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Introduction

In this paper an increasing sequence $\mathcal{E}^0, \mathcal{E}^1, \ldots$ of the classes of recursive functions is examined. Each class $\mathcal{E}^n$ is closed under the operations of substitution and under the operation of limited recursion. The initial functions are primitive recursive ones. Therefore $\mathcal{E}^n \subset \mathcal{R}$ where $\mathcal{R} =$ the class of primitive recursive functions. Strictly speaking $\mathcal{R} = \sum_n \mathcal{E}^n$. Hence in the definiton of the class $\mathcal{R}$ the operation of recursion cannot be eliminated or exchanged into the operation of limited recursion.

The classes $\mathcal{E}^0$ and $\mathcal{E}^3$ shall be examined in particular. For each function $f \in \mathcal{E}^0$ there exists a number k_0 such that $f(n) < n + k_0$. However, each recursive enumerable set is enumerable by some function of the class $\mathcal{E}^0$. We start with the investigation of the class $\mathcal{E}^3$. It is the class of elementary computable functions of Kalmar.

§ 1. Preliminary notions

1. Pairing functions. We shall call the *pairing functions* every triplet of functions P, Q, R, defined over non-negative integers which satisfies the following conditions:

$$P(Qz, Rz) = z, \tag{1}$$

$$Q(P(x,y)) = x, \tag{2}$$

$$R(P(x,y)) = y. \tag{3}$$

The functions which satisfy these three formulae establish a one-one correspondence between the set of pairs of non-negative integers and the whole set of non-negative integers. The functions Q and R are inverse to the functions P. The first element of the pair represented by z is the value of the function Qz, and the second element is the value of the function Rz.

For instance, the following three functions are pairing functions with the above mentioned properties:

$$P(x,y) = 2^x \cdot (2y+1) - 1,$$

$Qz =$ *the largest integer* u *such that* $z+1$ *is divisible by* 2^u

$$Rz = \frac{\dfrac{z+1}{2^{Qz}} - 1}{2}.$$

Another example of the pairing functions present the functions I, K, L:

$$I(x,y) = (x+y)^2 + x,$$
$$Kz = z - [\sqrt{z}]^2,$$
$$Lz = [\sqrt{z}] - Kz,$$

where $[\sqrt{z}]$ is the integral part of the root.

We have:

$$I(Kz, Lz) = z, \qquad K(I(x,y)) = x, \qquad L(I(x,y)) = y.$$

The pairing functions permit us to form functions of triplets, *e. g.*

$$T(x,y,z) = P(x, P(y,z)) \quad \text{or} \quad S(x,y,z) = I(x, I(y,z))$$

which establish a correspondence between the set of triplets and the set of numbers. The functions inverse to T are:

$$T_1 u = Qu; \qquad T_2 u = QRu; \qquad T_3 u = RRu.$$

Analogically, all finite complexes of numbers can, of course, be represented by natural numbers.

The letters P, Q, R and I, K, L will subsequently denote any triplet of pairing functions.

2. Universal functions. All functions considered in the following are defined over the set of non-negative integers and assume only the integral values. We shall use the names "integer" and "number" only in the sense of "non-negative integer". The capital letters $\mathcal{X}, \mathcal{Y}, \mathcal{R}, \mathcal{E}^n$ will denote classes of functions. If $\mathcal{X}$ is a class of functions, then $\mathcal{X}_n$ is the class of functions of n arguments which belong to the class $\mathcal{X}$.

The class $\mathcal{X}$ is *closed under the operations of substitutions* if $\mathcal{X}$ is closed under the following three operations:

1. The operation of the substitution (superposition) of functions. If $\mathcal{X}$ includes the functions f and g symbolised by the expressions

$$f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n), \qquad g(y_1, \ldots, y_m),$$

then it includes the function called the *substitution of the function* g in the function f for the k-th variable, and symbolised thus:

$$f(x_1, \ldots, x_{k-1}, g(y_1, \ldots, y_m), x_{k+1}, \ldots, x_n).$$

2. The operation of the identification of variables. If the class $\mathcal{X}$ includes the function $f(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_m)$ then it also includes the function which is obtained by the identification of the variables x_j and x_k and their replacing by the variable y, and is symbolised by $f(x_1, \ldots, y, \ldots, y, \ldots, x_m)$. The variable y differs from all variables x_i.

3. The operation of the substitution of a constant. If the class $\mathfrak{X}$ includes the function $f(x_1,\ldots,x_k,\ldots,x_m)$, then it also includes the function $f(x_1,\ldots,0,\ldots,x_m)$, obtained from the function f through the substitution of zero for the k-th variable.

For instance the class $\mathscr{A}$ of linear functions, *i. e.* the class of the functions f of the form:

$$f(x_1,\ldots,x_n)=k_1x_1+k_2x_2+\ldots+k_nx_n+k_0\,, \qquad \text{where} \quad k_1,\ldots,k_n>0,$$

is the smallest class containing the initial functions $x+1$, $x+y$ and closed under the operations of substitution.

The function $F(x_1,\ldots,x_n,t)$ of $n+1$ arguments is a *universal function* for the class $\mathfrak{X}_n$ provided that, for each function $f(x_1,\ldots,x_n)$ of n-arguments, the function f belongs to $\mathfrak{X}_n$ if and only if there exists a number t such that, for each $x_1,\ldots,x_n$, $f(x_1,\ldots,x_n)=$ $=F(x_1,\ldots,x_n,t)$.

E.g. the function $F(x,t)=(Qt+1)x+Rt$ is universal for the class $\mathscr{A}_1$, $F(x_1,x_2,t)=(T_1t+1)x_1+(T_2t+1)x_2+T_3t$ is universal for the class $\mathscr{A}_2$.

It is evident that, if a class $\mathfrak{X}$ contains pairing functions and is closed under the operations of substitution, then the universal function $F_n(x_1,\ldots,x_n,t)$ for the class $\mathfrak{X}_n$ can be defined by means of the universal function $F(x,t)$ for the class $\mathfrak{X}_1$. Namely we can set

$$F_2(x_1,x_2,t)=F\big(I(x_1,x_2),t\big).$$

Generally

$$F_n(x_1,\ldots,x_n,t)=F\Big(I\big(x_1,I(x_2,\ldots,I(x_{n-1},x_n)\ldots)\big),t\Big).$$

If a class $\mathfrak{X}$ is closed under the operations of substitution and contains the function $x+1$, then the universal function for the class $\mathfrak{X}_n$ does not belong to the class $\mathfrak{X}$. If $F(x,t)$ is universal for the class $\mathfrak{X}_1$, when F belongs to the class $\mathfrak{X}$, then the function $F(x,x)+1$ belongs to $\mathfrak{X}_1$. Hence for some t: $F(x,x)+1=F(x,t)$. This leads to a contradiction: $F(t,t)+1=F(t,t)$.

3. The relations of a given class. We shall identify the sets with the relations of one argument. The notion of relation will be related to the notion of the function of a given class. We say that the relation of n elements $R(x_1,\ldots,x_n)$ is a *relation of the class* $\mathfrak{X}$ if and only if there exists such a function f of the class $\mathfrak{X}$ that the following equivalence is true:

$$R(x_1,\ldots,x_n)\equiv f(x_1,\ldots,x_n)=0 \qquad \text{for all} \quad x_1,\ldots,x_n.$$

Let $x\dot{-}y$ denote the subtraction defined over non-negative integers, $x\dot{-}y=x-y$ if $x\geqslant y$ and $x\dot{-}y=0$ if $x<y$.

Theorem 1.1. *If $\mathfrak{X}$ is a class of functions closed under the operations of substitution and $\mathfrak{X}$ includes the functions $x+1$, $x+y$, $x\dot{-}y$, then the set of the relations of the class $\mathfrak{X}$ is closed under the operations of the propositional calculus.*

Proof. If $R(x_1,\ldots,x_n)$ and $S(y_1,\ldots,y_m)$ are relations of the class $\mathfrak{X}$, then there exist functions f and g which satisfy the equivalences

$$R(x_1,\ldots,x_n)\equiv f(x_1,\ldots,x_n)=0, \qquad S(y_1,\ldots,y_m)\equiv g(y_1,\ldots,y_m)=0.$$

In conformity with the following obvious laws of the arithmetic of non-negative integers:

$$a=0\,.\,b=0\,.\!\equiv.\,a+b=0, \qquad \sim(a=0)\equiv 1\dot{-}a=0,$$

we can deduce the equivalences

$$\sim R(x_1,\ldots,x_n)\,.\!\equiv.\,1\dot{-}f(x_1,\ldots,x_n)=0,$$
$$R(x_1,\ldots,x_n)\,.\,S(y_1,\ldots,y_m)\,.\!\equiv.\big(f(x_1,\ldots,x_n)+g(y_1,\ldots,y_m)\big)=0.$$

These equivalences show that if the relations R and S belong to the class $\mathfrak{X}$, then the relation which is the complement of the relation R, and the relation which is the logical product of the relations R and S, belong to the class $\mathfrak{X}$. Since all other finite logical operations (the operations of the propositional calculus) can be defined by means of conjunction (logical product) and negation (complement), therefore finite logical operations do not lead outside the class $\mathfrak{X}$.

Corollary 1.2. *Instead of the functions $x+1$, $x+y$, $x\dot{-}y$, the function x^y can also be used for the purpose of defining the logical operations of propositional calculus on relations.*

Proof. We have

$$\sim(a=0)\equiv 0^a=0, \qquad a=0\,.\,b=0\,.\!\equiv.\,a^{0^b}=0.$$

Hence

$$\sim R(x_1,\ldots,x_n)\,.\!\equiv.\,0^{f(x_1,\ldots,x_n)}=0,$$
$$R(x_1,\ldots,x_n)\,.\,S(y_1,\ldots,y_m)\,.\!\equiv.\big(f(x_1,\ldots,x_n)\big)^{0^{g(y_1,\ldots,y_m)}}=0.$$

Let $\mathcal{P}$ be the class of polynomials defined as follows: $\mathcal{P}$ is the smallest class which includes the initial functions $x+1$, $x+y$, $x\dot{-}y$, $x\cdot y$ and is closed under the operations of substitution.

For instance, the relations $<$, $\leqslant$, $=$, $\neq$, are of the class $\mathcal{P}$ since we have the equivalences:

$$x<y\,.\equiv.\big((x+1)\dot{-}y\big)=0,$$
$$x\leqslant y\,.\equiv.\,x\dot{-}y=0,$$
$$x=y\,.\equiv.\,x\leqslant y\,.\,y\leqslant x\,.\equiv.\,(x\dot{-}y)+(y\dot{-}x)=0,$$
$$x\neq y\,.\equiv.\sim(x=y)\,.\equiv.\,1\dot{-}\big((x\dot{-}y)+(y\dot{-}x)\big)=0\,.$$

§ 2. Elementary functions

Definition. The notion of elementary functions has been introduced by Kalmar[1]). This class is much wider than the classes discussed in the preceding section. In most cases, the elementary functions are sufficient for practical applications. According to the definition of Kalmar, slightly simplified, the *class of elementary functions* $\mathcal{E}$ is the smallest class

1° including as initial functions $x+1$, $x+y$, $x\dot{-}y$,

2° closed under the following operations:

the operations of substitution,

the operations of limited summation and limited multiplication, which state that if $F(x_1,\dots,x_n,t)$ belongs to the class $\mathcal{E}$, then this class includes the functions f and g, which are defined as follows:

$$f(x_1,\dots,x_n,y)=\sum_{i\leqslant y}F(x_1,\dots,x_n,i),$$
$$g(x_1,\dots,x_n,y)=\prod_{i\leqslant y}F(x_1,\dots,x_n,i).$$

Here the operations $\sum\limits_{i\leqslant y}$ and $\prod\limits_{i\leqslant y}$ symbolise finite sums and products[2]):

$$\sum_{i\leqslant y}F(\mathfrak{u},i)=F(\mathfrak{u},0)+F(\mathfrak{u},1)+\dots+F(\mathfrak{u},y),$$
$$\prod_{i\leqslant y}F(\mathfrak{u},i)=F(\mathfrak{u},0)\cdot F(\mathfrak{u},1)\cdot\ldots\cdot F(\mathfrak{u},y).$$

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[1]) Cf. Peter [4], p. 60.

[2]) To simplify the notation we use German letters to denote the complexes of variables, *e. g.* $\mathfrak{u}$ is an abbreviation of $u_1,\dots,u_\lambda$.

