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DIAGONALIZATION AND THE RECURSION THEOREM

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In 1938 Kleene showed that if f is a recursive function then, for some number c, $\varphi_c \simeq \varphi_{f(c)}$, where φ_e is the partial recursive function with index e. Since that time other fixed-point theorems have been found with similar proofs. All of these theorems tend to strain one's intuition; in fact, many people find them almost paradoxical. The most popular proofs of these theorems only serve to aggravate the situation because they are completely unmotivated, seem to depend upon a low combinatorial trick, and are so barbarically short as to be nearly incapable of rational analysis. It is our intention, one, to put Kleene's proof on classically intuitive grounds by explaining how it can be viewed as a natural modification of an ordinary diagonal argument and, two, to present a formulation of Kleene's theorem sufficiently abstract to yield all known similar theorems as corollaries.*

In a typical diagonal argument one has a class of sequences (with terms from a set S), which he arranges as the rows of a square matrix, and a mapping α of S into S. This mapping induces an operation α^* on the class of arbitrary sequences of elements of S in the natural way—if $\langle s(i) : i \in I\rangle$ is such a sequence then $\alpha^*(\langle s(i) : i \in I\rangle) = \langle \alpha(s(i)) : i \in I\rangle$. One then applies α^* to the sequence of diagonal elements of the matrix and shows that the resulting sequence is not a row of the matrix, thus diagonalizing himself out of the class of sequences he began with. A good example is the matrix whose rows are all infinite periodic sequences of 0's and 1's (binary expansions of rationals) with the mapping $\alpha(0) = 1$, $\alpha(1) = 0$.

Usually, as in the example just given, the rows of the matrix are closed under the operation α^*. Hence, if the diagonalization succeeds, it is usually true that the diagonal sequence itself is not one of the rows. But what if the diagonalization fails, that is, what if the diagonal sequence is one of the rows? Then the image of the diagonal sequence under α^* will also be one of the rows, which means some member of the diagonal sequence must be left unchanged under the action of α^*. In other words, α has a fixed point!

To understand Kleene's theorem in these terms, first assume f is a

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recursive function which is well-defined on the partial recursive functions; i.e., assume $\varphi_e \simeq \varphi_{e'} \rightarrow \varphi_{f(e)} \simeq \varphi_{f(e')}$. Let S be the set of partial recursive functions and let the n-th row of our matrix be $\varphi_{\varphi_n(0)}, \varphi_{\varphi_n(1)}, \varphi_{\varphi_n(2)}, \ldots$ where, if $\varphi\ (k)$ is undefined we mean by $\varphi_{\varphi_n(k)}$ the completely undefined partial recursive function. If $\varphi \in S$, define $\alpha(\varphi) = \varphi_{f(e)}$ where e is any index for φ. It is clear that the diagonal sequence is one of the rows (there exists a recursive function h such that $\varphi_{h(e)} \simeq \varphi_{\varphi_e(e)}$ so the diagonal sequence is the a-th row where a is any index for h) and that the rows of our matrix are closed under the induced operation α^*. So, for some number c, $\alpha(\varphi_c) = \varphi_c$; i.e., $\varphi_c \simeq \varphi_{f(c)}$. We can easily compute a value for c. The α^*-image of the diagonal sequence is $\varphi_{f(h(0))}, \varphi_{f(h(1))}, \ldots$; that is, $\varphi_{\varphi_d(0)}, \varphi_{\varphi_d(1)}, \ldots$, where d is any index for the composition of f over h. Thus the d-th term of this sequence is a fixed point; that is, c can be any number with $\varphi_c \simeq \varphi_{\varphi_d(d)}$. Since $\varphi_{h(d)} \simeq \varphi_{\varphi_d(d)}$ we may take $c = h(d)$.

Now suppose f is not well-defined on the partial recursive functions. Then we cannot define α as above; instead, we take α to be a binary relation on S.[1] If θ, ψ are partial recursive functions, we say θ is α-related to ψ if and only if there exists an index e such that θ is φ_e and ψ is $\varphi_{f(e)}$. Then any row of our matrix is α-related to some other row in the sense that each of its terms is α-related to the corresponding term in the other row. So, since the diagonal sequence is one of the rows, it follows immediately that some element of the diagonal sequence is α-related to itself. Thus, for some number c, $\varphi_c \simeq \varphi_{f(e)}$. A moment's reflection reveals that we may once again take c to be $h(d)$.

