\beta) < f_1(\alpha)$, a contradiction.

Hence, $\beta \geq \alpha$, and since $A_2$ is an $\mathcal{O}$-segment and $\beta \in A_2$, we have $\alpha \in A_2$. Assume $\beta > \alpha$. By strict monotonicity, $f_2(\alpha) < f_2(\beta)$. Since $f_1$ is surjective, there is some $\gamma \in A_1$ such that $f_1(\gamma) = f_2(\alpha)$. Since $f_2(\alpha) = f_1(\gamma)$, $f_2(\beta) = f_1(\alpha)$, and $f_2(\alpha) < f_2(\beta)$, we have $f_1(\gamma) < f_1(\alpha)$. By strict monotonicity, we have $\gamma < \alpha$. By the induction hypothesis, $f_1(\gamma) = f_2(\gamma)$, and since $f_1(\gamma) = f_2(\alpha)$, then $f_2(\gamma) = f_2(\alpha)$. Since $f_2$ is injective, we have $\alpha = \gamma$, a contradiction. Hence, $\alpha = \beta$ and $f_1(\alpha) = f_2(\alpha)$. Therefore, we have shown that $A_1 \subseteq A_2$ and for every $\alpha \in A_1$, $f_1(\alpha) = f_2(\alpha)$. Using a symmetric argument, we can show that $A_2 \subseteq A_1$ and for every $\alpha \in A_2$, $f_1(\alpha) = f_2(\alpha)$. Hence, $A_1 = A_2$ and $f_1 = f_2$. $\square$

Given a set $B \subseteq \mathcal{O}$, for every $\beta \in B$, let $B(\beta) = \{\gamma \in B \mid \gamma < \beta\}$. Sets of the form $B(\beta)$ are called *proper segments* of $B$. Observe that $B(\beta) = B \cap \mathcal{O}(\beta)$. Using proposition 6.10, we prove the following crucial result.

**Proposition 6.11** Every subset $B \subseteq \mathcal{O}$ has a unique ordering function $f : A \to B$.

*Proof.* First, the following claim is shown.

*Claim*: If every proper segment $B(\beta)$ of a set $B \subseteq \mathcal{O}$ has an ordering function, then $B$ has an ordering function.

*Proof of claim.* The idea is to construct a function $g : B \to \mathcal{O}$ and to show that $g$ is strictly monotonic and that its range is an $\mathcal{O}$-segment. Then, the inverse of $g$ is an ordering function for $B$. By the hypothesis, for every $\beta \in B$, we have an ordering function $f_\beta : A_\beta \to B(\beta)$ for each proper segment $B(\beta)$ of $B$. By axiom (2) (in definition 6.1), $B(\beta)$ is countable. Since $f_\beta$ is bijective, $A_\beta$ is also countable, and therefore, it is a proper $\mathcal{O}$-segment. Hence, for every $\beta \in B$, there is a unique ordinal $\gamma$ such that $A_\beta = \mathcal{O}(\gamma)$, and we define the function $g : B \to \mathcal{O}$ such that $g(\beta) = \gamma$.

We show that $g$ is strictly monotonic. Let $\beta_1 < \beta_2$, $\beta_1, \beta_2 \in B$. Since the function $f_{\beta_2} : \mathcal{O}(g(\beta_2)) \to B(\beta_2)$ is surjective and $\beta_1 \in B(\beta_2)$ (since $\beta_1 < \beta_2$ and $\beta_2 \in B$), there is some $\alpha < g(\beta_2)$ such that $f_{\beta_2}(\alpha) = \beta_1$. Observe that the restriction of $f_{\beta_2}$ to $\mathcal{O}(\alpha)$ is an ordering function of $B(\beta_1)$. Since $f_{\beta_1} : A_{\beta_1} \to B(\beta_1)$ is also an ordering function for $B(\beta_1)$, by proposition 6.10, $\mathcal{O}(\alpha) = \mathcal{O}(g(\beta_1))$, and therefore, $g(\beta_1) = \alpha < g(\beta_2)$.

We show that $g(B)$ is an $\mathcal{O}$-segment. We have to show that for every $\gamma \in g(B)$, if $\alpha < \gamma$, then $\alpha \in g(B)$. Let $\beta \in B$ such that $\gamma = g(\beta)$. Since $f_\beta : \mathcal{O}(g(\beta)) \to B(\beta)$ and $\alpha < g(\beta)$, $f_\beta(\alpha) = \beta_0$ for some $\beta_0 \in B(\beta)$. The restriction of $f_\beta$ to $\mathcal{O}(\alpha)$ is an ordering function of $B(\beta_0)$. Since $f_{\beta_0} : \mathcal{O}(g(\beta_0)) \to B(\beta_0)$ is also an ordering function for $B(\beta_0)$, by proposition 6.10, $\alpha = g(\beta_0)$, and therefore $\alpha \in g(B)$.

Since the function $g : B \to \mathcal{O}$ is strictly monotonic and $g(B)$ is an $\mathcal{O}$-segment, say $A$, its inverse $g^{-1} : A \to B$ is an ordering function for $B$. This proves the claim. $\square$

Let $B \subseteq \mathcal{O}$. For every $\beta \in B$, note that every proper segment of $B(\beta)$ is of the form $B(\beta_0)$ for some $\beta_0 < \beta$. Using the previous claim, it follows by transfinite induction that every proper segment $B(\beta)$ of $B$ has an ordering function. By the claim, $B$ itself has an ordering function. By proposition 6.10, this function is unique. $\square$

An important property of ordering functions is continuity.

**Definition 6.12** A subset $B \subseteq \mathcal{O}$ is *closed* iff for every countable nonempty set $M$,

$$M \subseteq B \Rightarrow \bigsqcup M \in B.$$

An ordering function $f : A \to B$ is *continuous* iff $A$ is closed and for every nonempty countable set $M \subseteq A$,

$$f(\bigsqcup M) = \bigsqcup f(M).$$

**Proposition 6.13** The ordering function $f : A \to B$ of a set $B$ is continuous iff $B$ is closed.

*Proof*. Let $f : A \to B$ be the ordering function of $B$. First, assume that $f$ is continuous. Since $f$ is bijective, for every nonempty countable subset $M \subseteq B$, there is some nonempty countable subset $U \subseteq A$ such that $f(U) = M$. Since $f$ is continuous, $f(\bigsqcup U) = \bigsqcup f(U) = \bigsqcup M$, and therefore $\bigsqcup M \in f(A) = B$, and $B$ is closed.