E. g., $x\cdot y$, x^y, $x!$ are elementary functions:

$$x\cdot y=\Big(\sum_{i\leqslant y}(x+i)\dot{-}i\Big)\dot{-}x=\big((x+0)\dot{-}0+(x+1)\dot{-}1+\dots+(x+y)\dot{-}y\big)\dot{-}x.$$

Set

$$f(x,y)=x^{y+1}=\prod_{i\leqslant y}(x+i)\dot{-}i\,,$$
$$x^y=\big(1\dot{-}(1\dot{-}y)\big)\cdot f(x,y\dot{-}1)+(1\dot{-}y).$$

Set

$$g(x)=\prod_{i\leqslant x}(i+1),\qquad x!=g(x\dot{-}1).$$

Other elementary functions will be defined by means of certain operations which can in turn be defined by means of the operations specified in the definition of the class $\mathcal{E}$.

The operation of limited minimum. This operation leads from a function $F(\mathfrak{u},x)$ of $n+1$ arguments to a function $f(\mathfrak{u},y)=\mu x\leqslant y[F(\mathfrak{u},x)=0]$ of $n+1$ arguments, defined as follows:

$$\mu x\leqslant y[F(\mathfrak{u},x)=0]=\begin{cases}\text{the smallest } x\leqslant y \text{ such that } F(\mathfrak{u},x)=0, \text{ when such an } x \text{ exists,}\\ 0 \text{ when there is no } x\leqslant y \text{ for which } F(\mathfrak{u},x)=0.\end{cases}$$

Theorem 2.1. *The class $\mathcal{E}$ is closed under the operation of limited minimum.*

Proof. If $F(\mathfrak{u},y)$ belongs to the class $\mathcal{E}$, and if the function g satisfies the condition

$$g(\mathfrak{u},x)=\mu y\leqslant x[F(\mathfrak{u},y)=0]$$

then the function g also belongs to the class $\mathcal{E}$. Namely the function g can be defined by means of the operation of limited summation in the following way: we first define the auxiliary function f,

(i) $$f(\mathfrak{u},y)=\sum_{i\leqslant y}\big(1\dot{-}F(\mathfrak{u},i)\big)$$

and then we set

(ii) $$g(\mathfrak{u},x)=\Big(\sum_{y\leqslant x}\big(1\dot{-}f(\mathfrak{u},y)\big)\Big)\cdot\Big(1\dot{-}\Big(\Big(\sum_{y\leqslant x}\big(1\dot{-}f(\mathfrak{u},y)\big)\Big)\dot{-}x\Big)\Big).$$

Let us notice in the first place that $f(\mathfrak{u},y)=0$ if and only if for every $i\leqslant y$ we have $F(\mathfrak{u},i)\neq 0$. The conditions $f(\mathfrak{u},y)=0$ and $1\dot{-}f(\mathfrak{u},y)=1$ are equivalent. Hence

$$\sum_{y\leqslant x}\big(1\dot{-}f(\mathfrak{u},y)\big)$$

is the sum of as many units as there are numbers $y=0,1,2,\ldots,x$ for which it is true that if $i\leqslant y$ we have $F(\mathfrak{u},i)\neq 0$. If among the numbers $y=0,1,2,\ldots,x$ there exists such smallest number $a\leqslant x$ that $F(\mathfrak{u},a)=0$, then, of course, it is true only regarding the numbers $y<a$ that for every $i\leqslant y$ we have $F(\mathfrak{u},i)\neq 0$. Consequently, it is true only for $y<a$ that $1\dot{-}f(\mathfrak{u},y)=1$, hence

$$\sum_{y\leqslant x}\big(1\dot{-}f(\mathfrak{u},y)\big)=a,$$

because this sum is the sum of as many units as there are numbers smaller than a. Since $a\leqslant x$, this sum is smaller than $x+1$. Hence the second factor of the definition of the function g is in such a case equal to one, and $g(\mathfrak{u},x)=a$. If such a number $a\leqslant x$ that $F(\mathfrak{u},a)=0$ does not exist, then regarding every $y\leqslant x$ it is true that for every $i\leqslant y$, $F(\mathfrak{u},i)\neq 0$, *i. e.* for every $y\leqslant x$, $1\dot{-}f(\mathfrak{u},y)=1$, and so

$$\sum_{y\leqslant x}\big(1\dot{-}f(\mathfrak{u},y)\big)=x+1,$$

being the sum of $x+1$ units. The second factor of the definition of the function g is in such a case equal to zero, and $g(\mathfrak{u},x)=0$, which is also in agreement with the meaning of the operation of limited minimum, as described above.

Notice that in (i) and (ii) the operation of minimum was defined by means of the operation which leads from a function $F(\mathfrak{u},x)$ to a function $f(\mathfrak{u},y)$ defined as follows:

$$f(\mathfrak{u},y)=\sum_{i\leqslant y}1\dot{-}F(\mathfrak{u},i).$$

Let us call this operation *the narrowed operation of limited summation*. This observation can be expressed in the following form of a general conclusion, which will be used subsequently:

Corollary 2.1a. *If the class $\mathfrak{X}$ is closed under the operations of substitution and of narrowed limited summation, and contains the function: $x\dot{-}y$, and $\sigma(x,y)=x\cdot(1\dot{-}y)$, then $\mathfrak{X}$ is closed under the operation of limited minimum.*

Elementary relations. The relations of the class $\mathcal{E}$ will be called *elementary relations*. We say that the *relation $T(u,y)$ is defined by means of the relation R, by the operation of the limited existential quantifier* provided that the following equivalence is true:

$$T(\mathfrak{u},y).\equiv.\textstyle\sum_{x\leqslant y}R(\mathfrak{u},x).\equiv.\sum_x\{x\leqslant y\,.\,R(\mathfrak{u},x)\}.$$

The relation $S(\mathfrak{u},y)$ is defined by means of the operation of the limited general quantifier if the equivalence

$$S(\mathfrak{u},y).\equiv.\textstyle\prod_{x\leqslant y}R(\mathfrak{u},x).\equiv.\prod_x\{x\leqslant y.\rightarrow.R(\mathfrak{u},x)\}$$

is true.

Theorem 2.2 a. *The class of elementary relations is closed under the logical operations of limited quantifiers.*

Proof. If R is an elementary relation, *i. e.* if there exists such an elementary function f that

$$R(\mathfrak{u},x)\equiv f(\mathfrak{u},x)=0$$

and if the relations T and S are defined as above, then there exist such elementary functions g and h that

(α) $$T(\mathfrak{u},y)\equiv h(\mathfrak{u},y)=0,$$

(β) $$S(\mathfrak{u},y)\equiv g(\mathfrak{u},y)=0.$$

Such functions can be defined directly by means of the operations of summation and multiplication:

$$h(\mathfrak{u},y)=\prod_{x\leqslant y}f(\mathfrak{u},x),\qquad g(\mathfrak{u},y)=\sum_{x\leqslant y}f(\mathfrak{u},x).$$

As can easily be proved, these two functions satisfy the equivalences (α) and (β).

We can generally prove

Theorem 2.2 b. *If the class $\mathfrak{X}$ is closed under the operations of limited minimum and substitutions, then the class of relations of the class $\mathfrak{X}$ is closed under the operation of limited existential quantifier.*

Proof. If $f\in\mathfrak{X}$ and $f(\mathfrak{u},x)=0\equiv R(\mathfrak{u},x)$, then the function

$$h(\mathfrak{u},y)=f\big(\mathfrak{u},\mu x\leqslant y\,[f(\mathfrak{u},x)=0]\big)$$

belongs to the class $\mathfrak{X}$.

It is easy to show that

$$h(\mathfrak{u},y)=0.\equiv.\textstyle\sum_{x\leqslant y}f(\mathfrak{u},x)=0.$$

The class $\mathfrak{X}$ is also closed under the operation of limited existential quantifier.

Similarly the class $\mathcal{X}$ is closed under the logical operations of the propositional calculus, then the class $\mathcal{X}$ is closed under the operations of limited universal quantifiers. This follows from the well-known logical laws of de Morgan:

$$\prod_{x\leqslant y} R(\mathfrak{u},x) . \equiv . \sim\left(\sum_{x\leqslant y} \sim R(\mathfrak{u},x)\right).$$

It follows from Theorem 1.1 that the class of elementary relations is closed under the operations of the propositional calculus. Thus the relations: "smaller than", "equal to", "divisible by", and the class of prime numbers, are elementary ones, since they can be defined as follows:

$$x\leqslant y \equiv x \dot{-} y = 0, \qquad x=y . \equiv . x\leqslant y . y\leqslant x,$$

$$x<y \equiv x\leqslant y . \sim(x=y), \qquad x|y . \equiv . \sum_{z\leqslant y} x\cdot z = y,$$

$$\text{prime}\,(x) \equiv \{\prod_{y\leqslant x} y|x : \rightarrow : y=1 . \vee . y=x\} . x\geqslant 2.$$

Hence for instance, the following functions are elementary ones:

$$\left[\frac{x}{y}\right] = \mu z\leqslant x\,[x+1\leqslant y\cdot(z+1)]$$

(the integer part of the fraction x/y);

$$r(x,y) = x \dot{-} \left(y\cdot\left[\frac{x}{y}\right]\right)$$

(the remainder of the division of x by y);

$$x^{+} = x^{x}; \qquad x^{++} = x^{x^{+}}$$

(auxiliary functions);

$$[\sqrt[y]{x}] = \mu z\leqslant x [z^{y}\leqslant x . \{\prod_{t\leqslant x} t^{y}\leqslant x . \rightarrow . t\leqslant z\}]$$

(the integer part of a root).

The operations of maximum. In the sequel we shall use the following two operations of maximum:

$$(\max x\leqslant z)[R(\mathfrak{u},x)] = \mu x\leqslant z [R(\mathfrak{u},x) . \{\prod_{t\leqslant z} R(\mathfrak{u},t) . \rightarrow . t\leqslant x\}]$$

(the largest $x\leqslant z$ for which the relation $R(\mathfrak{u},x)$ holds), and

$$\operatorname{Max} F(\mathfrak{u},x) \text{ for } x\leqslant z = F(\mathfrak{u}, \mu x\leqslant z[\prod_{t\leqslant z} F(\mathfrak{u},t)\leqslant F(\mathfrak{u},x)])$$

(the maximum value of the function $F(\mathfrak{u},x)$ for $x\leqslant z$).

As follows from the definitions, these operations do not lead outside the class $\mathcal{E}$.

Prime numbers. Prime numbers will play an important role in our further considerations. The sequence of prime numbers will be defined in an elementary way. The following functions and relations are elementary ones:

$$\exp(y,k) = (\max x\leqslant y)[k^{x}|y]$$

(the largest exponent x for which y is still divisible by k^{x});

$$xNy . \equiv : \text{prime}(x) . \text{prime}(y) . x>y . \prod_{z\leqslant x} \{y<z . \text{prime}(z) . \rightarrow z=x\}$$

(x is the next prime after y);

$$x \operatorname{Pr} k . \equiv . \sum_{v\leqslant(k+2)^{++}} \{\exp(v,2)=2 . \text{prime}(x) . \exp(v,x)$$

(i) $$= k+2 . \prod_{t\leqslant(k+2)^{+}} \prod_{z>t} [\text{prime}(z) . \text{prime}(t) . z|v . t|v .$$

$$\rightarrow . \sum_{w\leqslant t} \{w|v . wNz . \exp(v,w)=\exp(v,z)+1\}]\}$$

(x is the k-th prime number);

(ii) $$p_{k} = \mu x\leqslant (k+2)^{+}[x \operatorname{Pr} k]$$

(the k-th prime number).

By the last definition: $2=p_0$, $3=p_1$, etc. The number v, which appears in the definition (i) is a number which has the form

$$v = 2^{2}\cdot 3^{3}\cdot 5^{4}\cdot\ldots\cdot p_{k}^{k+2}.$$

In conformity with the definition, this number satisfies the condition that in its decomposition into prime factors every successive prime number has the exponent which is the successor of the exponent of the preceding prime number. Since in conformity with the theory of numbers the inequality

$$p_{k}\leqslant (k+2)^{k+2} = (k+2)^{+}$$

is true, then (ii) is an adequate definition of the sequences p_k.

Representation of finite sequences by numbers. The functions defined above make it possible to represent, in many reasonings, a finite sequence of numbers by one number. This method is often applied in the theory of computable functions. The finite sequence of numbers: $m_0, \ldots, m_k$ is represented by the number $m = 2^{m_0} \cdot 3^{m_1} \cdot \ldots \cdot p_k^{m_k}$. The numbers $m_0, \ldots, m_k$ are the exponents of powers in which the corresponding prime numbers appear in the decomposition of the number m: $m_i = \exp(m, p_i)$. Hence, instead of the expression:

there exists a finite sequence of such numbers $m_0, \ldots, m_k$ that

$$R(m_0, \ldots, m_k),$$

we can say:

there exists such a non-negative number m that

$$R\big(\exp(m, p_0), \ldots, \exp(m, p_k)\big).$$

This representation of finite sequences is used above all for the elementary formulation of the recursive definitions[3]).

