Functions all the way down!

Lambda Calculus and Church Encoding

Ivan Lazar Miljenovic

Maths PhD Journal Club

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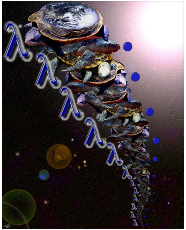


Outline

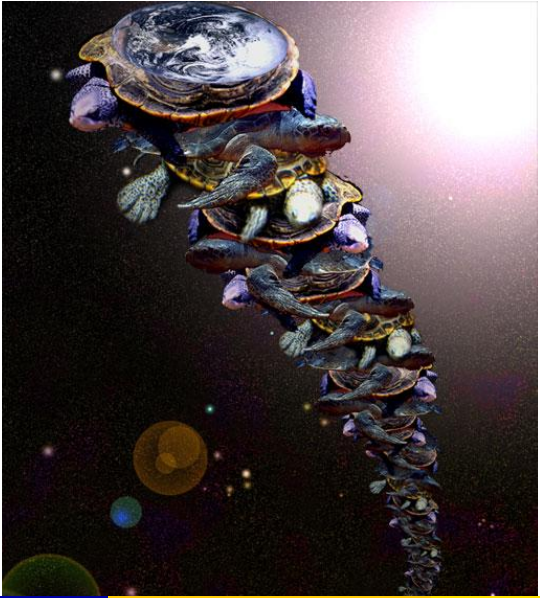
1. Introduction
2. Theory of Computing
3. The λ-Calculus
4. Church Encoding

Introduction

1 Introduction

- Turtles all the way down...
- Programming languages and Functions
- Functions all the way down

Turtles all the way down...



Turtles all the way down...

From Stephen Hawking's *A Brief History of Time*

A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the center of a vast collection of stars called our galaxy.
At the end of the lecture, a little old lady at the back of the room got up and said: "What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise." The scientist gave a superior smile before replying, "What is the tortoise standing on?" "You're very clever, young man, very clever," said the old lady. "But it's *turtles all the way down!*"

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Most "standard" programming languages There is data, and then functions that act on that data.

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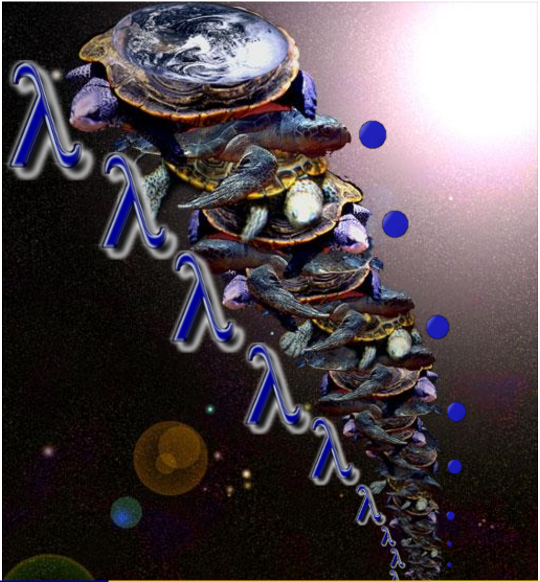
What about replacing data with functions?

Functions all the way down



Theory of Computing

2 Theory of Computing

- The Entscheidungsproblem
- Solutions to the Entscheidungsproblem

The Entscheidungsproblem

The decidability problem

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Given a description of a formal language and a mathematical statement in that language, determine if the statement is true or false.

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Original Statement

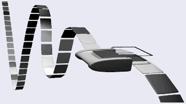
Generalisation of his question "Is mathematics *decidable*".

Solutions to the Entscheidungsproblem

The Halting Problem

January, 1937

On Computable Numbers, with an Application to the Entscheidungsproblem by Alan Turing



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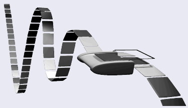

Theorem (Church-Turing Thesis)

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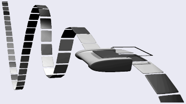

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- *Every effectively calculable function is a computable function.*

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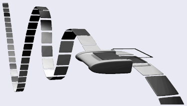

Theorem (Church-Turing Thesis)

- *Every effectively calculable function is a computable function.*
- *λ-Calculus, Turing machines, etc. are equivalent.*

The λ-Calculus

3 The λ-Calculus

- What is the λ-Calculus?
- Definition of the λ-Calculus
- Informal Examples
- Usage of the λ-Calculus

What is the λ-Calculus?