Conversely, assume that $B$ is closed. Let $U \subseteq A$ be a nonempty countable subset of $A$. Since $f$ is bijective, $f(U)$ is a nonempty countable subset of $B$. Since $B$ is closed, $\bigsqcup f(U) \in B$. Since $B = f(A)$, there is some $\alpha \in A$ such that $f(\alpha) = \bigsqcup f(U)$. Since $f(\alpha) = \bigsqcup f(U)$, for every $\delta \in U$, we have $f(\delta) \leq f(\alpha)$, and by strict monotonicity of $f$, this implies that $\delta \leq \alpha$. Hence $\bigsqcup U \leq \alpha$. Since $A$ is an $\mathcal{O}$-segment, $\bigsqcup U \in A$. Hence, $A$ is closed. For all $\delta \in U$, $\delta \leq \bigsqcup U$, and so $f(\delta) \leq f(\bigsqcup U)$. Then, $f(\bigsqcup U)$ is an upper bound for

$f(U)$, and so $\bigsqcup f(U) \leq f(\bigsqcup U)$. Also, since $\bigsqcup U \leq \alpha$, we have $f(\bigsqcup U) \leq f(\alpha) = \bigsqcup f(U)$. But then, $\bigsqcup f(U) = f(\bigsqcup U)$, and $f$ is continuous. $\square$

An ordering function that is continuous and whose domain is the entire set $\mathcal{O}$ is called a *normal function*. Normal functions play a crucial role in the definition of $\Gamma_0$.

**Proposition 6.14** The ordering function $f : A \to B$ of a set $B$ is a normal function iff $B$ is closed and unbounded.

*Proof*. By axiom (2) and (3) (in definition 6.1), a subset $M$ of $\mathcal{O}$ is bounded iff it is countable. Since an ordering function $f : A \to B$ is bijective, it follows that $B$ is unbounded iff $A$ is unbounded. But $A$ is an $\mathcal{O}$-segment, and $\mathcal{O}$ is the only unbounded $\mathcal{O}$-segment (since a proper $\mathcal{O}$-segment is bounded). Hence, the ordering function $f$ has domain $\mathcal{O}$ iff $B$ is unbounded. This together with proposition 6.13 yields proposition 6.14. $\square$

We now show that normal functions have fixed points.

**Proposition 6.15** Let $f : \mathcal{O} \to \mathcal{O}$ be a continuous function. For every $\alpha \in \mathcal{O}$, let $f^0(\alpha) = \alpha$, and $f^{n+1}(\alpha) = f(f^n(\alpha))$ for all $n \geq 0$. If $\alpha \leq f(\alpha)$ for every $\alpha \in \mathcal{O}$, then $\bigsqcup_{n\geq 0} f^n(\alpha)$ is the least fixed point of $f$ that is $\geq \alpha$, and $\bigsqcup_{n\geq 0} f^n(\alpha')$ is the least fixed point of $f$ that is $> \alpha$.

*Proof*. First, observe that a continuous function is monotonic, by applying the continuity condition to each set $\{\alpha, \beta\}$ with $\alpha \leq \beta$. Since $f$ is continuous,

$$\begin{aligned} f(\bigsqcup_{n\geq 0} f^n(\alpha)) &= \bigsqcup_{n\geq 0} f(f^n(\alpha)) \\ &= \bigsqcup_{n\geq 0} f^{n+1}(\alpha) \\ &= \bigsqcup_{n\geq 1} f^n(\alpha) \\ &= \bigsqcup_{n\geq 0} f^n(\alpha), \end{aligned}$$

since $\alpha \leq f(\alpha)$. Hence, $\bigsqcup_{n\geq 0} f^n(\alpha)$ is a fixed point of $f$ that is $\geq \alpha$. Let $\beta$ be any fixed point of $f$ such that $\alpha \leq \beta$. We show by induction that $f^n(\alpha) \leq \beta$. For $n = 0$, this follows from the fact that $f^0(\alpha) = \alpha$ and the hypothesis $\alpha \leq \beta$. If $f^n(\alpha) \leq \beta$, since $f$ is monotonic we have, $f(f^n(\alpha)) \leq f(\beta)$, that is, $f^{n+1}(\alpha) \leq \beta$, since $f^{n+1}(\alpha) = f(f^n(\alpha))$ and $f(\beta) = \beta$ (because $\beta$ is a fixed point of $f$). Hence, $\bigsqcup_{n\geq 0} f^n(\alpha) \leq \beta$, which shows that $\bigsqcup_{n\geq 0} f^n(\alpha)$ is the least fixed point of $f$ that is $\geq \alpha$.

From above, $\bigsqcup_{n\geq 0} f^n(\alpha')$ is the least fixed point of $f$ that is $\geq \alpha'$, and since $\beta \geq \alpha'$ iff $\beta > \alpha$, the second part of the lemma holds. $\square$

**Corollary 6.16** For every normal function $f$, for every $\alpha \in \mathcal{O}$, $\bigsqcup_{n\geq 0} f^n(\alpha)$ is the least fixed point of $f$ that is $\geq \alpha$, and $\bigsqcup_{n\geq 0} f^n(\alpha')$ is the least fixed point of $f$ that is $> \alpha$.

*Proof*. Since a normal function is continuous and $\alpha \leq f(\alpha)$ for all $\alpha$, the corollary follows from proposition 6.15. □

Using the concept of a normal function, we are going to define addition and exponentiation of ordinals.

## 6.4 Addition and Exponentiation of Ordinals

For every $\alpha \in \mathcal{O}$, let $B_\alpha = \{\beta \in \mathcal{O} \mid \alpha \leq \beta\}$. Let $f_\alpha$ be the ordering function of $B_\alpha$ given by proposition 6.11. It is easy to see that $B_\alpha$ is closed and unbounded. Hence, by proposition 6.14, $f_\alpha$ is a normal function. We shall write $\alpha + \beta$ for $f_\alpha(\beta)$. The following properties of $+$ can be shown:

$\alpha \leq \alpha + \beta$.