The operation of limited recursion. The class $\mathfrak{X}$ is *closed* under the operation of *limited recursion* if it satisfies the following condition:

if g, h, j are functions of the class $\mathfrak{X}$, and if the function f satisfies the conditions

$$\text{(I)}\quad \begin{aligned} &\text{(a)}\quad f(\mathfrak{u}, 0) = g(\mathfrak{u}),\\ &\text{(b)}\quad f(\mathfrak{u}, x+1) = h\big(\mathfrak{u}, x, f(\mathfrak{u}, x)\big),\\ &\text{(c)}\quad f(\mathfrak{u}, x) \leqslant j(\mathfrak{u}, x),\end{aligned}$$

then the function f also belongs to the class $\mathfrak{X}$.

Conditions (a) and (b) are ordinary conditions satisfied by those recursive definitions of function which are used in arithmetic. Condition (a) determines the initial value of f, and condition (b) makes the value for the next number dependent on its value for the preceding number by means of the function h. In this way the

[3]) Another method of representing finite sequences has been given by Gödel. For every finite sequence of numbers $a_0, \ldots, a_n$ there exist such two numbers u, v that:

$$r\big(u, 1 + v(z+1)\big) = a_z.$$

Cf. Robinson [19], p. 707, and Gödel [1].

value of the function f is well defined for each value of the variable x. Thus the function $f(\mathfrak{u}, x)$ is unambiguously defined by conditions (a) and (b). Condition (c) limits this possibility of defining to those functions which do not exceed other functions contained in the class $\mathfrak{X}$.

Examples. If a class $\mathfrak{X}$ is closed under the operation of limited recursion, and includes the functions $S(x) = x+1$, and x^y, then it also includes the functions $x+y$ and $x \cdot y$, since these functions can be defined by means of limited recursion. Setting:

$$U_2(x, y) = y^{x^0} = y,$$
$$g_1(y) = U_2(y, y), \qquad h_1(y, x, z) = U_2\big(y, U_2(x, S(z))\big),$$
$$j_1(y, x) = S(y)^{S(x)},$$

we can easily prove that the function $f(y, x) = y + x$ satisfies Scheme I:

$$\text{(A)}\quad \begin{aligned} &\text{(a)}\quad f(y, 0) = g_1(y),\\ &\text{(b)}\quad f(y, x+1) = h_1\big(y, x, f(x, y)\big),\\ &\text{(c)}\quad f(y, x) \leqslant j_1(y, x).\end{aligned}$$

Having the function $y+x$, we set in a similar way

$$g_2(y) = U_2(y, 0), \qquad h_2(y, x, z) = U_2(x, y+z)$$

and see that multiplication satisfies the conditions:

$$\text{(B)}\quad \begin{aligned} &\text{(a)}\quad y \cdot 0 = g_2(y),\\ &\text{(b)}\quad y \cdot (x+1) = h_2(y, x, y \cdot x),\\ &\text{(c)}\quad y \cdot x \leqslant j_1(y, x).\end{aligned}$$

Applying hereafter the operation of recursion we shall neither write out the identity functions U_1, U_2, $\big(U_1(x, y) = x = U_2(y, x)\big)$, which were needed for formal reasons only, nor define the auxiliary functions g, h, but shall write the recursion formulae directly, *e.g.* replacing the formulae (A) and (B) by the following simpler formulae, equivalent to them:

$$\begin{aligned} y + 0 &= y, & y \cdot 0 &= 0,\\ y + (x+1) &= S(y+x), & y \cdot (x+1) &= y + (y \cdot x),\\ y + x &\leqslant Sx^{Sy}, & y \cdot x &\leqslant Sx^{Sy}.\end{aligned}$$

These formulae have, of course, the same meaning as the formulae

(A) and (B). Further, if we define a function of one argument, $f(x)$, then Scheme I can easily be reduced to a simpler Scheme II:

$$\text{(II)}\quad \begin{aligned} &\text{(a)}\quad f(0)=g(0),\\ &\text{(b)}\quad f(x+1)=h\big(x,f(x)\big),\\ &\text{(c)}\quad f(x)\leqslant j(x). \end{aligned}$$

The definition of the function $x!$ can serve as an example of applying Scheme II:

$$0!=S(0),$$
$$(x+1)!=x!\cdot S(x),$$
$$x!\leqslant x^x.$$

Theorem 2.3. *The class $\mathcal{E}$ is closed under the operation of limited recursion.*

Proof. If the functions f,g,h,j satisfy the conditions (a), (b) and (c) of Scheme I, then it is true that

(1) $y=f(\mathfrak{u},x)$ *if and only if there exists a finite sequence of numbers* $m_0,\dots,m_x$ *such that*

$$m_0=g(\mathfrak{u}),\ m_1=h(\mathfrak{u},0,m_0),\ m_2=h(\mathfrak{u},1,m_1),\dots,m_x=h(\mathfrak{u},x-1,m_{x-1}),$$

and $y=m_x$.

The numbers $m_0,\dots,m_x$ thus satisfy the conditions $m_0=g(\mathfrak{u})$, $m_{k+1}=h(\mathfrak{u},k,m_k)$, which exactly determines each of them. In conformity with the above-mentioned method of the representation of finite sequences we may represent the numbers $m_0,\dots,m_x$ by the number $m=2^{m_0}\cdot 3^{m_1}\cdot\ldots\cdot p_x^{m_x}$. Thus we have $m_i=\exp(m,p_i)$; consequently the inductive condition $m_{k+1}=h(\mathfrak{u},k,m_k)$ is equivalent to the following ones:

$$\exp(m,p_{k+1})=h\big(\mathfrak{u},k,\exp(m,p_k)\big).$$

Thus the following equivalence results from equivalence (1).

Putting

$$F(x,\mathfrak{u})=p_x^{(x+2)\cdot\operatorname{Max} j(\mathfrak{u},z)\ \text{for}\ z\leqslant x},$$

$$(2)\quad \begin{aligned} y=f(\mathfrak{u},x)\equiv \textstyle\sum_{m\leqslant F(x,\mathfrak{u})}\{&\exp(m,2)=g(\mathfrak{u}).\\ &\cdot y=\exp(m,p_x).\textstyle\prod_{k<x}[\exp(m,p_{k+1})\\ &=h\big(\mathfrak{u},k,\exp(m,p_k)\big)]\}. \end{aligned}$$

As p_x is the largest prime of the number m, and $m_i\leqslant j(\mathfrak{u},i)$, therefore $\exp(m,p_i)\leqslant\operatorname{Max} j(\mathfrak{u},i)$ *for* $i\leqslant x$. Thus the estimation of the number m is sufficient. Let us set $R(y,\mathfrak{u},x)$ for the relation on the right side of the equivalence (2). The relation R is an elementary one if the functions g,h,j are elementary. From the equivalence (2) it follows that the function f may be defined by the operation of limited minimum by means of the relation R in the following way:

$$f(\mathfrak{u},x)=\mu y\leqslant j(\mathfrak{u},x)[R(y,\mathfrak{u},x)].$$

Thus the function f is an elementary one.

Equivalent definitions of the class of elementary functions. Let $\mathcal{E}'$ be the smallest class of functions

1° including the initial functions $x+1$, $x\dot{-}y$, x^y,

and

2° closed under the following operations:
the operations of substitution,
the operation of limited minimum.

Theorem 2.4. *The class $\mathcal{E}'$ is identical with the class $\mathcal{E}$.*

Proof. The first inclusion $\mathcal{E}'\subset\mathcal{E}$ has been proved by showing that x^y belongs to the class $\mathcal{E}$, and that this class is closed under the operation of limited minimum (Theorem 2.1.).

To show that $\mathcal{E}\subset\mathcal{E}'$ it is necessary to prove that $x+y$ belongs to $\mathcal{E}'$, and that $\mathcal{E}'$ is closed under the operations of limited summation and limited multiplication. Let us notice first that, by Corollary 1.2 and Theorem 2.2 b, the class $\mathcal{E}'$ is closed under the operations of the propositional calculus and under the operations of limited quantifiers.

Further, the relations "smaller than", "larger than", and "equal to" are also the relations of the class $\mathcal{E}'$:

$$x\leqslant y.\equiv.x\dot{-}y=0,\qquad x=y.\equiv.x\leqslant y.y\leqslant x.$$

Hence it can easily be seen that the class $\mathcal{E}'$ is closed under the two operations of maximum, since these have been defined by means of the operations of limited minimum and of limited quantifiers. Thus the following functions and relations belong to the class $\mathcal{E}'$:

$$x\cdot y=\mu z\leqslant(x+1)^{y+1}[(2^x)^y=2^z];$$
$$x+y=\mu z\leqslant(x+1)\cdot(y+1)[2^x\cdot 2^y=2^z].$$

Further: x^+, x^{++}, $x|y$, prime(x), $\exp(x,y)$, xNy, p_x, are functions and relations of the class $\mathcal{E}'$, since they are defined by means of functions and operations which have been proved to belong to $\mathcal{E}'$; this can easily be seen from the structure of their definitions. Hence it follows that the class $\mathcal{E}'$ is closed under the operation of limited recursion, since in the proof of Theorem 2.3 the recursion scheme was reduced to the operations of limited minimum, limited quantifiers, and the functions mentioned above. It can now easily be proved by means of the recursion scheme that the operations of limited summation and limited multiplication do not lead outside the class $\mathcal{E}'$. Namely, let us suppose that the functions f and g are defined by limited summation and limited multiplication respectively, by means of the function F belonging to the class $\mathcal{E}'$:

$$f(\mathfrak{u},x)=\sum_{i\leqslant x}F(\mathfrak{u},i),\qquad g(\mathfrak{u},x)=\prod_{i\leqslant x}F(\mathfrak{u},i).$$

It can easily be proved that the functions f and g satisfy the conditions (a), (b) and (c) of the operation of limited recursion:

$$f(\mathfrak{u},0)=F(\mathfrak{u},0),$$
$$f(\mathfrak{u},x+1)=f(\mathfrak{u},x)+F(\mathfrak{u},x+1),$$
$$f(\mathfrak{u},x)\leqslant(x+1)\cdot\operatorname{Max}F(\mathfrak{u},z)\ \text{ for } z\leqslant x,$$

$$g(\mathfrak{u},0)=F(\mathfrak{u},0),$$
$$g(\mathfrak{u},x+1)=g(\mathfrak{u},x)\cdot F(\mathfrak{u},x+1),$$
$$g(\mathfrak{u},x)=\big(\operatorname{Max}F(\mathfrak{u},z)\ \text{ for } z\leqslant x\big)^{x+1}.$$

So the functions f and g also belong to the class $\mathcal{E}'$. Therefore the class $\mathcal{E}'$ is closed under all the operations of the class $\mathcal{E}$ and includes the initial functions of the class $\mathcal{E}$, hence $\mathcal{E}\subset\mathcal{E}'$. We have thus proved that the classes $\mathcal{E}$ and $\mathcal{E}'$ are identical.

Let $\mathcal{E}''$ be the smallest class of functions

1° including $x+1$ and x^y as the initial functions,

and

2° closed under the following operations:
the operations of substitution,
the operation of limited recursion.

Theorem 2.5. *The class $\mathcal{E}''$ is identical with the class $\mathcal{E}$.*

Proof. The inclusion $\mathcal{E}''\subset\mathcal{E}$ results from Theorem 2.3. The inverse inclusion $\mathcal{E}\subset\mathcal{E}''$ remains to be proved. We shall prove it in an indirect way, by showing first that $\mathcal{E}'\subset\mathcal{E}''$, and by using Theorem 2.4. First of all, it can easily be shown that both the remaining initial function of the class $\mathcal{E}'$, namely $x\dot{-}y$, and the following functions: $P(x)=x\dot{-}1$, $x+y$, $x\cdot y$, belong to the class $\mathcal{E}''$. These functions satisfy the conditions of the recursion scheme:

$$P(0)=0,\qquad y\dot{-}0=y,$$
$$P(x+1)=x,\qquad y\dot{-}(x+1)=P(y\dot{-}x),$$
$$P(x)\leqslant x,\qquad y\dot{-}x\leqslant y,$$
$$y+0=y,\qquad y\cdot 0=0,$$
$$y+(x+1)=(y+x)+1,\qquad y\cdot(x+1)=y\cdot x+y,$$
$$y+x\leqslant(y+1)^{x+1}.\qquad y\cdot x\leqslant y^x.$$

Now we prove that the class $\mathcal{E}''$ is closed under the following narrowed operation of limited summation: if $F\in\mathcal{E}''$ and

$$f(\mathfrak{u},x)=\sum_{i\leqslant x}1\dot{-}F(\mathfrak{u},i)$$

then $f\in\mathcal{E}''$, since it can be defined by limited recursion:

$$f(\mathfrak{u},0)=1\dot{-}F(\mathfrak{u},0),$$
$$f(\mathfrak{u},x+1)=f(\mathfrak{u},x)+\big(1\dot{-}F(\mathfrak{u},x+1)\big),$$
$$f(\mathfrak{u},x)\leqslant x+1.$$

Thus from Corollary 2.1 a it follows that the class $\mathcal{E}''$ is closed under the operation of limited minimum. Hence $\mathcal{E}'\subset\mathcal{E}''$, and by Theorem 2.4 $\mathcal{E}\subset\mathcal{E}''$. The classes $\mathcal{E}$ and $\mathcal{E}''$ are also identical.