A much simpler situation is the following. Suppose the multiplication table of a semigroup S has the property that its main diagonal is one of its rows. Then since the rows are closed under multiplication on the left by a fixed element of S, given any $s \in S$ there must exist a $t \in S$ with $st = t$.[2] Two examples of such multiplication tables appear below.

	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

Theorem 1.[3] *Let $\langle S, \circ, *, \cdot, \square, \equiv, \delta\rangle$ be a structure in which S is a set of objects, $\circ$, $*$, and $\cdot$ are binary operations on S, $\square$ is a partial binary operation on S, $\equiv$ is an equivalence relation on S, and δ is a special object in S such that $\delta \square f$ is defined for all $f \in S$. Suppose further that $\delta \square f \equiv f * f$ for all $f \in S$ and $(f \circ g) * h \equiv f \cdot (g \square h)$ whenever $f, g, h \in S$ and $g \square h$ is defined. Then, given any $f \in S$, there exists a $t \in S$ such that $t \equiv f \cdot t$, namely $t = \delta \square (f \circ \delta)$.*

1. This idea is due to Carl Jockusch, Jr.

2. In fact, as was pointed out to the author by Judson Temple, one can take $t = s^3$.

3. The present statement of this theorem profited considerably from a suggestion of E. G. K. Lopez-Escobar.

Proof. $t = \delta \,\Box\, (f \circ \delta) \equiv (f \circ \delta) * (f \circ \delta) \equiv f \cdot (\delta \,\Box\, (f \circ \delta)) = f \cdot t$.

To understand Theorem 1 as a diagonal argument, one supposes there are two multiplications defined on S, $*$ and $\Box$ ($\Box$ may be only a partial multiplication). The equation $\delta \,\Box f \equiv f * f$ says that, modulo the equivalence relation $\equiv$, the diagonal of the multiplication table for $*$ is the "δ-th" row of the multiplication table for $\Box$. The equation $(f \circ g) * h \equiv f \cdot (g \,\Box\, h)$ means that if one applies f to the "g-th" row of the $\Box$-table, the result is, again modulo $\equiv$, the "$f \circ g$-th" row of the $*$-table. It follows that, given any $f \in S$, there exists a term t of the "δ-th" row of the $\Box$-table such that $f \cdot t \equiv t$. For the result of applying f to that row is, up to equivalence, the "$f \circ \delta$-th" row of the $*$-table. So, since the "δ-th" $\Box$-row is, modulo $\equiv$, the $*$-diagonal, the "$f \circ \delta$-th" $*$-row must intersect the $*$-diagonal in an element equivalent to the corresponding term of the "δ-th" $\Box$-row. Hence, f must not map the "$f \circ \delta$-th" term of the "δ-th" $\Box$-row outside its own equivalence class; i.e., $f \cdot t \equiv t$ where $t \equiv \delta \,\Box\, (f \circ \delta)$. Notice that, although $(f \circ \delta) * (f \circ \delta) \equiv \delta \,\Box\, (f \circ \delta)$, $(f \circ \delta) * (f \circ \delta)$ need not be a fixed point of f; nowhere did we assume that $\cdot$ (or, for that matter, any of the binary operations) was well-defined with respect to $\equiv$.

We give five applications. The first four are known theorems; the fifth is included as an illustration of the generality of Theorem 1. In each application some of the operations $\circ, *, \cdot, \Box$ are identified. We do not know of a reasonable application in which all four operations are distinct. In our discussion of semigroups preceding Theorem 1 all four operations were the same; the equation $(f * g) * h \equiv f * (g * h)$ is just the associative law.

In Applications 1, 2, and 4 the reader will notice that the operations $*$, $\cdot$, and $\Box$ are identified. In this case, one should think of S as a collection of names or indices for functions mapping S into S. $\circ$ is a composition and $*$ is evaluation. The assumption $(f \circ g) * h \equiv f * (g * h)$ is simply the definition of composition; δ is the "self-evaluation" map. Let N be the set of non-negative integers for each application.