$\beta < \gamma \Rightarrow \alpha + \beta < \alpha + \gamma$ (right strict monotonicity).

If $\alpha \leq \beta$, then there is a unique $\gamma$ such that $\alpha + \gamma = \beta$.

For every limit ordinal $\beta \in \mathcal{O}$, $\bigsqcup \mathcal{O}(\beta) = \beta$, and $\alpha + \beta = \bigsqcup\{\alpha + \gamma \mid \gamma \in \mathcal{O}(\beta)\}$.

$\alpha + 0 = \alpha$.

$\alpha + \beta' = (\alpha + \beta)'$.

$\beta \leq \alpha + \beta$.

$0 + \beta = \beta$

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

$\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$ (left weak monotonicity).

It should be noted that addition of ordinals is *not* commutative. Indeed, $0' + \omega = \bigsqcup \mathbf{N} = \omega$, but $\omega < \omega + 0'$ by right strict monotonicity. Also,

**Definition 6.17** An ordinal $\alpha \in \mathcal{O}$ is a *principal additive ordinal* iff $\alpha \neq 0$ and for every $\beta < \alpha$, $\beta + \alpha = \alpha$.

Clearly, $1 = 0'$ is the smallest additive principal ordinal, and it is not difficult to show that $\omega$ is the least additive principal ordinal greater than 1. Note that $\alpha + 1 = \alpha'$.

If $\alpha$ is an additive principal ordinal, then $\mathcal{O}(\alpha)$ is closed under addition.

**Proposition 6.18** The set of additive principal ordinals is closed and unbounded.

*Proof*. First, we show unboundedness. Given any ordinal $\alpha$, let $\beta_0 = \alpha'$, $\beta_{n+1} = \beta_n + \beta_n$, $M = \{\beta_n \mid n \in \mathbf{N}\}$, and $\beta = \bigsqcup M$. Since $\beta_0 = \alpha' > 0$, we have $\beta_n > 0$ for all $n \geq 0$, and by right strict monotonicity of $+$, $\beta_n < \beta_n + \beta_n = \beta_{n+1}$. Hence, $\alpha < \beta_n < \beta$ for all $n \geq 0$, and $\beta > 0$. If $\eta < \beta$, then there is some $n \geq 0$ such that $\eta < \beta_n$. Hence, for all $m \geq n$, $\eta + \beta_m \leq \beta_m + \beta_m = \beta_{m+1} < \beta$. Hence, $\bigsqcup\{\eta + \beta_n \mid n \in \mathbf{N}\} \leq \beta$. But we also have $\beta \leq \eta + \beta = \bigsqcup\{\eta + \beta_n \mid n \in \mathbf{N}\} \leq \beta$. Hence, $\eta + \beta = \beta$ for all $\eta < \beta$. Therefore, $\beta$ is an additive principal ordinal.

Next, we show closure. Let $M$ be a nonempty set of additive principal ordinals. Since for every $\beta \in M$, $\beta > 0$, we have $\bigsqcup M > 0$. Let $\eta < \bigsqcup M$. Then, there is some $\alpha \in M$ such that $\eta < \alpha$. For every $\beta \in M$, if $\beta \geq \alpha$, then $\eta < \beta$, and since $\beta$ is additive principal, $\eta + \beta = \beta$. Hence, $\bigsqcup\{\eta + \beta \mid \beta \in M\} = \bigsqcup M$ for all $\eta < \bigsqcup M$, which shows that $\bigsqcup M$ is additive principal. $\square$

By proposition 6.14, the ordering function of the set of additive principal ordinals is a normal function.

**Definition 6.19** The ordering function of the set of additive principal ordinals is a normal function whose value for every ordinal $\alpha$ is denoted by $\omega^\alpha$.

The following properties hold.

$0 < \omega^\alpha$.

$\beta < \omega^\alpha \Rightarrow \beta + \omega^\alpha = \omega^\alpha$.

$\alpha < \beta \Rightarrow \omega^\alpha < \omega^\beta$.

For every additive principal ordinal $\beta$, there is some $\alpha$ such that $\beta = \omega^\alpha$.

For every limit ordinal $\beta$, $\omega^\beta = \bigsqcup\{\omega^\alpha \mid \alpha \in \mathcal{O}(\beta)\}$.

$\alpha < \beta \Rightarrow \omega^\alpha + \omega^\beta = \omega^\beta$.

$\omega^0 = 1$.

$\omega^1 = \omega$.

The following result known as the *Cantor Normal Form* for the (countable) ordinals is fundamental.

**Proposition 6.20** (Cantor Normal Form) For every ordinal $\alpha \in \mathcal{O}$, if $\alpha > 0$ then there are unique ordinals $\alpha_1 \geq \ldots \geq \alpha_n$, $n \geq 1$, such that

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}.$$

*Proof*. First, we show the existence of the representation. We proceed by transfinite induction. If $\alpha$ is an additive principal ordinal, then $\alpha = \omega^{\alpha_1}$ for some $\alpha_1$ since $\gamma \mapsto \omega^\gamma$ is the ordering function of the additive principal ordinals. Otherwise, there is some $\delta < \alpha$ such that $\delta + \alpha \neq \alpha$. Then, since $\alpha \leq \delta + \alpha$ (by proposition 6.9), $\delta > 0$ and $\delta + \alpha > \alpha$. Since $\delta < \alpha$, there is some $\eta > 0$ such that $\alpha = \delta + \eta$. We must have $\eta < \alpha$, since otherwise, by right monotonicity, we would have $\delta + \alpha \leq \delta + \eta = \alpha$, contradicting $\delta + \alpha > \alpha$. Hence, $\alpha = \delta + \eta$, with $0 < \delta, \eta < \alpha$. By the induction hypothesis, $\delta = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\eta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, for some ordinals such that $\alpha_1 \geq \ldots \geq \alpha_m$ and $\beta_1 \geq \ldots \geq \beta_n$. If we had $\alpha_i < \beta_1$ for all $i$, $1 \leq i \leq m$, then we would have $\delta + \eta = \eta$ (using the fact that for additive principal ordinals, if $\alpha < \beta$, then $\omega^\alpha + \omega^\beta = \omega^\beta$), that is, $\alpha = \eta$, contradicting the fact that $\eta < \alpha$. Hence, there is a largest $k$, $1 \leq k \leq m$ such that $\alpha_k \geq \beta_1$. Consequently, $\alpha_1 \geq \ldots \geq \alpha_k \geq \beta_1 \geq \ldots \geq \beta_n$, and since $\omega^{\alpha_j} + \omega^{\beta_1} = \omega^{\beta_1}$ for $k+1 \leq j \leq m$, we have