Let $\mathcal{E}'''$ denote the smallest class

1° including $x+1$, $x\dot{-}y$, $x\cdot y$, x^y as the initial functions;

2° closed under the following operations:
the operations of substitution,
the operation of limited summation.

Theorem 2.6. *The class $\mathcal{E}'''$ is identical with the class $\mathcal{E}$.*

Proof. $\mathcal{E}'''$ is included in $\mathcal{E}$ *ex definitione*. Conversely $\mathcal{E}\subset\mathcal{E}'''$, because $\mathcal{E}'''$ is closed under the operation of limited minimum, according to Corollary 2.1 a. Hence, from Theorem 2.4, it follows that $\mathcal{E}=\mathcal{E}'''$.

The above theorems show that on the basis of the operations of substitution and on the basis of the initial functions $x+1$, $x+y$, $x\dot{-}y$, $x\cdot y$, x^y, the limited operations of summation, minimum and recursion are equivalent. It can also be proved that on this basis each of the above-mentioned operations is equivalent to each of the following: the operation of limited multiplication, the two operations of maximum, and the operation of minimal value:

$$\operatorname{Min} F(\mathfrak{u},x) \text{ for } x\leqslant z=\mu y\leqslant F(\mathfrak{u},0)\big[\textstyle\sum_{x\leqslant z} y=F(\mathfrak{u},z)\big].$$

The proofs of these easy theorems are left to the readers.

Thus the class of elementary functions may be defined as including the above-mentioned initial functions and closed under the operations of substitution and under one of the seven operations enumerated above. These operations will henceforth be called *elementary operations.*

§ 3. Classes based on limited recursion

Besides the class $\mathcal{E}$ we shall examine other classes of functions closed under the operation of limited recursion. For this purpose we shall prove some theorems of a more general character.

We say that a *class $\mathcal{X}$ of functions is inductively definable* by means of the initial functions: $f_1,\dots,f_n$, and operations $O_1,\dots,O_s$ provided that $\mathcal{X}$ is the smallest class containing the functions $f_1,\dots,f_n$ and closed under the operations $O_1,\dots,O_s$. The class $\mathcal{X}$ is said to be *inductively defined* if it is inductively defined by means of certain operations mentioned in this paper.

Theorem 3.1. *If $\mathcal{X}$ is inductively definable by means of the operations of substitution as well as the operations $O_{k_1},\dots,O_{k_s}$ and $O_{k_1},\dots,O_{k_s}$ do not lead outside the functions of one argument, and if $\mathcal{X}$ includes the pairing functions I,K,L, then the class $\mathcal{X}_1$ is inductively definable.*

To simplify the proof we shall assume that $\mathcal{X}$ includes the initial functions $f_1(x,y),\dots,f_k(x,y)$, of two arguments, and $f_{k+1}(x),\dots,f_{k+t}(x)$, of one argument; we shall prove that class $\mathcal{X}_1$ is identical with the class $\mathcal{X}^*$, being the smallest class including the initial functions[4] $f_1(Kx,Lx),\dots,f_k(Kx,Lx)$, $f_{k+1}(x),\dots,f_{k+t}(x)$, Kx, Lx, $I(x,x)$, $I(x,0)$, $I(0,x)$ and closed under the operations $O_{k_1},\dots,O_{k_s}$ and O_1,O_2.

[4]) In addition we assume that $I(0,0)=0$. On the other hand the condition $I(Kz,Lz)=z$ is not necessary for this proof.

O_1: If $f(x)$ and $g(x)$ belong to $\mathcal{X}^*$, then $f(g(x))$ also belongs to $\mathcal{X}^*$.

O_2: If $f(x)$ belongs to $\mathcal{X}^*$, then $I(Kx,f(Lx))$ and $I(f(Kx),Lx)$ also belongs to $\mathcal{X}^*$.

The inclusion $\mathcal{X}^*\subset\mathcal{X}_1$ is self-evident. The initial functions of the class $\mathcal{X}^*$ belong to the class $\mathcal{X}_1$, and the operations assumed in the definition of the class $\mathcal{X}^*$ do not lead outside $\mathcal{X}_1$.

The inverse inclusion will be obtained by means of several lemmata. If f is a function of one argument, then the n-th iteration of the function will be symbolised as f^n. Thus $f^0(x)=x$, $f^{n+1}(x)==f(f^n(x))$.

We set $C_nx=KL^{n-1}x$, hence $C_1x=Kx$, $C_2x=KLx$. Further:

$$\langle x_1,\dots,x_n,y\rangle=I\Big(x_1,I\big(x_2,(\dots I(x_n,y)\dots)\big)\Big).$$

From the properties of the pairing functions I,K,L it follows that

$$C_i\langle x_1,\dots,x_n,y\rangle=x_i \quad \text{for} \quad 0<i\leqslant n,$$
$$L^n\langle x_1,\dots,x_n,y\rangle=y, \quad C_nL^mx=C_{n+m}x,$$
$$\langle x_1,\dots,x_n,\langle y_1,\dots,y_m,z\rangle\rangle=\langle x_1,\dots,x_n,y_1,\dots,y_m,z\rangle.$$

Lemma 1. *If $f,g\in\mathcal{X}^*$, then the function $I(f(x),g(x))$ also belongs to $\mathcal{X}^*$.*

Proof. By the operation O_2 the functions $I(Kx,g(Lx))$ and $I(f(Kx),Lx)$ belong to $\mathcal{X}^*$. Hence the functions

$$I\big(f(KI(x,x)),LI(x,x)\big)=I(f(x),x),$$

$$I\Big(KI(f(x),x),g\big(LI(f(x),x)\big)\Big)=I(f(x),g(x))$$

also belong to $\mathcal{X}^*$ by the operation O_1.

Lemma 2. *If $j_1,\dots,j_k$ belong to $\mathcal{X}^*$, then the function*

$$\langle (j_k(x),\dots,j_1(x)\rangle$$

also belongs to $\mathcal{X}^*$.

Proof. Since $j_2,j_1\in\mathcal{X}^*$, then, by Lemma 1, the function

$$\langle j_2(x),j_1(x)\rangle=I(j_2(x),j_1(x))$$

also belongs to $\mathcal{X}^*$. Let us suppose that our lemma is true for $k=n$, *i.e.* that $\langle j_n(x),\dots,j_1(x)\rangle$ belongs to $\mathcal{X}^*$. By Lemma 1, the function

$$\langle j_{n+1}(x),\dots,j_1(x)\rangle = I\big(j_{n+1}(x),\langle j_n(x),\dots,j_1(x)\rangle\big)$$

also belongs to $\mathcal{X}^*$, so that our lemma is true for $k=n+1$.

Thus by induction we find that for every k the function $\langle j_k(x),\dots,j_1(x)\rangle$ belongs to $\mathcal{X}^*$.

Lemma 3. *If $j_1,\dots,j_k\in\mathcal{X}^*$, then the functions*

$$f(x)=\langle j_1(x),\dots,j_k(x),0\rangle,$$
$$g(x)=\langle j_1(x),\dots,j_i(x),0,j_{i+1}(x),\dots,j_k(x),0\rangle$$

also belong to $\mathcal{X}^*$.

Proof. To the class $\mathcal{X}^*$ belong the functions $\langle j_k(x),0\rangle$ (from $I(x,0)$ by substitution). Hence also $f(x)$, and

$$h(x)=\langle j_{i+1}(x),\dots,j_k(x),0\rangle=\langle j_{i+1}(x),\dots,j_{k-1}(x),\langle j_k(x),0\rangle\rangle$$

by lemma 2. Similarly $I\big(0,h(x)\big)$ from $I(0,x)$, and

$$g(x)=\langle j_1(x),\dots,j_i(x),I\big(0,h(x)\big)\rangle$$

by lemma 2.

Lemma 4. *If $f(x_1,\dots,x_n)$ is a function of the class $\mathcal{X}$, then $f'(x)=f(C_1x,\dots,C_nx)$ is a function of the class $\mathcal{X}^*$.*

Proof by induction with respect to the order of the function f in the class $\mathcal{X}$.

If f is an initial function, of two arguments, of the class $\mathcal{X}$, $f(x,y)=f_i(x,y)$, $1\leqslant i\leqslant k$, then the function $f_i(Kx,Lx)$ is an initial function of the class $\mathcal{X}^*$. This class includes also the functions Kx, $I(Kx,KLx)$ (by O_2) and

$$f_i\big(KI(Kx,KLx),LI(Kx,KLx)\big)=f_i(Kx,KLx)=f_i\big(C_1x,C_2(x)\big)=f_i'(x).$$

Let us now suppose that for all the functions g and h, of orders lower than n, the corresponding functions g' and h' belong to the class $\mathcal{X}^*$, and that f is a function of the order n. We shall prove that f' also belongs to $\mathcal{X}^*$. The function f is obtained from certain functions of orders lower than n by means of the operations of substitution or any of the operations $O_{k_1},\dots,O_{k_s}$. Thus we have to examine the following cases:

A. The function f is obtained by means of the operation of the substitution of the function

$$f(x_1,\dots,x_k,y_1,\dots,y_m)=g\big(x_1,\dots,x_n,h(y_1,\dots,y_m),x_{n+1},\dots,x_k\big).$$

From the inductive hypothesis it follows that the functions

$$g'(x)=g(C_1x,\dots,C_{k+1}x),$$
$$h'(x)=h(C_1x,\dots,C_mx)$$

belong to $\mathcal{X}^*$. $\mathcal{X}^*$ includes the functions C_tx for any t. Hence from Lemma 3 it follows that the function $\langle C_{n+1}x,\dots,C_{n+m}x,0\rangle$ also belongs to $\mathcal{X}^*$. Thus $\mathcal{X}^*$ includes

$$h'(\langle C_{n+1}x,\dots,C_{n+m}x,0\rangle)=h(C_{n+1}x,\dots,C_{n+m}x)$$

and

$$j(x)=\langle C_1x,\dots,C_nx,h(C_{n+1}x,\dots,C_{n+m}x),C_{n+m+1}x,\dots,C_{m+k}x,0\rangle$$

(by Lemma 3), and also the function

$$g'\big(j(x)\big)=g\big(C_1x,\dots,C_nx,h(C_{n+1}x,\dots,C_{n+m}x),C_{n+m+1}x,\dots,C_{m+k}x\big)$$
$$=f'(x).$$

B. f is obtained from the function g by the identification of the k-th and j-th variables, *i. e.*

$$f(x_1,\dots,x_n)=g(x_1,\dots,x_j,\dots,x_{k-1},x_j,x_{k+1},\dots,x_n).$$

By the inductive hypothesis $g'\in\mathcal{X}^*$; then $\mathcal{X}^*$ also includes (by Lemma 3) the function

$$j(x)=\langle C_1x,\dots,C_jx,\dots,C_{k-1}x,C_jx,C_{k+1}x,\dots,C_nx,0\rangle$$

and consequently the function $f'(x)=g'\big(j(x)\big)$.

C. f is obtained from g by the substitution of a constant

$$f(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)=g(x_1,\dots,x_{k-1},0,x_{k+1},\dots,x_n)$$

and $g'\in\mathcal{X}^*$. We set

$$j(x)=\langle C_1x,\dots,C_{k-1}x,0,C_{k+1}x,\dots,C_nx,0\rangle,$$

$j\in\mathcal{X}^*$, by Lemma 3, therefore the function $f'(x)=g'\big(j(x)\big)$ also belongs to $\mathcal{X}^*$.