Application 1 (Kleene's fixed-point theorem for Church's λ-calculus). Let S be the set of all terms of the λ-calculus and let $\equiv$ denote λ-convertibility. If F, G are terms define $F * G = F \cdot G = F \,\Box\, G = (FG)$, $F \circ G = \lambda x(F(Gx))$. Let δ be $\lambda x(xx)$. Then $\delta \,\Box\, G = (\delta G) = (\lambda x(xx)G) \equiv (GG) = G * G$ and $(F \circ G) * H = (F \circ GH) = (\lambda x(F(Gx))H) \equiv (F(GH)) = F \cdot (G \,\Box\, H)$. Applying Theorem 1 we find for every term F there exists a term T such that $(FT) = T$. Namely, $T = (\lambda x(xx)\lambda y(F(\lambda x(xx)y)))$. Notice that T does not have a normal form, in the sense of Church.

Application 2 (Kleene's recursion theorem [1, p. 352]). Identify $*$, $\cdot$, and $\Box$. Let S be N and let $\psi_0(u, v), \psi_1(u, v), \psi_2(u, v), \ldots$ be a standard enumeration of all partial recursive functions of one or two arguments (here u may be a dummy variable). Let $s(m,n)$, $t(m,n)$ be recursive functions such that, for all $m, n \in N$, $\psi_{s(m,n)}(v) \simeq \psi_m(n, v)$, $\psi_{t(m,n)}(u, v) \simeq \psi_m(s(n, u), v)$. If $m, n \in N$ define $m * n = s(m,n)$, $m \circ n = t(m,n)$. Let $m \equiv n$ denote $\psi_m \simeq \psi_n$ and let δ be an integer such that $\psi_\delta(u, v) \simeq \psi_u(u, v)$ for all u, v. We find $\psi_{\delta \Box n}(v) \simeq \psi_{s(\delta, n)}(v) \simeq \psi_\delta(n, v) \simeq \psi_n(n, v) \simeq \psi_{s(n,n)}(v) \simeq \psi_{n*n}$ so that $\delta \,\Box\, n \equiv n * n$. Also, $\psi_{(m \circ n)*p}(v) \simeq \psi_{s(m \circ n, p)}(v) \simeq \psi_{m \circ n}(p, v) \simeq \psi_{t(m,n)}(p, v) \simeq \psi_m(s(n, p),$

$v) \simeq \psi_m(n \,\square\, p, v) \simeq \psi_{s(m, n \square p)}(v) \simeq \psi_{m \cdot (n \square p)}(v)$ so that $(m \circ n) * p \equiv m \cdot (n \,\square\, p)$. So, by Theorem 1, given a partial recursive function $\psi_a(u,v)$ there exists an integer c with $\psi_c(v) \simeq \psi_a(c, v)$, namely $c = \delta \,\square\, (a \circ \delta) = s(\delta, t(a, \delta))$.