$$\begin{aligned}
\alpha &= \delta + \eta \\
&= \omega^{\alpha_1} + \cdots + \omega^{\alpha_k} + \omega^{\alpha_{k+1}} + \cdots + \omega^{\alpha_m} + \omega^{\beta_1} + \cdots + \omega^{\beta_n} \\
&= \omega^{\alpha_1} + \cdots + \omega^{\alpha_k} + \omega^{\beta_1} + \cdots + \omega^{\beta_n}.
\end{aligned}$$

Assume $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m} = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$. Uniqueness is shown by induction on $m$. Note that $\alpha + \omega^{\alpha'_1} = \omega^{\alpha'_1}$, which implies that $\alpha < \omega^{\alpha'_1}$ (by right strict monotonicity, since $\omega^{\alpha'_1} > 0$), and similarly, $\alpha < \omega^{\beta'_1}$. If we had $\beta'_1 \leq \alpha_1$, we would have $\omega^{\beta'_1} \leq \omega^{\alpha_1} \leq \alpha$, contradicting the fact that $\alpha < \omega^{\beta'_1}$. Hence, $\alpha_1 < \beta'_1$. Similarly, we have $\beta_1 < \alpha'_1$. But then, $\alpha_1 \leq \beta_1$ and $\beta_1 \leq \alpha_1$, and therefore, $\alpha_1 = \beta_1$. Hence, either $m = n = 1$, or $m, n > 1$ and $\omega^{\alpha_2} + \cdots + \omega^{\alpha_m} = \omega^{\beta_2} + \cdots + \omega^{\beta_n}$. We conclude using the induction hypothesis. $\square$

As we shall see in the next section, there are ordinals such that $\omega^\alpha = \alpha$, and so, we cannot ensure that $\alpha_i < \alpha$. However, if $n > 1$, by right strict monotonicity of $+$, it is true that $\omega^{\alpha_i} < \alpha$, $1 \leq i \leq n$. We are now ready to define some normal functions that will lead us to the definition of $\Gamma_0$.

## 6.5 $\alpha$-Critical Ordinals

For each $\alpha \in \mathcal{O}$, we shall define a subset $Cr(\alpha) \subseteq \mathcal{O}$ and its ordering function $\varphi_\alpha$ inductively as follows.

**Definition 6.21** For each $\alpha \in \mathcal{O}$, the set $Cr(\alpha) \subseteq \mathcal{O}$ and its ordering function $\varphi_\alpha : A_\alpha \to Cr(\alpha)$ are defined inductively as follows.

(1) $Cr(0) =$ the set of additive principal ordinals, $A_0 = \mathcal{O}$, and for every $\alpha \in \mathcal{O}$, $\varphi_0(\alpha) = \omega^\alpha$, the ordering function of $Cr(0)$.

(2) $Cr(\alpha') = \{\eta \in A_\alpha \mid \varphi_\alpha(\eta) = \eta\}$, the set of fixed points of $\varphi_\alpha$, and $\varphi_{\alpha'} : A_{\alpha'} \to Cr(\alpha')$ is the ordering function of $Cr(\alpha')$.

(3) For every limit ordinal $\beta \in \mathcal{O}$,

$$Cr(\beta) = \{\eta \in \bigcap_{\alpha<\beta} A_\alpha \mid \forall\alpha < \beta,\ \varphi_\alpha(\eta) = \eta\},$$

and $\varphi_\beta : A_\beta \to Cr(\beta)$ is the ordering function of $Cr(\beta)$.

The elements of the set $Cr(\alpha)$ are called *$\alpha$-critical ordinals*. The following proposition shows that for $\alpha > 0$ the $\alpha$-critical ordinals are the common fixed points of the normal functions $\varphi_\beta$, for all $\beta < \alpha$.

**Proposition 6.22** For all $\alpha, \eta \in \mathcal{O}$, if $\alpha = 0$ then $\eta \in Cr(0)$ iff $\eta$ is additive principal, else $\eta \in Cr(\alpha)$ iff $\eta \in \bigcap_{\beta<\alpha} A_\beta$ and $\varphi_\beta(\eta) = \eta$ for all $\beta < \alpha$.

*Proof*. We proceed by transfinite induction. The case $\alpha = 0$ is clear since $Cr(0)$ is defined as the set of additive principal ordinals. If $\alpha$ is a successor ordinal, there is some $\beta$ such that $\alpha = \beta'$. By the induction hypothesis, $\eta \in Cr(\beta)$ iff $\eta \in \bigcap_{\gamma<\beta} A_\gamma$ and $\varphi_\gamma(\eta) = \eta$ for all $\gamma < \beta$. By the definition of $Cr(\beta')$, $\eta \in Cr(\beta') = Cr(\alpha)$ iff $\eta \in A_\beta$ and $\varphi_\beta(\eta) = \eta$. Hence, since $\alpha = \beta'$, $\eta \in Cr(\alpha)$ iff $\eta \in \bigcap_{\gamma<\alpha} A_\gamma$ and $\varphi_\gamma(\eta) = \eta$ for all $\gamma < \alpha$. If $\alpha$ is a limit ordinal, the property to be shown is clause (3) of definition 6.21. $\square$

The following important result holds.

**Proposition 6.23** Each set $Cr(\alpha)$ is closed and unbounded.

*Proof*. We show by transfinite induction that $Cr(\alpha)$ is closed and unbounded and that $A_\alpha = \mathcal{O}$.

