Famous Mistakes in Mathematics

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The Time Machine is still working

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  Extremal axioms: logical, mathematical and cognitive aspects.
- The project will be conducted (2016–2018) at the Department of Logic and Cognitive Science of the Adam Mickiewicz University in Poznań, Poland.

- Two modest scholarships will be offered in the years 2017–2018 for PhD students willing to participate in the project.
- For applications, check the announcements of the National Scientific Center by the end of 2016.

To warm up: Siberian proof of the FLT

From: www.sciteclibrary.ru/eng/catalog/pages/6253.html
FLT states that the equation $x^n + y^n = z^n$, where $z \neq 0$, $x < z$, $y < z$, $n \geqslant 3$ does not have solutions in natural numbers x, y, z. Let us divide both sides of the equation by z^n. We get: $(\frac{x}{z})^n + (\frac{y}{z})^n = 1$. Now, let us think (sic!) about events A and B with the following probabilities:
$P(A) = (\frac{x}{z})^n$ and $P(B) = (\frac{y}{z})^n$. Let also $P(A \cup B) = 1$. Then
$P((A \cup B)^c) = 0$ (where X^c denotes the event complementary to X). By De Morgan Laws we have:
$0 = P((A \cup B)^c) = P(A^c \cap B^c) = P(A^c) \cdot P(B^c) = (1 - (\frac{x}{z})^n) \cdot (1 - (\frac{y}{z})^n)$.
Hence either $x = z$ or $y = z$. The initial equation has thus only trivial solutions and therefore the FLT is proved.

The argument is discussed, with a few comments by Alexander Bogomolny on his page http://www.cut-the-knot.org/ctk/ErrDisc.shtml

Erro Ergo Disco

- Most mathematical mistakes remain hidden, drowned in oblivion. Currently accepted style of publications follows Gauss rather than Euler (the context of discovery is hidden).
- Only these mistakes remain remembered which became famous. Most valuable among them are such mistakes which gave rise to new mathematical problems, theorems, theories.

- Euclid's *Pseudaria*. A lost text, mentioned by Proclus.
- De Morgan's *A Budget of Paradoxes*.
- Lecat's list (1935) of about 500 mistakes made by 300 famous mathematicians.
- Internet resources concerning mathematical mistakes.

Nothing as it seems?

- The term *cognitive bias* refers (in cognitive psychology) to a systematic pattern of deviation from norm or rationality in judgment.
- Does it apply to mathematical thinking?

- Old dilemma: mathematical discovery or mathematical creation?
  - What is a mistake in mathematical discovery?
  - What is a mistake in mathematical creation?
- Is it possible to have mutually incompatible (or even mutually contradictory) mathematical intuitions?
  - Newton and Leibniz
  - Axiom of Choice and Axiom of Determinacy

Problems, ideas, concepts, methods, language,. . .

Terminology:

- fallacy, lapsus, error, mistake, flaw,
- contradiction,
- ambiguity,
- incomplete proof, non sequitur,
- wrong assumption,
- false analogy, hasty generalization,
- false conjecture,
- correct results formerly rejected by mathematicians
- sophisms (mistakes made on purpose).

Beyond our interest today: paradox, fraud, puzzle, conundrum, brain-teaser, howlers.

Impossibility of classification

Types of mistakes:

- Material and formal mistakes
- Incomplete proofs
- False analogy
- Disputable: false conjectures

Sources of mistakes:

- Incompetence or oversight
- Wrong suggestions (e.g. drawings, induction)
- Lack of solid logical foundations
- Great complexity of a problem

Euclid and his followers

- *Proclus* (410–485) formulated commentaries concerning false proofs (of the Vth postulate) and then gave his own proof, false as well.
- *Alhazen* (965–1040) introduced *Lambert quadrilateral* (a quadrilateral three of whose angles are right angles) and used concepts related to motion in geometry trying to prove the postulate by reductio ad absurdum. He also obtained some theorems of hyperbolic and elliptic geometry.
- *Omar Chajjiam* (1048–1131) was the first who did not commit petitio principii but postulated to infer the postulate from a more intuitive one. He did not accept motion in geometry. He introduced *Saccheri quadrilateral* (a quadrilateral with two equal sides perpendicular to the base) and recognized three possibilities obtained by omitting the fifth postulate.

- *Nasir ad-Din Tusi* (1201–1274) tried to prove the postulate by an apagogic method. He mentioned hyperbolic and elliptic geometries but he rejected them.
- *Giordano Vitale* (1633–1711) has observed that the line whose all points are in equal distance from a given straight line must be a straight line.
- *Girolamo Saccheri* (1667–1733) tried a reductio ad absurdum proof. Since the fifth postulate is equivalent to the statement that the sum of angles of a triangle equals π, he considered two cases: when the sum is smaller than π and when it is bigger than π. As a consequence of the first case, straight lines would be limited, which contradicted Euclid (but notice that this is exactly the case in elliptic geometry). In the second case Saccheri has obtained several consequences which he considered counterintuitive. One of such consequences is the existence of a triangle with maximum area (and this is exactly the case in hyperbolic geometry).