D. If f is obtained from g and h by means of any of the operations $O_{k_1},\dots,O_{k_s}$, then f,g,h are functions of one argument. We have

$$g'(x)=g(Kx),\qquad h'(x)=h(Kx),$$

and

$$g(x)=g'\big(I(x,x)\big),\qquad h(x)=h'\big(I(x,x)\big),\qquad f'(x)=f(Kx).$$

Hence if $g', h' \epsilon \mathcal{X}^*$, then $g, h \epsilon \mathcal{X}^*$, and $f, f' \epsilon \mathcal{X}^*$ according to the definition of $\mathcal{X}^*$.

Having proved Lemma 4, we shall prove by induction the inclusion

$$\mathcal{X}_1 \subset \mathcal{X}^*.$$

If $f(x)$ is a function of the order 0 in the class $\mathcal{X}$, then $f(x) = f_{k+l}(x)$ $(1 \leqslant l \leqslant t)$, and f belongs to the class $\mathcal{X}^*$ as its initial function.

Let us now suppose that all the functions of the class $\mathcal{X}_1$, which are of orders lower than n in the class $\mathcal{X}$, belong to $\mathcal{X}^*$. We shall prove that the function $f(x)$, of the order n in the class $\mathcal{X}$, also belongs to $\mathcal{X}^*$.

We distinguish two cases:

A. f is obtained from functions of one argument by means of the operations $O_1, O_{k_1}, \ldots, O_{k_s}$, hence $f \epsilon \mathcal{X}^*$, because the operations $O_1, O_{k_1}, \ldots, O_{k_s}$ do not lead outside the class $\mathcal{X}^*$;

B. the function f is obtained from a certain function of two arguments, $F(x,y)$, by means of one of the operations of substitution. Also:

$$f(x) = F(x,x), \text{ or } f(x) = F(x,0), \text{ or } f(x) = F(0,x).$$

By Lemma 4, the function $F'(x) = F(C_1 x, C_2 x)$ belongs to $\mathcal{X}^*$. $\mathcal{X}^*$ includes the functions

$$I(Kx, I(Lx, 0));$$
$$I(KI(x,x), I(LI(x,x), 0)) = \langle x, x, 0 \rangle,$$

similarly $\langle x, 0, 0 \rangle$ and $\langle 0, x, 0 \rangle$.

It can easily be shown that

$$F(x,x) = F'(\langle x, x, 0 \rangle),$$
$$F(x,0) = F'(\langle x, 0, 0 \rangle),$$
$$F(0,x) = F'(\langle 0, x, 0 \rangle).$$

Thus, in each of the possible cases the function f belongs to the class $\mathcal{X}^*$. Hence the classes $\mathcal{X}_1$ and $\mathcal{X}^*$ are identical.

Theorem 3.2. *The class $\mathcal{E}_1$ is inductively definable.*

Proof. The operation of limited summation can be formulated in such a manner that it leads from the function $F(x)$ to the function $f(x)$ defined as follows:

$$f(x) = \sum_{i \leqslant Kx} F(I(i, Lx)).$$

Thus from Theorem 3.1 it follows that the class $\mathcal{E}_1$ can be based on this operation and on the operations O_1 and O_2. Also the class $\mathcal{E}_k$ is inductively definable for each $k > 0$.

We say that the function $f(x)$ *increases faster* than each function of the class $\mathcal{X}$, provided that for every function $g \epsilon \mathcal{X}$ there exists such a number n_0 that for every $x \geqslant n_0$ we have the inequality $f(x) > g(x)$.

The function $f(x_1, \ldots, x_n, \ldots, x_k)$ is called *non-decreasing with respect to the n-th argument*, if for every $x_1, \ldots, x_k$ and y the inequality $x_n \leqslant y$ implies that

$$f(x_1, \ldots, x_n, \ldots, x_k) \leqslant f(x_1, \ldots, y, \ldots, x_k).$$

A function which is non-decreasing with respect to each of its arguments is called simply *non-decreasing*. A function is called *increasing* if it satisfies the above condition provided that weak inequalities $\leqslant$ are replaced by strong ones $<$.

Further, we may say that the function g *dominates* the function f provided that $f(u) \leqslant g(u)$ for each u.

Theorem 3.3. *If $\mathcal{X}$ and $\mathcal{Y}$ are inductively definable by means oofthe operations of substitution, and besides at most by means of the operation of limited recursion, and if the class $\mathcal{X}$ includes non-decreasing functions which dominate the initial functions of the class $\mathcal{Y}$, then every function f of the class $\mathcal{Y}$ is dominated by a certain non-decreasing function f', belonging to the class $\mathcal{X}$ and having the same number of arguments as the function f.*

Proof by induction with respect to the order of the function f in the class $\mathcal{Y}$.

For the initial functions this property is assumed.

If for any functions g, h, j, of orders lower than n, there exist such non-decreasing functions g', h', j', that

(i) $$g(x,y) \leqslant g'(x,y),$$

(ii) $$h(y) \leqslant h'(y),$$

(iii) $$j(x,u) \leqslant j'(x,u)$$

and if f is a function of the order n in the class $\mathcal{Y}$, then, if f is obtained by the substitution of functions $f(x,y) = g(x, h(y))$ then f is dominated by the non-decreasing function $g'(x, h'(y))$, since from (i), (ii), and from the fact that g' is a non-decreasing function it results that

$$f(x,y)=g(x,h(y))\leqslant g'(x,h(y))\leqslant g'(x,h'(y)).$$

If, however, f is obtained from a function g by the identification of the variables x and y for example, or by the substitution of a constant for the variable y for example, then f is dominated by the non-decreasing function f', obtained from g' by the application of the same operation. It results from the inequality (i) that

$$g(y,y)\leqslant g'(y,y), \qquad g(x,0)\leqslant g'(x,0).$$

Finally, if f is obtained from the functions g,h,j by limited recursion, and if the condition (c) of the operation of limited recursion has the form

$$f(x,u)\leqslant j(x,u),$$

then it results from the inequality (iii) that the function j' dominates the function f.

Theorem 3.4. *If $\mathcal{X}$ and $\mathcal{Y}$ are inductively definable by means of the operations of substitution, and besides at most by means of the operation of limited recursion, and if the class $\mathcal{X}$ includes non-decreasing functions which dominate the initial functions of the class $\mathcal{Y}$, and if f increases faster than any function of the class $\mathcal{X}_1$, then f increases faster than any function of the class $\mathcal{Y}_1$.*

Proof. By Theorem 3.3 every function $g\in\mathcal{Y}_1$ is dominated by a certain function $g'\in\mathcal{X}_1$, which means that

$$g(x)\leqslant g'(x)$$

for every x. If f increases faster than g', then there exists such an n that for $x\geqslant n$ we have

$$g'(x)<f(x).$$

Hence for $x\geqslant n$ we also have $g(x)<f(x)$, which means that f increases faster than g.

Theorem 3.5. *If $\mathcal{X}$ and $\mathcal{Y}$ include the successor-function and are inductively definable by means of among others, the operations of substitution, and if for every function of the class $\mathcal{X}_1$ there exists a function of the class $\mathcal{Y}_1$ which dominates it, then the class $\mathcal{X}$ does not include the universal function of the class $\mathcal{Y}$.*

Proof. If $F(n,x)$ is a universal function for the class $\mathcal{Y}_1$, then if $\mathcal{X}$ included $F(n,x)$, then $\mathcal{X}_1$ would include the function $F(x,x)+1$.

The latter is dominated by a certain function $g\in\mathcal{Y}_1$: $g(x)\geqslant F(x,x)+1$. Since F is the universal function for $\mathcal{Y}_1$, then there exists such an n that $g(x)=F(n,x)$ for each x. From this, however, setting $x=n$, we obtain the false inequality

$$F(n,n)=g(n)\geqslant F(n,n)+1.$$

For instance the universal function for the class $\mathcal{X}_n$ is not a member of the class $\mathcal{X}$ [5]). Yet a universal function for a class which is closed only under the operations of substitution can always be defined by limited recursion:

Theorem 3.6. *If $\mathcal{X}$ includes the pairing functions I,K,L and is inductively defined by means of the operations of substitution, $\mathcal{Y}$ is inductively defined by means of the operations of substitution and limited recursion, and $\mathcal{Y}$ includes such a function $h(n,x)$ that $f(x)<h(n,x)$ if f is of the order n in the class $\mathcal{X}_1$, then the class $\mathcal{Y}$ includes a universal function for the class $\mathcal{X}_1$.*

Proof. By Theorem 3.1 the class $\mathcal{X}_1$ is inductively definable by means of the operations O_1,O_2. The function $F(n,x)$, universal for the class $\mathcal{X}_1$, can be defined as follows:

(a) $\left.\begin{array}{l} F(0,x)= \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ F(k,x)= \end{array}\right\}$ the initial functions for the class $\mathcal{X}_1$;

(b) for $n\geqslant k$

$$F(n+1,x)=\begin{cases} F(T_1n,F(T_2n,x)), & \text{when } T_3n=0, \\ I(Kx,F(T_1n,Lx)), & \text{when } T_3n=1, \\ I(F(T_1n,Kx),Lx), & \text{when } T_3n>1; \end{cases}$$

(c) $F(n,x)\leqslant h(n,x)$.

The case $T_3n=0$ corresponds to the use of the operation O_1. The cases $T_3n=1$ and $T_3n>1$ correspond to the use of the operation O_2.

It can easily be seen that conversely, if $\mathcal{Y}$ includes the function $F(n,x)$, universal for $\mathcal{X}_1$, then the class $\mathcal{Y}$ also includes the function $h(n,x)$ which has the property formulated in the theorem, namely:

$$h(n,x)=\sum_{i\leqslant x}F(n,i)+1.$$

---

[5]) Cf. p. 6.

§ 4. Classes $\mathcal{E}^n$

Let us consider the following sequence of computable functions:

$$f_0(x,y)=y+1,$$
$$f_1(x,y)=x+y,$$
$$f_2(x,y)=(x+1)\cdot(y+1);$$

for $n\geqslant 2$

$$f_{n+1}(0,y)=f_n(y+1,y+1),$$
$$f_{n+1}(x+1,y)=f_{n+1}\big(x,f_{n+1}(x,y)\big).$$

We shall now prove certain properties of this sequence which will be needed in our further considerations.

Theorem 4.1. *$f_n(x,y)>y$ for $n>1$.*

Proof by induction.

If $n=2$, then $f_2(x,y)=(x+1)\cdot(y+1)>y$.

Let us now suppose that this theorem is satisfied for a given $n>1$ by any x and y. It holds also for $n+1$. It follows from the inductive hypothesis that

$$f_{n+1}(0,y)=f_n(y+1,y+1)>y+1>y.$$

Let us suppose that the inequality

$$f_{n+1}(x,y)>y$$

holds for a given x and for any y. Hence it holds also for $x+1$:

$$f_{n+1}(x+1,y)=f_{n+1}\big(x,f_{n+1}(x,y)\big)>f_{n+1}(x,y)>y.$$

Thus the theorem holds for any x and n.

Theorem 4.2. *$f_{n+1}(x+1,y)>f_{n+1}(x,y)$ for $n\geqslant 0$.*

Proof. For $n\geqslant 2$ we prove in virtue of Theorem 4.1:

$$f_{n+1}(x+1,y)=f_{n+1}\big(x,f_{n+1}(x,y)\big)>f_{n+1}(x,y).$$

For $n=0,1$ we verify directly: $x+1+y>x+y$; $(x+2)\cdot(y+1)>$ $>(x+1)\cdot(y+1)$.

Theorem 4.3. *$f_n(x,y+1)>f_n(x,y)$ for $n>0$.*

Proof. For $n=1,2$ we verify the theorem directly. For $n\geqslant 2$ we prove it by induction. Let us assume that this theorem is true for a given n and for any x and y. From this, and from Theorem 4.2, we obtain

$$f_{n+1}(0,y+1)=f_n(y+2,y+2)>f_n(y+1,y+2)$$
$$>f_n(y+1,y+1)=f_{n+1}(0,y).$$

Let us suppose that Theorem 4.3 is true for a given x and $n+1$, and for any y. We find that it is true also for $x+1$:

$$f_{n+1}(x+1,y+1)=f_{n+1}\big(x,f_{n+1}(x,y+1)\big)>f_{n+1}\big(x,f_{n+1}(x,y)\big)$$
$$=f_{n+1}(x+1,y).$$

Theorem 4.2 and Theorem 4.3 mean that each of the functions $f_n(x,y)$ for $n>0$ is strictly increasing with respect to both its arguments. As we shall see, the functions f_n for $n>3$ are not elementary, but they are computable. By means of the sequence f_n we can define the following sequence of classes of computable functions.