*Proof of closure*. For $\alpha = 0$ this follows from the fact the the set of additive principal ordinals is closed. Assume $\alpha > 0$, and let $M \subseteq Cr(\alpha)$ be a nonempty countable subset of $Cr(\alpha)$. By the induction hypothesis, for every $\beta < \alpha$, $Cr(\beta)$ is closed and $A_\beta = \mathcal{O}$. Hence, by proposition 6.13, $\varphi_\beta$ is continuous. Hence, $\varphi_\beta(\bigsqcup M) = \bigsqcup M$ for all $\beta < \alpha$. By proposition 6.22, since we also have $A_\beta = \mathcal{O}$ for all $\beta < \alpha$, this implies that $\bigsqcup M \in Cr(\alpha)$. Hence, $Cr(\alpha)$ is closed.

*Proof of Unboundedness*. For $\alpha = 0$, this follows from the fact that the set of additive principal ordinals in unbounded and that $A_0 = \mathcal{O}$. Assume $\alpha > 0$. Given any ordinal $\beta$, let $\gamma_0 = \beta'$, $\gamma_{n+1} = \bigsqcup\{\varphi_\eta(\gamma_n) \mid \eta < \alpha\}$, $M = \{\gamma_n \mid n \in \mathbf{N}\}$, and $\gamma = \bigsqcup M$. By the induction hypothesis, for every $\delta < \alpha$, $Cr(\delta)$ is unbounded, and so $\gamma_n$ is well defined for all $n \geq 0$. We have $\beta < \gamma_0 \leq \gamma$. For every $\delta < \alpha$, we have $\varphi_\delta(\gamma_n) \leq \gamma_{n+1} \leq \gamma$, and so

$\bigsqcup\{\varphi_\delta(\gamma_n) \mid \gamma_n \in M\} \leq \gamma$. By the induction hypothesis, for every $\delta < \alpha$, $Cr(\delta)$ is closed and unbounded and $A_\delta = \mathcal{O}$. Hence, $\varphi_\delta$ is continuous and

$$\varphi_\delta(\bigsqcup M) = \bigsqcup\{\varphi_\delta(\gamma_n) \mid \gamma_n \in M\}.$$

Hence, $\varphi_\delta(\gamma) \leq \gamma$. By proposition 6.9, we also have $\gamma \leq \varphi_\delta(\gamma)$. Hence, $\gamma = \varphi_\delta(\gamma)$ for all $\delta < \alpha$. By proposition 6.22, we have $\gamma \in Cr(\alpha)$, and $\gamma$ is an $\alpha$-critical ordinal $> \beta$. Hence $Cr(\alpha)$ is unbounded, and so $A_\alpha = \mathcal{O}$. $\square$

Proposition 6.23 has the following corollary.

**Proposition 6.24** For every $\alpha \in \mathcal{O}$, $A_\alpha = \mathcal{O}$ and $\varphi_\alpha$ is a normal function.

In view of proposition 6.24, since every function $\varphi_\alpha$ has domain $\mathcal{O}$, we can define the function $\varphi : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ such that $\varphi(\alpha, \beta) = \varphi_\alpha(\beta)$ for all $\alpha, \beta \in \mathcal{O}$. From definition 6.21 and proposition 6.24, we have the following useful properties.

**Proposition 6.25** (1) $\eta \in Cr(\alpha')$ iff $\varphi(\alpha, \eta) = \eta$.

(2) For a limit ordinal $\beta$, $Cr(\beta) = \bigcap_{\alpha<\beta} Cr(\alpha)$.

**Proposition 6.26** (1) If $\alpha < \beta$ then $Cr(\beta) \subseteq Cr(\alpha)$.

(2) Every ordinal $\varphi(\alpha, \beta)$ is an additive principal ordinal.

(3) $\varphi(0, \beta) = \omega^\beta$.

An ordinal $\alpha$ such that $\alpha \in Cr(\alpha)$ is particularly interesting. Actually, it is by no means obvious that such ordinals exist, but they do, and $\Gamma_0$ is the smallest. We shall consider this property in more detail.

It is interesting to see what are the elements of $Cr(1)$. By the definition, an ordinal $\alpha$ is in $Cr(1)$ iff $\omega^\alpha = \alpha$. Such ordinals are called *epsilon ordinals*, because their ordering function is usually denoted by $\epsilon$. The least element of $Cr(1)$ is $\epsilon_0$. It can be shown that $\epsilon_0$ is the least upper bound of the set

$$\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\cdot^{\cdot^{\cdot^\omega}}}}, \ldots\}.$$

This is already a rather impressive ordinal. What are the elements of $Cr(2)$? Well, denoting the ordering function of $Cr(1)$ by $\epsilon$, $\alpha \in Cr(2)$ iff $\epsilon_\alpha = \alpha$. We claim that the smallest of these ordinals is greater than

$$\epsilon_0, \epsilon_1, \ldots, \epsilon_\omega, \ldots, \epsilon_{\epsilon_0}, \ldots, \epsilon_{\epsilon_1}, \ldots, \epsilon_{\epsilon_{\epsilon_0}}, \ldots$$

Amazingly, the ordinal $\Gamma_0$ *dwarfs* the ordinals just mentioned, and many more!

The following proposition gives a rather explicit characterization of $\varphi_{\alpha'}$ in terms of fixed points. It also shows that the first element of $Cr(\alpha')$ is farther down than the first element of $Cr(\alpha)$ on the ordinal line (in fact, much farther down).

**Proposition 6.27** For each $\alpha, \beta \in \mathcal{O}$, let $\varphi_\alpha^0(\beta) = \beta$, and $\varphi_\alpha^{n+1}(\beta) = \varphi_\alpha(\varphi_\alpha^n(\beta))$ for every $n \geq 0$. Then, we have

$$\begin{aligned}\varphi_{\alpha'}(0) &= \bigsqcup_{n\geq 0} \varphi_\alpha^n(0),\\ \varphi_{\alpha'}(\beta') &= \bigsqcup_{n\geq 0} \varphi_\alpha^n(\varphi_{\alpha'}(\beta)+1),\\ \varphi_{\alpha'}(\beta) &= \bigsqcup_{\gamma<\beta} \varphi_{\alpha'}(\gamma),\end{aligned}$$

for a limit ordinal $\beta$. Furthermore, $\varphi_\alpha(0) < \varphi_{\alpha'}(0)$ for all $\alpha \in \mathcal{O}$.