- *Johann Heinrich Lambert* (1728–1777) made use of the Lambert quadrilateral and eliminated the possibility that the fourth angle in it is obtuse. He proved several theorems under the assumption that this angle is acute – among others that the sum of angles of a triangle is growing when the area of the triangle is diminishing. He did not think that these results contradict intuitions, he even speculated about possible models.
- *Carl Friedrich Gauss* (1777–1855) has considered a system of geometry with the negation of the fifth postulate, but he did not publish his considerations.
- *Nicolai Lobachevsky* (1792–1856) has published in 1829 a work in which the fourth angle in the Lambert quadrilateral is acute.
- *János Bolyai* (1802–1860) independently of Lobachevsky has published in 1831 a paper concerning the same system of geometry.
- Lobachevsky, Bernhard Riemann (1826–1866) and Henri Poincaré (1854–1912) have developed elliptic and hyperbolic geometry.
- *Eugenio Beltrami* (1835-1899) has proved in 1868 the independence of the Vth postulate from the other postulates of the Euclidean system.

Lessons from history

- Euclid's Vth postulate
- Galileo: brachistochrone
- Origins of calculus, infinitesimals, infinite series

- Radicals: naive intuitions abolished
- Fermat's Last Theorem: several failures

- Quadrature of the circle
- Early rejection of negative and imaginary numbers

Infinitesimals: the struggle continues

An example of a recent quarrel:

- Schubring, G. 2005. *Conflicts between Generalization, Rigor, and Intuition. Number Concepts Underlying the Development of Analysis in 17–19th Century France and Germany*. Springer Verlag, New York.
- Błaszczyk, P., Katz, M.G., Sherry, D. 2013. Ten misconceptions from the history of analysis and their debunking. *Foundations of Science* **18**, no. 1, 43–74.
- Schubring, G. 2015. Comments on a Paper on Alleged Misconceptions Regarding the History of Analysis. *Foundations of Science*. DOI 10.1007/s10699-015-9424-0.
- Błaszczyk, P., Kanovei, V., Katz, M.G., Sherry, D. 2015. Comments on Schubring's *Conflicts*. To appear in *Foundations of Science*.

New mathematics, new mistakes

- Henri Lebesgue: projections of Borel sets
- Italian school of algebraic geometry
- Algebra: uniqueness of factorization (Lamé)

- Henri Poincaré: three bodies problem
- Malfatti circles
- Perko pair
- Four colors theorem

- Number theory: several (false) conjectures
- Mertens conjecture
- Borsuk's conjecture
- Nikolai Chebotaryov: factorization of $x^n - 1$

Mathematical cranks

- Underwood Dudley: *Mathematical cranks*, *The Trisectors*
- Siberian "proof" of the FLT
- Trisectors, Fermatists, Goldbachers,. . .

- Wolfgang Mückenheim: fight with actual infinity
- Louis de Branges: a proof of the Riemann Hypothesis

- Numerology
- Do you have more examples?

Frege was not alone

- Gottlob Frege: unrestricted comprehension axiom
- Lewis Carroll: heuristic rules involving resolution

- Ernst Zermelo and Skolem's Paradox: a flawed proof
- Ernst Zermelo and Kurt Gödel: a misinterpretation

- Rudolf Carnap: *Gabelbarkeitssatz*
- Abraham Fraenkel: axiom of restriction

Mathematical Agnosticism

- Claim: beliefs about the status of mathematics (creation or discovery) do not influence mathematical practice of working mathematicians.
- Question: to what extent are cognitive metaphors responsible for the intuitions of professional mathematicians?

- Mathematical thinking and thinking in natural language. Is it possible to investigate mathematical thinking experimentally?
- Are mathematical abilities irrelevant from the evolutionary point of view?

What next?

- Is it possible to recover mathematical intuitions by an analysis of the source mathematical texts?
- Danger of misinterpretation: one should not impose the modern understanding of mathematical concepts on their earlier understanding, which is to be recovered.

- How to organize didactic experiments which could improve mathematical intuitions? Is it possible to correct wrong mathematical intuitions possessed by adults?
- Mathematical ignorance as a social disease. What causes math anxiety? Creative kids versus rigid school rules? How to cure mathematical inabilities?
- Pitfalls of intuition. Prejudices, illusions, paradoxes. Examples of fruitful sophisms.

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Thank you for your attention