Let $\mathcal{E}^n$ be the smallest class

1° including $x+1$, $U_1(x,y)=x$, $U_2(x,y)=y$, $f_n(x,y)$ as the initial functions,

and

2° closed under the following operations:
the operations of substitution,
the operation of limited recursion.

The sequence of the classes $\mathcal{E}^n$ has the following properties:

Theorem 4.4. *$\mathcal{E}^3$ is a class of elementary functions.*

Proof. The function f_3 is an elementary one:

$$f_3(0,y)=(y+2)^2,\qquad f_3(1,y)=\big((y+2)^2+2\big)^2\qquad \text{etc.}$$

Generally, f_3 can easily be defined by means of limited recursion:

$$g(0,y)=y,$$
$$g(x+1,y)=\big(g(x,y)+2\big)^2,$$
$$g(x,y)<(y+2)^{2^{2^{2x}}},$$
$$f_3(x,y)=g(2^x,y).$$

In the class $\mathcal{E}^3$ the function x^y can also be defined:

$$x+0=x,\qquad x\cdot 0=0=U_2(x,0),$$
$$x+(y+1)=(x+y)+1,\qquad x\cdot(y+1)=x\cdot y+x,$$
$$x+y\leqslant f_3(x,y),\qquad x\cdot y\leqslant f_3(x,y),$$

$$x^0=1,$$
$$x^{y+1}=x^y\cdot x,$$
$$x^y\leqslant f_3(x,y).$$

Thus the initial functions of the class $\mathcal{E}^3$ belong to $\mathcal{E}$, and *vice versa*. By Theorem 2.5, the classes $\mathcal{E}^3$ and $\mathcal{E}$ are closed under the same operations. Consequently they are identical.

Theorem 4.5. *The class $\mathcal{E}^0$ includes the pairing functions $Qx=Ex=x\dot{-}[\sqrt{x}]^2$, R and Q^nx, and is closed under the operation of limited minimum.*

Proof. By means of the operations of the class $\mathcal{E}^0$ we can define the following functions:

1. $O(x)=U_2(x,0)$;
2. $U_1(x,y,z)=U_1(x,U_1(y,z))$;
3. $U_2(x,y,z)=U_1(U_2(x,y),z)$,
4. $U_3(x,y,z)=U_2(x,U_2(y,z))$;
5. $P(x)=x\dot{-}1$: $\quad P(0)=O(0)$, $\quad P(x+1)=U_1(x,P(x))$, $\quad P(x)\leqslant U_1(x,x)$;
6. $x\dot{-}y$: $\quad x\dot{-}0=U_1(x,x)$, $\quad x\dot{-}(y+1)=U_3(x,y,P(x\dot{-}y))$, $\quad x\dot{-}y\leqslant U_1(x,y)$;
7. $\sigma(x,y)=x\cdot 0^y=x\cdot(1\dot{-}y)$: $\quad \sigma(x,0)=U_1(x,x)$, $\quad \sigma(x,y+1)=O\big(U_3(x,y,\sigma(x,y))\big)$, $\quad \sigma(x,y)\leqslant U_1(x,y)$;
8. $\tau(x,y)=x+0^y$: $\quad \tau(x,0)=x+1$, $\quad \tau(x,y+1)=U_1(x,y,\tau(x,y))$, $\quad \tau(x,y)\leqslant U_1(x,y)+1$;
9. $r(x,y)=$ the remainder of the division of x by y:
$$r(0,y)=O(y),$$
$$r(x+1,y)=U_2\Big(x,\sigma\Big(r(x,y)+1,1\dot{-}\big(y\dot{-}(r(x,y)+1)\big)\Big)\Big),$$
$$r(x,y)\leqslant U_2(x,y);$$

10. $\left[\dfrac{0}{y}\right]=O(y)$,
$$\left[\frac{x+1}{y}\right]=\tau\left(\left[\frac{x}{y}\right],r(x+1,y)\right),$$
$$\left[\frac{x}{y}\right]\leqslant U_1(x,y);$$
11. $[\sqrt{0}]=0$,
$$[\sqrt{x+1}]=\tau\left([\sqrt{x}],\ ([\sqrt{x}]+1)\dot{-}\left[\frac{x+1}{[\sqrt{x}]+1}\right]\right),$$
$$[\sqrt{x}]\leqslant U_1(x,x);$$
12. $E(0)=0$,
$$E(x+1)=\sigma\big(E(x)+1,1\dot{-}r(x+1,[\sqrt{x+1}])\big),$$
$$E(x)\leqslant U_1(x,x);$$
13. $Qx=Ex$;
14. $Q^0(x)=x$,
$$Q^{n+1}(x)=Q(Q^n(x)),$$
$$Q^n(x)\leqslant U_1(x,n);$$
15. $W(0,y)=0(y)$,
$$W(x+1,y)=\tau\big(W(x,y),2\dot{-}\tau(1\dot{-}(y\dot{-}Qx),Qx\dot{-}y)\big),$$
$$W(x,y)\leqslant U_1(x,y).$$

The function W satisfies the equality:

$$W(x+1,y)=W(x,y)+0^{(Qx\dot{-}y)}.$$

The value of the function $W(x,y)$ equals the number of those numbers $s<x$ for which $Qs=y$,

$$Rx=W(x,Qx).$$

The functions Qx,Rx are pairing functions (other than those defined in § 1). The function Qx value every number infinitely many times. The function Rx indicates for how many numbers $s<x$ the equality $Qx=Qs$ is true. The function $P(x,y)$, which corresponds to them, does not, of course, belong to the class $\mathcal{E}^0$.

Let us now suppose that F belongs to $\mathcal{E}^0$. Consequently, $\mathcal{E}^0$ also includes the function which is defined in the following manner:

$$f(u,0)=1\dot{-}F(u,0),$$
$$f(u,x+1)=\tau\big(f(u,x),F(u,x+1)\big),$$
$$f(u,x)\leqslant U_2(u,x+1).$$

It can easily be seen that

$$f(u,x)=\sum_{i\leqslant x}1\dot{-}F(u,i).$$

Hence this narrowed operation of summation does not lead outside $\mathcal{E}^0$. Hence it results from Corollary 2.1a that $\mathcal{E}^0$ is closed under the operation of limited minimum.

Theorem 4.6. *For every n, $\mathcal{E}^n$ is closed under the operation of limited minimum. The relations of the class $\mathcal{E}^n$ are closed under the operations of limited quantifiers and under the operations of the propositional calculus.*

Proof. The definitions given above can be repeated in each class $\mathcal{E}^n$, since $\mathcal{E}^0\subset\mathcal{E}^n$. Hence from Theorem 2.2 b we find that each class $\mathcal{E}^n$ is closed under the operations of limited quantifiers. Further, by the equivalences

$$1\dot{-}x=0.\equiv.\sim(x=0),$$
$$\sigma(x,1\dot{-}y)=0.\equiv:x=0.\vee.y=0$$

the relations of the class $\mathcal{E}^n$ are closed under the operations of the propositional calculus.

Theorem 4.7. $\mathcal{E}^n\subset\mathcal{E}^{n+1}$.

Proof. We shall first show that for $n>0$ the following inequality holds:

$$(\alpha)\qquad f_{n+1}(x,y)>f_n(x,y).$$

For $n=1$ we have

$$f_2(x,y)=(x+1)\cdot(y+1)>x+y=f_1(x,y).$$

For $n\geqslant 2$ we reason as follows: if $x=0$, then we have the inequality (α),

$$f_{n+1}(0,y)=f_n(y+1,y+1)>f_n(0,y)$$

since f_n is a strictly increasing function (Theorems 4.2 and 4.3).

Let us now suppose that for every y

$$f_{n+1}(k,y)>f_n(k,y).$$

Hence, since f_{n+1} is a strictly increasing function, we obtain

$$f_{n+1}(k+1,y)=f_{n+1}\big(k,f_{n+1}(k,y)\big)$$
$$>f_{n+1}\big(k,f_n(k,y)\big)>f_n\big(k,f_n(k,y)\big)=f_n(k+1,y)$$

and we also obtain the inequality (α) by induction.

This inequality helps us to prove that in the class $\mathcal{E}^{n+1}$ the functions f_i (for $i\leqslant n$) can be defined by means of limited recursion. Indeed, it follows from the inequality (α) that for $0<i\leqslant n$

$$(\beta)\qquad f_i(x,y)<f_{n+1}(x,y).$$

For instance, if $n>1$ (or $n>2$), then we can easily define $x+y$ (or $x\cdot y$ and x^y) as we have done in the class $\mathcal{E}^3$. Hence the inclusions $\mathcal{E}^0\subset\mathcal{E}^1\subset\mathcal{E}^2\subset\mathcal{E}^3\subset\mathcal{E}^{n+1}$ for $n\geqslant 3$.

Now let us suppose that for $n>2$ and for $i<n$ the function f_i has been defined in the class $\mathcal{E}^{n+1}$, we shall show that in such a case f_{i+1} also belongs to the class $\mathcal{E}^{n+1}$. The function f_{i+1} satisfies the conditions:

$$(\gamma)\qquad \begin{aligned} f_{i+1}(0,y)&=f_i(y+1,y+1),\\ f_{i+1}(x+1,y)&=f_{i+1}\big(x,f_{i+1}(x,y)\big),\\ f_{i+1}(x,y)&\leqslant f_{n+1}(x,y).\end{aligned}$$

These conditions define the function f_{i+1} in a computable manner. But this is not the case with the simple scheme of limited recursion[6]). The function f_{i+1}, hovewer, can be defined in the class $\mathcal{E}^{n+1}$ by means of the operation of minimum, in a way similar to that in which the functions satisfying the conditions of the ordinary scheme of limited recursion (Theorem 2.3) are defined by means of the operation of minimum, namely by using the sequence p_n of prime numbers. The only difference is that here we use a double sequence of prime numbers, defined as $p(x,y)=p_{P(x,y)}$, where $P(x,y)$ is a pairing function.

[6]) The function f_{i+1} is defined by means of the operation of limited recursion with entanglement. Cf. Peter [3], p. 622.

Putting: $$F(x,y)=p\left(x,f_{n+1}(x,y)^{(x+2)\cdot f_{n+1}(x+y)}\right),$$

$$f_{i+1}(x,y)=\mu z\leqslant f_{n+1}(x,y)\Big[\sum\nolimits_{m\leqslant F(x,y)}\Big\{z+1=\exp\big(m,p(x,y)\big)$$

$$\cdot\prod\nolimits_{v\leqslant m}\Big\{\exp\big(m,p(0,v)\big)\neq 0.\rightarrow.\exp\big(m,p(0,v)\big)$$

$$=f_i(v+1,v+1)+1\Big\}\cdot\prod\nolimits_{t,v,w\leqslant m}\Big\{t\neq 0\cdot$$

$$t=\exp\big(m,p(w+1,v)\big).\rightarrow.$$

$$t=\exp\Big(m,p\Big(w,\exp\big(m,p(w,v)\big)\dot{-}1\Big)\Big)\Big\}\Big\}\Big].$$

The number m, used in the above definition, has the following property: the prime number $p(w,v)$ appears in the decomposition of m into primes with positive exponent if and only if $f_{i+1}(w,v)$ is necessary for the computation of the value of the function $f_{i+1}(x,y)$ according to the definition (γ). It can also easily be proved that if $\exp\big(m,p(w,u)\big)\neq 0$, then

$$\exp\big(m,p(w,u)\big)=f_{i+1}(w,u)+1.$$

Hence, if $f_i\in\mathcal{E}^{n+1}$, then $f_{i+1}\in\mathcal{E}^{n+1}$ for $i\leqslant n$. All the f_i for $i\leqslant n$ can be consecutively defined in the class $\mathcal{E}^{n+1}$. Thus the class $\mathcal{E}^{n+1}$ includes all the initial functions of the earlier classes, and consequently includes all those classes too.

It can easily be proved that for $n\geqslant 2$ the classes $\mathcal{E}^n$ are closed under the operation of limited summation. Likewise, for $n\geqslant 3$, the classes $\mathcal{E}^n$ are closed under the operation of limited multiplication etc. Each successive class is closed under a more limited operation. The class $\mathcal{E}^4$ is closed under the operation

$$\underset{i\leqslant x}{\Omega}F(i,u)=F(x,u)^{F(x-1,u)^{\cdot^{\cdot^{F(0,u)}}}}.$$

The proofs of these easy theorems are left to the readers.