*Proof*. Since $\varphi_\alpha$ is a normal function, by proposition 6.15, $\bigsqcup_{n\geq 0} \varphi_\alpha^n(0)$ is the least fixed point of $\varphi_\alpha$, and for every $\beta \in \mathcal{O}$, $\bigsqcup_{n\geq 0} \varphi_\alpha^n(\varphi_{\alpha'}(\beta)+1)$ is the least fixed point of $\varphi_\alpha$ that is $> \varphi_{\alpha'}(\beta)$. Since $\varphi_{\alpha'}$ enumerates the fixed points of $\varphi_\alpha$, $\varphi_{\alpha'}(\beta') = \bigsqcup_{n\geq 0} \varphi_\alpha^n(\varphi_{\alpha'}(\beta)+1)$.

Assume that $\beta$ is a limit ordinal. From the proof of proposition 6.4, we know that $\beta = \bigsqcup\{\gamma \mid \gamma < \beta\}$. Since $\varphi_{\alpha'}$ is continuous, we have

$$\varphi_{\alpha'}(\beta) = \varphi_{\alpha'}(\bigsqcup\{\gamma \mid \gamma < \beta\}) = \bigsqcup_{\gamma<\beta} \varphi_{\alpha'}(\gamma).$$

Since $0 < \varphi_\alpha(0)$, it is easily shown that $\varphi_\alpha^n(0) < \varphi_\alpha^{n+1}(0)$ for all $n \geq 0$ (using induction and the fact that $\varphi_\alpha$ is strictly monotonic), and so, $\varphi_\alpha^n(0) < \varphi_{\alpha'}(0)$. Since $\varphi_\alpha^1(0) = \varphi_\alpha(0)$, the first element of $Cr(\alpha)$, we have $\varphi_\alpha(0) < \varphi_{\alpha'}(0)$. $\square$

Proposition 6.27 justifies the claim we made about $\epsilon_0$, and also shows that the first element of $Cr(2)$ is the least upper bound of the set

$$\{\epsilon_0, \epsilon_{\epsilon_0}, \epsilon_{\epsilon_{\epsilon_0}}, \ldots, \epsilon_{\epsilon_{\cdot_{\cdot_{\cdot_{\epsilon_0}}}}}, \ldots\}$$

It is hard to conceive what this limit is! Of course, things get worse when we look at the first element of $Cr(3)$, not to mention the notational difficulties involved. Can you imagine what the first element of $Cr(\epsilon_0)$ is? Well, $\Gamma_0$ is farther away on the ordinal line!

The following proposition characterizes the order relationship between $\varphi(\alpha_1, \beta_1)$ and $\varphi(\alpha_2, \beta_2)$.

**Proposition 6.28** (i) $\varphi(\alpha_1, \beta_1) = \varphi(\alpha_2, \beta_2)$ iff either

(1) $\alpha_1 < \alpha_2$ and $\beta_1 = \varphi(\alpha_2, \beta_2)$, or

(2) $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, or

(3) $\alpha_2 < \alpha_1$ and $\varphi(\alpha_1, \beta_1) = \beta_2$.

(ii) $\varphi(\alpha_1, \beta_1) < \varphi(\alpha_2, \beta_2)$ iff either

(1) $\alpha_1 < \alpha_2$ and $\beta_1 < \varphi(\alpha_2, \beta_2)$, or

(2) $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$, or

(3) $\alpha_2 < \alpha_1$ and $\varphi(\alpha_1, \beta_1) < \beta_2$.

*Proof (sketch).* We sketch the proof of (ii). By the definition of $\varphi$, $\varphi(\alpha_2, \beta_2) \in Cr(\alpha_2)$. If $\alpha_1 < \alpha_2$, by proposition 6.22, $\varphi(\alpha_2, \beta_2)$ is a fixed point of $\varphi_{\alpha_1}$, and so,

$$\varphi(\alpha_1, \varphi(\alpha_2, \beta_2)) = \varphi(\alpha_2, \beta_2).$$

Since $\varphi_{\alpha_1}$ is strictly monotonic, $\varphi(\alpha_1, \beta_1) < \varphi(\alpha_1, \varphi(\alpha_2, \beta_2))$ iff $\beta_1 < \varphi(\alpha_2, \beta_2)$. The case where $\alpha_2 < \alpha_1$ is similar. For $\alpha_1 = \alpha_2$, the assertion follows from the fact that $\varphi_{\alpha_1}$ is strictly monotonic. □

Using proposition 6.9, since each function $\varphi_\alpha$ is an ordering function, we have the following useful property.

**Proposition 6.29** For all $\alpha, \beta \in \mathcal{O}$, $\beta \leq \varphi(\alpha, \beta)$.

By proposition 6.28 and 6.29, we also have the following.

**Corollary 6.30** For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{O}$, if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, then $\varphi(\alpha_1, \beta_1) \leq \varphi(\alpha_2, \beta_2)$.

The following can be shown by transfinite induction.

**Proposition 6.31** (i) For every $\alpha \in \mathcal{O}$, $\alpha \leq \varphi(\alpha, 0)$. Furthermore, if $\beta \in Cr(\alpha)$, then $\alpha \leq \beta$.

(ii) If $\alpha \leq \beta$, then $\varphi(\alpha, \beta) \leq \varphi(\beta, \alpha)$.

*Proof.* We show $\alpha \leq \varphi(\alpha, 0)$ by transfinite induction. This is clear for $\alpha = 0$. If $\alpha > 0$, for every $\beta < \alpha$, by strict monotonicity and proposition 6.22, $\varphi(\beta, 0) < \varphi(\beta, \varphi(\alpha, 0)) = \varphi(\alpha, 0)$, since $\varphi(\alpha, 0) > 0$ is a fixed point of $\varphi_\beta$. By the induction hypothesis, we have $\beta \leq \varphi(\beta, 0)$, and so $\beta < \varphi(\alpha, 0)$ for all $\beta < \alpha$. By proposition 6.4, this implies that $\alpha \leq \varphi(\alpha, 0)$.