Application 3 (Roger's version of Kleene's recursion theorem [2, p. 180]). Let S be N and, for $e \in S$, let φ_e be the partial recursive function of one argument having Gödel number e. Let $h(n, m)$ be a recursive function such that $\varphi_{h(n,m)} \simeq \varphi_{\varphi_n(m)}$ if $\varphi_n(m)$ is defined and such that $\varphi_{h(n,m)}$ is totally undefined if $\varphi_n(m)$ is not defined. Let $g(n, m)$ be a recursive function such that $\varphi_{g(n,m)}(v)$ is undefined unless $\varphi_m(v)$ and $\varphi_n(\varphi_m(v))$ are both defined, in which case $\varphi_{g(n,m)}(v) = \varphi_n(\varphi_m(v))$. If $n, m \in N$ let $n * m = n \cdot m = h(n, m)$ and $n \circ m = g(n, m)$. If $\varphi_n(m)$ is defined, set $n \,\square\, m = \varphi_n(m)$; otherwise, $n \,\square\, m$ is not defined. Let $n \equiv m$ mean $\varphi_n \simeq \varphi_m$ and let δ be an integer such that $\varphi_\delta(n) = h(n, n)$ for all n. Then $\varphi_{\delta \square n} \simeq \varphi_{\varphi_\delta(n)} \simeq \varphi_{h(n,n)} \simeq \varphi_{n*n}$, so $\delta \,\square\, n. \equiv n * n$. Also $\varphi_{(n \circ m)*p} \simeq \varphi_{h(n \circ m, p)} \simeq \varphi_{\varphi_{n \circ m}(p)} \simeq \varphi_{\varphi_{g(n,m)}(p)} \simeq \varphi_{\varphi_n(\varphi_m(p))} \simeq \varphi_{\varphi_n(m \square p)} \simeq \varphi_{h(n, m \square p)} \simeq \varphi_{n \cdot (m \square p)}$ if $\varphi_n(m \,\square\, p)$ is defined, $\varphi_{(n \circ m)*p}$ and $\varphi_{n \cdot (m \square p)}$ are totally undefined if not, so $(n \circ m) * p \equiv n \cdot (m \,\square\, p)$ whenever $m \,\square\, p$ is defined. So, given any recursive function f, there exists a number c such that $\varphi_{f(c)} \simeq \varphi_c$; namely, $c = \varphi_\delta(d)$ where $\varphi_d \simeq f \circ \varphi_\delta$ (that is, $c = \delta \,\square\, (a \circ \delta) = \varphi_\delta(g(a, \delta)) = h(g(a, \delta), g(a, \delta))$, where $\varphi_a \simeq f$). Since h and g are recursive, a fixed point c can be found effectively from any Gödel number a of f.

Application 4 (Feferman's fixed-point theorem for elementary number theory). Identify $*$, $\cdot$, and $\square$. Let $S = \{\Phi_0, \Phi_1, \Phi_2, \ldots\}$ where $\Phi_0, \Phi_1, \Phi_2, \ldots$ is the customary enumeration of all formulas of elementary number theory with at most one free variable v (cf. [1, §52]) and, if Ψ is such a formula, let $\ulcorner\Psi\urcorner = e$ where $\Psi = \Phi_e$. If $\Phi, \Psi \in S$ let $\Phi \circ \Psi$ be the formula $(\mathrm{E}u)(\Phi(u)\ \&\ A(\ulcorner\Psi\urcorner, u, v))$ in which A is a formula such that, for any formula Ψ and any $n, m \in N$, $\vdash A(\ulcorner\underline{\Psi}\urcorner, \mathsf{n}, \mathsf{m})$ iff $n = \ulcorner\underline{\Psi(\mathsf{m})}\urcorner$ (whenever a formula Φ has no free variables $\Phi(v)$ and $\Phi(\mathsf{n})$ are to be interpreted as Φ). If $\Phi, \Psi \in S$ let $\Phi * \Psi$ be $\Phi(\ulcorner\underline{\Psi}\urcorner)$ and let $\Phi \equiv \Psi$ mean $\vdash \Phi \equiv \Psi$. Let δ be a formula such that, for any $n \in N$, $\vdash \delta(\mathsf{n}) \equiv \Phi_n(\mathsf{n})$; it is well-known that δ exists (cf. [1, p. 206, Lemma 21]). We have $\delta \,\square\, \Psi = \delta(\ulcorner\underline{\Psi}\urcorner) \equiv \Psi(\ulcorner\underline{\Psi}\urcorner) = \Psi * \Psi$, $(\Phi \circ \Psi) * \theta \equiv \Phi(\ulcorner\underline{\Psi(\ulcorner\underline{\theta}\urcorner)}\urcorner) = \Phi \cdot (\Psi \,\square\, \theta)$, for all $\Phi, \Psi, \theta \in S$. So, by Theorem 1, given any formula Φ there exists a formula θ such that $\Phi(\ulcorner\underline{\theta}\urcorner) \equiv \theta$. Namely, $\theta = \delta \,\square\, (\Phi \circ \delta) = \delta(\ulcorner(\mathrm{E}u)(\Phi(u)\ \&\ A(\ulcorner\underline{\delta}\urcorner, u, v))\urcorner)$. Notice that θ is a sentence; i.e., θ has no free variables.