Let $\mathcal{W}^n$ be the smallest class which includes the functions $I,K,L,x+y,x^2,f_n$, and is closed under the operations of substitution. By Theorem 3.1, the class $\mathcal{W}_1^n$ is inductively definable, namely it is the smallest class which includes the functions $x^2,Kx,Lx,f_n(Kx,Lx)$, and is closed under the operations: 1) O_1 (of superposition), and 2) of summation of functions (*i.e.* if $f,g\in\mathcal{W}_1^n$ then $\mathcal{W}_1^n$ includes also the function $f(x)+g(x)$).

Theorem 4.8. *If f is a function of the order k in the class $\mathcal{W}_1^n$ (where $n\geqslant 2$), then:*

(δ) $$f(x)<f_{n+1}(k,x).$$

Proof based on Theorems 4.1, 4.2, and 4.3. The initial functions satisfy the above inequality, since

$$x^2<(x+2)^2=f_3(0,x)\leqslant f_{n+1}(0,x),\qquad Kx,Lx\leqslant x<f_{n+1}(0,x),$$
$$f_n(Kx,Lx)\leqslant f_n(x,x)<f_n(x+1,x+1)=f_{n+1}(0,x).$$

Let us suppose that g and h are functions of the orders l and k in the class $\mathcal{W}_1^n$, and satisfy the inequality (δ). Let $k\geqslant l$, then

$$g(x)<f_{n+1}(k,x),\qquad h(x)<f_{n+1}(k,x).$$

Hence, since f_{n+1} is a strictly increasing function, we obtain

$$g\big(h(x)\big)<f_{n+1}\big(k,h(x)\big)<f_{n+1}\big(k,f_{n+1}(k,x)\big)=f_{n+1}(k+1,x).$$

Further, using the inequality (α) of Theorem 4.7, we obtain

$$g(x)+h(x)<2f_{n+1}(k,x)$$
$$<\big(f_{n+1}(k,x)+2\big)^2=f_3\big(0,f_{n+1}(k,x)\big)\leqslant f_{n+1}(k+1,x).$$

Therefore, if f is a function of the order $k+1$ and is obtained from the functions g and h through one of the operations 1 or 2, then

$$f(x)<f_{n+1}(k+1,x).$$

Hence we prove our theorem by induction.

Theorem 4.9. *The function $f_{n+1}(x,x)$ increases faster than any function of the class $\mathcal{E}^n$.*

Proof. For $n\geqslant 2$ it follows from Theorem 4.8 that if f is of the order k in the class $\mathcal{W}_1^n$, and $x\geqslant k$, then

$$f(x)<f_{n+1}(x,x).$$

Hence $f_{n+1}(x,x)$ increases faster than any function of the class $\mathcal{W}_1^n$, and according to Theorem 3.4 increases faster than any function of the class $\mathcal{E}_1^n$, because the initial functions of the class $\mathcal{E}^n$ are dominated by the increasing function $f_n(x+1,y+1)$ of the class $\mathcal{W}^n$. For $n=0,1$ we can verify our theorem directly. It can easily be proved that if f is of the order k in the class $\mathcal{E}^0$, then $f(x)<x+2^k+1$. Similarly, if f is of the order k in the class $\mathcal{E}^1$, then $f(x)<(x+1)\cdot 2^k$. Hence the function $2x$ increases faster than any function $f \epsilon \mathcal{E}^0$, and $(x+1)^2$ increases faster than any function $f \epsilon \mathcal{E}^1$. Thus we have $\mathcal{E}^n \subset \mathcal{E}^{n+1}$, $\mathcal{E}^n \neq \mathcal{E}^{n+1}$.

Theorem 4.10. *The class of the relations of $\mathcal{E}^n$ for $n>2$ is the smallest class which includes the initial relations*

$$R_1(x) \equiv x=0, \qquad R_2(x,y) \equiv x=y+1, \qquad R_3(x,y,z) \equiv x=y^z,$$
$$R_4(x,y,z) \equiv x=y \dot{-} x, \qquad R_5(x,y,z) \equiv x=f_n(y,z),$$
$$R_6(x,y,z) \equiv x \leqslant f_n(y+1,z+1), \qquad R_6'(x,y) \equiv R_6(x,y,y),$$

and is closed under the operations of the propositional calculus and of limited quantifiers.

Proof. By the operation of limited existential quantifier we mean an operation which leads from the relation U to the relation V defined as follows:

$$V(x,y,\mathfrak{u}) \equiv \sum_z R_6(z,x,y) \cdot U(x,y,z,\mathfrak{u}).$$

In a special case the relation V can be of one variable.

Let $\mathcal{E}_*^n$ denote the smallest class of relations including the above-mentioned initial relations and closed under the operation specified above. It follows directly from the theorems proved before that $\mathcal{E}_*^n$ is included in the class of relations of the class $\mathcal{E}^n$.

To prove the inverse inclusion we need the following lemma: Let us say that the expresion $A(x,y_1,\ldots,y_m)$ *estimates x by means of $y_1,\ldots,y_m$* provided that the expression A has the form

$$\sum_{z_1,\ldots,z_k} R_6(x,z_1,z_2) \,.\, R_6(z_1,z_3,z_4) \ldots\ldots R_6(z_k,y_i,y_j)$$

and each of the variables $x,z_1,\ldots,z_k$ appears as the first argument in one of the expressions which symbolise the relation R_6. Moreover, the variable x appears only in one of the expressions representing the relation R_6; and if the expression $R_6(z_i,z_j,z_h)$ appears in the expression A then $i<j\leqslant k$ and $i<h\leqslant k$.

It is evident that if the expression $A(x,y_1,\ldots,y_m)$ estimates x by means of $y_1,\ldots,y_m$ and $B(x,y_1,\ldots,y_m,\mathfrak{u})$ symbolises a relation of the class $\mathcal{E}_*^n$, then the expression

$$\sum_x A(x,y_1,\ldots,y_m) \,.\, B(x,y_1,\ldots,y_m,\mathfrak{u})$$

denotes also the relation of the class $\mathcal{E}_*^n$.

Lemma 4.11. *For every function $f(y_1,\ldots,y_m)$ of the class $\mathcal{E}^n$ there exists an estimating expression $A(x_1,y_1,\ldots,y_m)$ such that the following implication holds:*

$$x \leqslant f(y_1,\ldots,y_m) \,.\!\rightarrow .\, A(x,y_1,\ldots,y_m).$$

Proof. $R_6(x,y_1,y_2)$ is the estimating expression for the initial functions of the class $\mathcal{E}^n$. If the functions g and h possess estimating expressions A and B, the following implications are satisfied:

$$x \leqslant g(y_1,\ldots,y_k) \,.\!\rightarrow .\, A(x,y_1,\ldots,y_k),$$
$$x \leqslant h(y,\mathfrak{u}) \,.\!\rightarrow .\, B(x,y,\mathfrak{u});$$

and then the expression $\sum_z B(x,z,\mathfrak{u}) \,.\, A(z,y_1,\ldots,y_k)$ is equivalent to the estimating expression for the function

$$f(y_1,\ldots,y_k,\mathfrak{u}) = h\big(g(y_1,\ldots y_k),\mathfrak{u}\big),$$

since the implication

$$x \leqslant h\big(g(y_1,\ldots,y_k),\mathfrak{u}\big) \,.\!\rightarrow .\, \sum_z B(x,z,\mathfrak{u}) \,.\, A(z,y_1,\ldots,y_k)$$

results from the implications given above. The remaining operations: identification of variables, substitution of a constant, and limited recursion, involve no difficulties whatever.

Having proved Lemma 4.11, we shall in turn prove by induction that for every function $f \epsilon \mathcal{E}^n$, when $n>2$, the relation $x=f(y_1,\ldots,y_m)$ belongs to the class $\mathcal{E}_*^n$. For the initial functions the corresponding relations are initial relations in the class $\mathcal{E}_*^n$. Let us suppose that the functions g and h have their corresponding relations G and H which belong to the class $\mathcal{E}_*^n$:

$$x=g(y,\mathfrak{u}) \,.\!\equiv .\, G(x,y,\mathfrak{u}),$$
$$x=h(y_1,\ldots,y_m) \,.\!\equiv .\, H(x,y_1,\ldots,y_m),$$

then we have the equivalence

$$x=g\big(h(y_1,\ldots,y_m),\mathfrak{u}\big)$$
$$.\!\equiv .\, \sum_z \{A(z,y_1,\ldots,y_m) \,.\, H(z,y_1,\ldots,y_m) \,.\, G(x,z,\mathfrak{u})\}$$

in which A is the estimating expression for the function h. Thus the above relation corresponds to a function obtained by substitution. Further

$$x=g(0,\mathfrak{u}).\equiv.\sum_z\{R_6(z,x,x).R_1(z).G(x,z,\mathfrak{u})\},$$
$$x=g(y,y).\equiv.\sum_z\{R_6(z,y,y).z=y.G(x,z,y)\},$$
$$x=\mu v\leqslant y[g(v,\mathfrak{u})=0].\equiv.\sum_z\{R_6(z,x,x).R_1(z):x\leqslant y.G(z,x,\mathfrak{u})$$
$$.\prod_v\{R_6(v,x,x).v<x.\rightarrow.\sim(G(z,v,\mathfrak{u}))\}.\vee.R_1(x)$$
$$.\prod_v\{R_6(v,y,y).v\leqslant y.\rightarrow.\sim(G(z,v,\mathfrak{u}))\}\}.$$

The relations $\leqslant$, $=$, $<$ may be defined in $\mathcal{E}^n_*$ by means of the relation R_4.

Since for $n>2$ the class $\mathcal{E}^n$ can be defined as closed under the operations of substitution and limited minimum, we have proved that for every function $f\epsilon\mathcal{E}^n$ the relation

$$x=f(y_1,\ldots,y_m)\equiv F(x,y_1,\ldots,y_m)$$

belongs to $\mathcal{E}^n_*$. Hence, $\mathcal{E}^n_*$ includes also the relation

$$f(y_1,\ldots,y_m)=0.\equiv.\sum_z\{R_6(z,y_1,y_1).R_1(z).F(z,y_1,\ldots,y_m)\}.$$

Thus the class $\mathcal{E}^n_*$ includes all the relations of the class $\mathcal{E}^n$.

Theorem 4.12. *For $n>2$, the class $\mathcal{E}^{n+1}$ includes the universal function for the class $\mathcal{E}^n_1$.*

Proof. If $n>2$, then the class $\mathcal{E}^n_1$ is the smallest class which includes certain initial functions and is closed under the operations: O_1 of superposition, of addition of functions $f(x)+g(x)$ and of limited minimum restricted to functions of one argument

$$h(x)=\mu z\leqslant Kx[g(I(z,Lx))=0].$$

Hence the function $F(n,x)$, universal for $\mathcal{E}^n_1$, can be defined as follows:

$$\left.\begin{array}{l}F(0,x)=\\ \cdots\cdots\\ F(k_0,x)=\end{array}\right\}\text{the initial functions for } \mathcal{E}^n_1;$$

for $n\geqslant k_0$:

$$F(n+1,x)=\begin{cases}F(T_1n,F(T_2n,x)), & \text{when } T_3n=0,\\ F(T_1n,x)+F(T_2n,x), & \text{when } T_3n=1,\\ \mu z\leqslant Kx[F(T_1n,I(z,Lx))=0], & \text{when } T_3n>1.\end{cases}$$

This recursive definition can be changed into a definition formulated by means of limited minimum; we can do this by using a double sequence of prime numbers $p(x,y)$, as we have done when proving Theorem 4.7. The numbers whose existence is supposed in the definitions of that type, can easily be estimated by means of the function f_{n+1}. The completion of the proof is left to the reader.

Let $\mathcal{R}$ be the class of primitive recursive functions.

Theorem 4.13 *The class $\mathcal{R}$ is the sum of the classes $\mathcal{E}^n$.*

Proof. Notice that $f_0,f_1,f_2\epsilon\mathcal{R}$, and if $f_n\epsilon\mathcal{R}$ then $f_{n+1}\epsilon\mathcal{R}$ according to the theorem of R. Peter[7]), because f_{n+1} is defined by means of the operation of recursion with entanglement which does not exceed the class $\mathcal{R}$. Hence, for each n, $f_n\epsilon\mathcal{R}$. The class $\mathcal{R}$ is closed under the operation of limited recursion, being closed under the operation of recursion. Hence $\sum_n\mathcal{E}^n\subset\mathcal{R}$.

The inverse inclusion is proved by means of the following lemma:

Lemma 4.14. *If f is a function of an order not higher than n in the class $\mathcal{R}_1$, then $f\epsilon\mathcal{E}^{n+3}$.*

Proof. We examine the definition of the class $\mathcal{R}_1$, as given by R. M. Robinson[8]). The initial functions of the class $\mathcal{R}_1$, namely $x+1$ and Ex, belong to $\mathcal{E}^3$.