$\beta \in Cr(\alpha)$ iff $\beta = \varphi(\alpha, \eta)$ for some $\eta$, and since $\alpha \leq \varphi(\alpha, 0)$, by monotonicity, we have $\alpha \leq \varphi(\alpha, 0) \leq \varphi(\alpha, \eta) = \beta$.

Assume $\alpha \leq \beta$. Since $\beta \leq \varphi(\beta, 0)$, we also have $\beta \leq \varphi(\beta, \alpha)$. By proposition 6.28, $\varphi(\alpha, \beta) \leq \varphi(\beta, \alpha)$, since $\alpha \leq \beta$ and $\beta \leq \varphi(\beta, \alpha)$. □

Another key result is the following.

**Proposition 6.32** For every additive principal ordinal $\gamma$, there exist unique $\alpha, \beta \in \mathcal{O}$ such that, $\alpha \leq \gamma$, $\beta < \gamma$, and $\gamma = \varphi(\alpha, \beta)$.

*Proof*. Recall that an additive principal ordinal is not equal to 0. By proposition 6.31, $\gamma \leq \varphi(\gamma, 0)$. Since $0 < \gamma$, by strict monotonicity of $\varphi_\gamma$, $\varphi(\gamma, 0) < \varphi(\gamma, \gamma)$, and so $\gamma < \varphi(\gamma, \gamma)$. Since $\mathcal{O}$ is well-ordered, there is a least ordinal $\alpha \leq \gamma$ such that $\gamma < \varphi(\alpha, \gamma)$. If $\alpha \neq 0$, the minimality of $\alpha$ implies that $\varphi(\eta, \gamma) = \gamma$ for all $\eta < \alpha$, and by proposition 6.22, $\gamma \in Cr(\alpha)$. If $\alpha = 0$, since $\gamma$ is an additive principal ordinal, by the definition of $Cr(0)$, $\alpha \in Cr(0)$. Hence, $\gamma \in Cr(\alpha)$. Hence, there is some $\beta$ such that $\gamma = \varphi(\alpha, \beta)$. Since $\gamma < \varphi(\alpha, \gamma)$, by strict monotonicity of $\varphi_\alpha$, we must have $\beta < \gamma$.

It remains to prove the uniqueness of $\alpha$ and $\beta$. If $\beta_1 < \gamma$, $\beta_2 < \gamma$, and $\gamma = \varphi(\alpha_1, \beta_1) = \varphi(\alpha_2, \beta_2)$, by proposition 6.28, we must have $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. □

Observe that the proof does not show that $\alpha < \gamma$, and indeed, this is not necessarily true. Also, for an ordinal $\gamma$, $\gamma = \varphi(\gamma, \beta)$ holds for some $\beta$ iff $\gamma \in Cr(\gamma)$. Such ordinals exist in abundance, as we shall prove next.

**Definition 6.33** An ordinal $\alpha \in \mathcal{O}$ is a *strongly critical ordinal* iff $\alpha \in Cr(\alpha)$.

**Proposition 6.34** An ordinal $\alpha$ is strongly critical iff $\varphi(\alpha, 0) = \alpha$.

*Proof*. If $\alpha \in Cr(\alpha)$, there is some $\beta$ such that $\alpha = \varphi(\alpha, \beta)$. By proposition 6.31, we have $\alpha \leq \varphi(\alpha, 0)$, and by strict monotonicity of $\varphi_\alpha$, we have $\beta = 0$. Conversely, it is obvious that $\varphi(\alpha, 0) = \alpha$ implies $\alpha \in Cr(\alpha)$. □

Let $\psi : \mathcal{O} \to \mathcal{O}$ be the function defined such that $\psi(\alpha) = \varphi(\alpha, 0)$ for all $\alpha \in \mathcal{O}$. We shall prove that $\psi$ is strictly monotonic and continuous. As a consequence, $\psi$ is a normal function for the set $\{\varphi(\alpha, 0) \mid \alpha \in \mathcal{O}\}$.

**Proposition 6.35** The function $\psi$ (also denoted by $\varphi(-, 0)$) defined such that $\psi(\alpha) = \varphi(\alpha, 0)$ for all $\alpha \in \mathcal{O}$ is strictly monotonic and continuous.

*Proof*. First, we prove the following claim.

*Claim*: $\psi$ satisfies the following properties:

$$\begin{aligned}\psi(0) &= \varphi(0,0),\\ \psi(\beta') &= \bigsqcup_{n\geq 0} \varphi_\beta^n(\psi(\beta)),\\ \psi(\beta) &= \bigsqcup_{\delta<\beta} \psi(\delta),\end{aligned}$$

for a limit ordinal $\beta$.

*Proof of claim.* By definition, $\psi(0) = \varphi(0,0)$, and the second identity follows from proposition 6.15, since $\varphi_\beta^1(0) = \varphi(\beta, 0) = \psi(\beta)$, which implies that $\varphi_\beta^n(\psi(\beta)) = \varphi_\beta^{n+1}(0)$ for all $n \geq 0$. By proposition 6.22, $\psi(\beta) = \varphi(\beta, 0) = \eta_0$, where $\eta_0$ is the least ordinal such that $\varphi(\gamma, \eta) = \eta$ for all $\gamma < \beta$. For every $\gamma < \beta$, since $\varphi_\gamma$ is continuous,

$$\begin{aligned}\varphi(\gamma, \bigsqcup_{\delta<\beta} \psi(\delta)) &= \bigsqcup_{\delta<\beta} \varphi(\gamma, \psi(\delta))\\ &= \bigsqcup_{\delta<\beta} \varphi(\gamma, \varphi(\delta, 0)).\end{aligned}$$

For $\delta > \gamma$, we have $\varphi(\gamma, \varphi(\delta, 0)) = \varphi(\delta, 0) = \psi(\delta)$, and since $\varphi$ is monotonic in both arguments,

$$\bigsqcup_{\delta<\beta} \varphi(\gamma, \varphi(\delta, 0)) = \bigsqcup_{\delta<\beta} \psi(\delta).$$