Types and Programming Languages

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- "…a formal system invented in the 1920s by Alonzo Church, in which all computation is reduced to the basic operations of function definition and application."

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Types and Programming Languages

- "...a formal system invented in the 1920s by Alonzo Church, in which all computation is reduced to the basic operations of function definition and application."
- "Its importance arises from the fact that it can be viewed simultaneously as a simple programming language *in which* computations can be described and as a mathematical object *about which* rigorous statements can be proved."

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1970 ...Lambdas are relegated to relative obscurity until Java makes them popular by not having them.

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Also use parentheses for grouping.

Definition of the λ-Calculus

Functions are:

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- Extend as far right as possible

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- Mathematics:

$$(\lambda\, x.\, x + 2)\; 3$$

- Multiple arguments:

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- Factorials:

$$\begin{aligned} g &= \lambda\, f.\, \lambda\, n.\, \begin{cases} 1 & n = 0 \\ n \times f(n-1) & n > 0 \end{cases} \\ Y &= \lambda\, h.\, (\lambda\, x.\, h\,(x\, x))\, (\lambda\, x.\, h\,(x\, x)) \\ \text{fact} &= \lambda\, n.\, Y\, g\, n \end{aligned}$$

Usage of the λ-Calculus

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Anonymous Functions Found in many other languages: C#, C++Ox, Matlab, etc.

Church Encoding

4 Church Encoding

- What is Church Encoding?
- Church Numerals
- Peano Arithmetic
  - Isomorphism between Church and Peano
- Other operations on Church Numerals
- Other Church Encodings
- Representation of Church Encodings
- Are Church Encodings practical?

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Encodings are available for:

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- Booleans
- Pairs
- Lists

Church Numerals

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$$\begin{aligned}
\mathbf{0} &\equiv \lambda f.\, \lambda x.\, x \\
\mathbf{1} &\equiv \lambda f.\, \lambda x.\, f\, x \\
\mathbf{2} &\equiv \lambda f.\, \lambda x.\, f\, (f\, x) \\
\mathbf{3} &\equiv \lambda f.\, \lambda x.\, f\, (f\, (f\, x)) \\
\ldots \\
\mathbf{n} &\equiv \lambda f.\, \lambda x.\, f^n\, x
\end{aligned}$$

where $f^0 = id$, $f^{n+1} = f \cdot f^n$.

Church Numerals

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$$
\begin{aligned}
\mathbf{0} &\equiv \lambda\, f.\, id \\
\mathbf{1} &\equiv \lambda\, f.\, f \\
\mathbf{2} &\equiv \lambda\, f.\, f \cdot f \\
\mathbf{3} &\equiv \lambda\, f.\, f \cdot f \cdot f \\
\ldots & \\
\mathbf{n} &\equiv \lambda\, f.\, f^n
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Exponentiation: $pow\; m\; n = n\; m$

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one :: Nat
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Definition (Fold on Peano Numbers)

```
fold                   :: (a → a) → a → Nat → a
fold succ zero Zero     = zero
fold succ zero (Succ n) = succ (fold succ zero n)
```

Peano Arithmetic

Operations on Nat

```
plus, mult, pow :: Nat → Nat → Nat

plus m n = fold Succ n m

mult m n = fold (add n) Zero m

pow  m n = fold (mult m) one n
```

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Converting between the two

```
type Church a = (a → a) → a → a

nat    :: Church Nat → Nat
nat c = c Succ Zero

church    :: Nat → Church a
church n = λ succ → λ zero → fold succ zero n
```

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Completely different formulation from before!

Other operations on Church Numerals

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. . . but they get very messy very quickly.

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Church Pairs:

$$\mathbf{pair} \equiv \lambda\, f.\, \lambda\, s.\, \lambda\, b.\, b\, f\, s$$
$$\mathbf{fst} \equiv \lambda\, p.\, p\, \mathbf{true}$$
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Church Lists: Too complicated to define here.

Representation of Church Encodings

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Church Numerals: Apply the function f on z a total of n times.

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Church Pairs: Hard-coded Church Booleans.

Church Lists: Combine elements using a combining function.

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OK, I lied: some optimizers, etc. internally use Church Encodings . . . but you should really know what you're doing!

Conclusion