Let us now suppose that the functions g and h, of an order not higher than k, satisfy the lemma: $g,h\epsilon\mathcal{E}^{k+3}$. Hence $g(x)+h(x)$ and $g(h(x))$ also belong to the class $\mathcal{E}^{k+3}$, since every class $\mathcal{E}^{n+3}$ includes the function $x+y$ and is closed under the operations of substitution. These functions belong *a fortiori* to $\mathcal{E}^{k+1+3}$, which also includes the function $h^x(0)$. Namely, if $h\epsilon\mathcal{E}^{k+3}$, then, by Theorem 3.3, there exists a certain non decreasing function $h'\epsilon\mathcal{W}^{k+3}$, such that h' dominates h, because the class $\mathcal{W}^{k+3}$ includes the increasing function $f_{k+3}(x+1,y+1)$ which dominates the initial functions of the class $\mathcal{E}^{k+3}$. Let h' be a function of the order l in the class $\mathcal{W}^{k+3}$. By Theorem 4.8 the inequality

$$(\delta)\qquad h(x)\leqslant h'(x)<f_{k+4}(l,x)$$

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7) See Peter [3], p. 622.

8) See Robinson [5], p. 940, Theorem 3.

holds for every x. Hence the following inequality holds:

$$(\varepsilon) \qquad h^y(0) < f_{k+4}(l+y, 0).$$

This inequality is satisfied for $y=0$, since

$$h^0(0) = 0 < 1 = f_2(0,0) < f_{k+4}(l, 0).$$

Suppose that the inequality (ε) holds for y. It follows from this, and from (δ) that it also holds for $y+1$:

$$h^{y+1}(0) = h(h^y(0)) < f_{k+4}(l, h^y(0)) < f_{k+4}(l, f_{k+4}(l+y, 0))$$
$$\leqslant f_{k+4}(l+y, f_{k+4}(l+y, 0)) = f_{k+4}(l+(y+1), 0).$$

We have thus proved by induction that the inequality (ε) holds for every y. Hence, if $f(x) = h^x(0)$, then the function f can be defined within the class $\mathcal{E}^{k+4}$ by means of limited recursion:

$$f(0) = 0,$$
$$f(x+1) = h(f(x)),$$
$$f(x) \leqslant f_{k+4}(l+x, 0).$$

Thus, if f is of the order $k+1$ in the class $\mathcal{R}_1$, then $f \epsilon \mathcal{E}^{k+1+3}$. Hence by recursion with respect to k we have proved Lemma 4.14.

From Lemma 4.14 it follows directly that every function f of the class $\mathcal{R}$ belongs to a class of the sequences $\mathcal{E}^n$. Let f be a function of k arguments, $f(x_1, \ldots, x_k)$, belonging to the class $\mathcal{R}$, then we can define the function f' as

$$f'(x) = f(C_1 x, C_2 x, \ldots, C_k x).$$

The function f' belongs to $\mathcal{R}_1$. Let f' be a function of the order n in the class $\mathcal{R}_1$. Then it follows from Lemma 4.14 that $f' \epsilon \mathcal{E}^{n+3}$. $\mathcal{E}^{n+3}$ also includes the function $f(x_1, \ldots, x_k) = f'(\langle x_1, \ldots, x_k, 0\rangle)$.

We shall say that the operations $O_{k_1}, \ldots, O_{k_s}$ are insufficient to obtain the class $\mathcal{X}$ if for every finite number of the functions $g_1, \ldots, g_k$, belonging to the class $\mathcal{X}$, there exists such a function f of the class $\mathcal{X}$ that cannot be obtained by means of the operations $O_{k_1}, \ldots, O_{k_s}$, if the functions $g_1, \ldots, g_k$ are taken as the starting point.

Theorem 4.15. *The operations of substitution and of limited recursion are insufficient to obtain all the primitively recursive functions.*

Proof. If $g_1, \ldots, g_k$ belong to $\mathcal{R}$, then it follows from Theorem 4.13 that there exists such an n that all the functions $g_1, \ldots, g_k$ belong to $\mathcal{E}^n$, because the classes $\mathcal{E}^n$ form an increasing sequence (Theorem 4.7). Since the operations of limited recursion and of substitution do not lead outside the class $\mathcal{E}^n$, then these operations are insufficient to obtain the function f_{n+1} which belongs to the class $\mathcal{E}^{n+1}$.

Numerous problems arise in connection with the classes $\mathcal{E}^n$, *e. g.*:

1. Are the operations of limited recursion and limited minimum equivalent on the basis of the operations of substitution, and of the finite number of the initial functions belonging to the classes $\mathcal{E}^0$ (or to $\mathcal{E}^1$ ($\mathcal{E}^2$))?

2. Analogous problems concerning the operations of limited summation and limited recursion.

In connection with the above problems:

3. Can the universal function for the class $\mathcal{E}^0_1$ belong to the classes $\mathcal{E}^1$ or $\mathcal{E}^2$?

4. Can the class of the relations of $\mathcal{E}^n$, for $n<3$, be characterised in a similar way to the class of the relations of $\mathcal{E}^n$ for $n \geqslant 3$ (Theorem 4.10.)? Can the relations of one argument, of the class $\mathcal{E}^n$, be characterised in an analogous way?

5. Is the operation mentioned on p. 34 under which the classes $\mathcal{E}^n$ are closed for $n>3$, equivalent to the operation of limited recursion?

6. Does the operation of double limited recursion lead outside the given class $\mathcal{E}^n$?

7. Let us define the following sequence of functions:

$$f'_0(x,y) = y+1, \qquad f'_1(x,y) = x+y;$$

for $n \geqslant 1$:

$$f'_{n+1}(0,y) = 1$$
$$f'_{n+1}(x+1, y) = f'_n(f'_{n+1}(x,y), y).$$

Let $\mathcal{F}^n$ denote the smallest class including the initial functions $x+1, U_1(x,y), U_2(x,y), f_n(x,y)$, and closed under the operations of substitution and limited recursion. Can the same theorems be proved for the classes $\mathcal{F}^n$ as for the classes $\mathcal{E}^n$?

8. Are the operations of substitution sufficient to obtain the class $\mathcal{E}^n$?

9. Let $\mathfrak{R}^2$ be the class of functions obtained by double recursion. Is the operation of single recursion sufficient to obtain the class $\mathfrak{R}^2$? Analogue problems for other classes of multiple-recursive functions.

§ 5. Applications of the class $\mathcal{E}^0$.

The functions of the class $\mathcal{E}^0$ can be used in the canonical form of the computable functions.

Let $\iota x[R(\ldots x\ldots)]$ denotes the unique x such that $R(\ldots x\ldots)$.

Theorem 5.1 *Every computable function can be presented in the form $f(\mathfrak{u})=A(\iota x[B(\mathfrak{u},x)=0])$, where A and B are functions of the class $\mathcal{E}^0$.*

Proof. From the theorem of J. Robinson[9]) it follows that the class $\mathcal{C}$ of computable functions is the smallest class containing $x+1, x+y, Ex$ as the initial functions and closed under the operations of substitution and of the operation of effective inversion $f^{-1}(y)=\mu x[f(x)=y]$ when f assumes all values.

The functions: $x+1$, $x+y$, Ex can easily be presented in the desired form:

$$x+1=\iota y[y=x+1],\qquad Ex=\iota y[y=Ex],$$
$$x+y=\iota z[z\geqslant y\,.\,z\geqslant x\,.\,z\dot{-}x=y].$$

By Theorem 4.5 the relations under the operation belong to the class $\mathcal{E}^0$. Now let us suppose that f and g have the form:

$$f(\mathfrak{u})=A(\iota x[B(\mathfrak{u},x)=0]),\qquad g(v,\mathfrak{y})=C(\iota z[D(v,\mathfrak{y},z)=0]),$$

where $A,B,C,D\in\mathcal{E}^0$. We shall show that a function obtained from the above functions by means of the operations permitted in the class $\mathcal{C}$, also has such a form.

a) The operation of substitution:

$$g(f(\mathfrak{u}),\mathfrak{y})=C\Big(\iota z\Big[D\Big(A\{\iota x[B(\mathfrak{u},x)=0]\},\mathfrak{y},z\Big)=0\Big]\Big)$$
$$=CR\Big(\iota v\,[D(A(Qv),\mathfrak{y},Rv)=0\,.\,B(\mathfrak{u},Qv)=0]\Big)$$

since the two numbers: $x=\iota x[B(\mathfrak{u},x)=0]$, and $z=\iota z[D(Ax,\mathfrak{y},z)=0]$ can be replaced by one, $v=P(x,z)$, such that $x=Qv$ and $z=Rv$, where P,Q,R are pairing functions.

9) See Robinson [6], p. 712, Theorem 4.

By Theorem 4.5, the relation under the ι-operation belongs to the class $\mathcal{E}^0$.

b) The operation of effective inversion.

Let us set:

$$S(z,v)\,.\!\equiv\!.\,B(0,Q^v z)=0\,.\prod_{j\leqslant v}\{j>0\,.\!\rightarrow\!.\,B(j,RQ^{v\dot{-}j}z)=0\}$$

and

$$x_t=\iota x[B(t,x)=0].$$

It can easily be seen that:

$$\text{(i)}\qquad \iota z[S(z,v)]=P\big(P(\ldots P(x_0,x_1),x_2),\ldots,x_v\big)$$

hence

$$x_v=R\big(\iota z[S(z,v)]\big),$$

and thus

$$f(t)=A(x_t)=AR\big(\iota z[S(z,t)]\big).$$

Further, it follows from the equality (i) that the function

$$h(v)=\iota z[S(z,v)]$$

is an increasing one. Hence it follows that

$$f^{-1}(y)=\mu t[f(t)=y]=\mu t\big[AR\big(\iota z[S(z,t)]\big)=y\big]$$
$$=Q\mu s[S(Rs,Qs)\,.\,ARRs=y]$$

because the smallest $s=P(z,t)$ corresponds to the smallest t if the number $z=\iota z[S(z,t)=0]=h(t)$ increases along with t.

Since, by Theorems 4.5 and 4.6, the relation S belongs to the class $\mathcal{E}^0$, therefore the class $\mathcal{E}^0$ contains such a relation T that

$$f^{-1}(y)=Q\mu s[T(s,y)];$$

this means that

$$f^{-1}(y)=Q\iota s\big[T(s,y)\,.\prod_{u<s}\sim\big(T(u,y)\big)\big].$$

The relation under the ι-operation in this case also belongs to the class $\mathcal{E}^0$, since the operations of limited quantifiers do not lead outside the class $\mathcal{E}^0$.

The operations of the identification of variables involve no difficulties whatever.

Theorem 5.2 *Each computable function can be expressed in the form*

$$f(\mathfrak{u})=Q(\iota x[A(x,\mathfrak{u})=0])$$

called the canonical form[10]), *where Q and A belong to the class $\mathcal{E}^0$, and Q is a constant function defining the first element of the pair.*

Proof. We know from the preceding theorem that f can be expressed as

$$f(\mathfrak{u})=A(\iota x[B(x,\mathfrak{u})=0]),$$

and hence

$$f(\mathfrak{u})=Q(\iota x[Qx=ARx.B(Rx,\mathfrak{u})=0]).$$

Theorem 5.3 *Every recursively enumerable set is enumerated by a certain function of the class $\mathcal{E}^0$.*

Proof. There exists an element a such that $a\epsilon X$. X is enumerated by a certain function f which can be given the following canonical form:

$$f(v)=Q(\mu x[F(x,v)=0]),$$

Q and F belong to $\mathcal{E}^0$ in conformity with Theorem 5.2.

The following equivalences are true:

$$u\epsilon X.\equiv.\sum_v u=f(v).\equiv.\sum_{v,x}u=Qx.x=\mu y[F(y,v)=0]$$
$$.\equiv.\sum_{v,x}u=Qx.F(x,v)=0.\prod_{y<x}F(y,v)\neq 0$$
$$.\equiv.\sum_{v,x}H(u,v,x)=0.\equiv.\sum_y H(u,Qy,Ry)=0$$

where H is a certain function which belongs to $\mathcal{E}^0$ in virtue of Theorem 4.5. It follows from the above equivalences that X is enumerated by the function

$$j(y,z)=\mu u\leqslant z+a[H(u,Qy,Ry)=0]$$

and consequently by the function $g(v)=j(Qv,Rv)$ as well. These functions belong to the class $\mathcal{E}^0$ because this class is closed under the operation of limited minimum, and because $z+a=z+1^{(1)}+\dots+1^{(a)}$, where a is constant.

10) Cf. Robinson [6], p. 716 and Kleene [2], p. 727.

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