Hence,

$$\varphi(\gamma, \bigsqcup_{\delta<\beta} \psi(\delta)) = \bigsqcup_{\delta<\beta} \psi(\delta),$$

for all $\gamma < \beta$, which shows that $\eta_0 \leq \bigsqcup_{\delta<\beta} \psi(\delta)$ (because $\eta_0$ is the least such common fixed point). On the other hand, $\psi(\delta) = \varphi(\delta, 0) \leq \varphi(\delta, \eta_0) = \eta_0$ for all $\delta < \beta$. Hence, $\bigsqcup_{\delta<\beta} \psi(\delta) \leq \eta_0$. But then, $\bigsqcup_{\delta<\beta} \psi(\delta) = \eta_0 = \psi(\beta)$. $\square$

We can now show that $\psi$ is continuous. Let $M$ be a nonempty countable subset of $\mathcal{O}$, and let $\beta = \bigsqcup M$. The case $\beta = 0$ is trivial. If $\beta = \alpha'$ for some $\alpha$, we must have $\beta \in M$, since otherwise $\beta$ would not be the least upper bound of $M$ (either $\gamma \leq \alpha$ for all $\gamma \in M$, or $\gamma > \alpha$ for some $\gamma \in M$, a contradiction in either case). But then, $\psi(\bigsqcup M) = \psi(\beta) = \bigsqcup_{\alpha\in M} \psi(\alpha)$, since $\psi$ is monotonic. If $\beta = \bigsqcup M$ is a limit ordinal, then $\beta = \bigsqcup M = \bigsqcup\{\delta \mid \delta < \beta\}$. Hence, for every $\alpha \in M$, there is some $\delta < \beta$ such that $\alpha < \delta$, and conversely, for every $\delta < \beta$, there is some $\alpha \in M$ such that $\delta < \alpha$. By monotonicity of $\psi$, this implies that

$$\bigsqcup_{\alpha\in M} \psi(\alpha) = \bigsqcup_{\delta<\beta} \psi(\delta).$$

By the claim,

$$\psi(\bigsqcup M) = \psi(\beta) = \bigsqcup_{\delta<\beta} \psi(\delta),$$

and therefore,

$$\psi(\bigsqcup M) = \bigsqcup_{\alpha\in M} \psi(\alpha),$$

showing that $\psi$ is continuous.

Finally, we show that $\psi$ is strictly monotonic. Since $\varphi$ is monotonic in both arguments, $\psi = \varphi(-,0)$ is monotonic. Assume $\alpha < \beta$. Then $\alpha < \alpha' \leq \beta$ and by proposition 6.27, $\psi(\alpha) < \psi(\alpha') \leq \psi(\beta)$. □

Proposition 6.35 implies that there are plenty of strongly critical ordinals.

**Proposition 6.36** The set of strongly critical ordinals is closed and unbounded.

*Proof*. First, we prove unboundedness. Since $\psi = \varphi(-,0)$ is a normal function, by proposition 6.22, for any arbitrary ordinal $\alpha$, $\psi$ has a least fixed point $> \alpha$. Since such fixed points are strongly critical ordinal, the set of strongly critical ordinals is unbounded.

Next, we prove that the set of strongly critical ordinals is closed. Let $M$ be a nonempty countable set of strongly critical ordinals. For each $\alpha \in M$, we have $\psi(\alpha, 0) = \alpha$. Hence, $\psi(M) = M$. Since $\psi = \varphi(-,0)$ is continuous, we have $\psi(\bigsqcup M) = \bigsqcup \psi(M) = \bigsqcup M$. This shows that $\bigsqcup M$ is a strongly critical ordinal, and therefore, the set of strongly critical ordinals is closed. □

From proposition 6.36, the ordering function of the set of strongly critical ordinals is a normal function. This function is denoted by $\Gamma$, and $\Gamma(0)$, also denoted $\Gamma_0$, is the least strongly critical ordinal. $\Gamma_0$ is the least ordinal such that $\varphi(\alpha, 0) = \alpha$. The following proposition shows that $\mathcal{O}(\Gamma_0)$ is closed under $+$ and $\varphi$.

**Proposition 6.37** For all $\alpha, \beta \in \mathcal{O}$, if $\alpha, \beta < \Gamma_0$, then $\alpha + \beta < \Gamma_0$, and $\varphi(\alpha, \beta) < \Gamma_0$.

*Proof (sketch)*. Since $\Gamma_0$ is an additive principal ordinal, closure under $+$ is clear. Let $\gamma_0 = 0$, $\gamma_{n+1} = \varphi(\gamma_n, 0)$, $U = \{\gamma_n \mid n \in \mathbf{N}\}$, and $\gamma = \bigsqcup U$. By proposition 6.15, we have $\gamma = \Gamma_0$. Now, if $\alpha, \beta < \Gamma_0$, since $\Gamma_0 = \bigsqcup U$, there is some $\gamma_n$ such that $\alpha, \beta < \gamma_n$. By proposition 6.28, we have $\varphi(\alpha, \beta) < \varphi(\gamma_n, 0)$, because $\beta < \gamma_n \leq \varphi(\gamma_n, 0)$. Hence, $\varphi(\alpha, \beta) < \gamma_{n+1} \leq \Gamma_0$. □

Proposition 6.37 shows that $\Gamma_0$ cannot be obtained from strictly smaller ordinals in terms of the function $+$ and the powerful functions $\varphi_\alpha$, $\alpha < \Gamma_0$. As Smoryński puts it in one of his articles [50],

> "$\Gamma_0$ is the first countable ordinal which cannot be described without reference (if only oblique) to the uncountable."

Indeed, referring to $\Gamma_0$ as the *least* ordinal $\alpha$ satisfying $\alpha = \varphi(\alpha, 0)$ is indirect and somewhat circular – the word "least" involves reference to all ordinals, including $\Gamma_0$. One could claim that the definition of $\Gamma_0$ as $\bigsqcup\{\gamma_n \mid n \in \mathbf{N}\}$, as in proposition 6.37, is "constructive", and does not refer to the uncountable, but this is erroneous, although the error is more subtle. Indeed, the construction of the function $\varphi(-, 0)$ is actually an iteration of the *functional* taking us from $\varphi(\alpha, -)$ to $\varphi(\alpha', -)$, and therefore, presupposes as domain of this functional a class of functions on ordinals and thus (on close examination) the uncountable. As logicians say, the definition of the ordinal $\Gamma_0$ is *impredicative*.