Application 5. Let S be the set of all partial recursive functions of one variable and, for $e \in N$, let φ_e be as in Application 3. For each $\psi \in S$ choose a number $\ulcorner\psi\urcorner \in N$ such that $\varphi_{\ulcorner\psi\urcorner} \simeq \psi$. Identify $\cdot$ with $\circ$ and $\square$ with $*$. When $n \in N$ we shall denote by $\lambda u n$ the constant function whose value for any argument is n. If $\varphi, \psi \in S$ let $\varphi \circ \psi(u) = \varphi(\psi(u))$ if both $\psi(u)$ and $\varphi(\psi(u))$ are defined; otherwise, let $\varphi \circ \psi(u)$ be undefined. Define $\varphi * \psi = \lambda u(\varphi(\ulcorner\psi\urcorner))$ if $\varphi(\ulcorner\psi\urcorner)$ is defined; otherwise, let $\varphi * \psi$ be φ_0, the completely undefined function. Let $\equiv$ be $\simeq$ and let $\delta \in S$ be such that $\delta(u) = \varphi_u(u)$ if $\varphi_u(u)$ is defined, $\delta(u)$ is undefined otherwise. We find $\delta \,\square\, \psi \simeq \lambda u(\delta(\ulcorner\psi\urcorner)) \simeq \lambda u(\varphi_{\ulcorner\psi\urcorner}(\ulcorner\psi\urcorner)) \simeq \lambda u(\psi(\ulcorner\psi\urcorner)) \simeq \psi * \psi$ if $\psi(\ulcorner\psi\urcorner)$ is defined; otherwise, $\delta \,\square\, \psi \simeq \psi * \psi \simeq \varphi_0$. Hence $\delta \,\square\, \psi \equiv \psi * \psi$ for all $\psi \in S$. Also $(\varphi \circ \psi) * \theta \simeq \lambda u(\varphi(\psi(\ulcorner\theta\urcorner))) \simeq \varphi \cdot \lambda u(\psi(\ulcorner\theta\urcorner)) \simeq \varphi \cdot (\psi \,\square\, \theta)$ if $\psi(\ulcorner\theta\urcorner)$ and $\varphi(\psi(\ulcorner\theta\urcorner))$ are both defined; otherwise,

$(\varphi \circ \psi) * \theta \simeq \varphi \cdot (\psi \,\Box\, \theta) \simeq \varphi_0$. Hence $(\varphi \circ \psi) * \theta \equiv \varphi \cdot (\psi \,\Box\, \theta)$ for all $\varphi, \psi, \theta \in S$. So, given any $\psi \in S$ there exists a $\theta \in S$ such that $\psi \circ \theta \simeq \theta$; namely, $\theta \simeq \lambda u(\delta(\ulcorner\psi \circ \delta\urcorner))$ if $\delta(\ulcorner\psi \circ \delta\urcorner)$ is defined and $\theta \simeq \varphi_0$ if not.

Actually, the completely undefined function φ_0 is always a fixed point, but it is not true that the fixed point θ given above is always φ_0. For example consider the case ψ is the identity function. Then $\ulcorner\psi \circ \delta\urcorner$ is just $\ulcorner\delta\urcorner$, so whether or not θ is φ_0 depends upon whether or not $\delta(\ulcorner\delta\urcorner)$ is defined. Now the function $\rho \in S$ determined by the conditions $\rho(u) = \varphi_u(u) + 1$ if $\varphi_u(u)$ is defined, $\rho(u)$ is undefined otherwise, is, of course, not defined at $\ulcorner\rho\urcorner$ no matter what choice one makes for $\ulcorner\rho\urcorner$. Therefore, one might suspect that δ has the same property. Not so! There exist recursive functions f, f_k $(k \geq 0)$ such that

$$\varphi_{f(e)}(u) = \begin{cases} \varphi_u(u) \text{ if } \varphi_u(u) \text{ is defined and } u \neq e \\ \text{undefined otherwise} \end{cases}$$

$$\varphi_{f_k(e)}(u) = \begin{cases} \varphi_u(u) \text{ if } \varphi_u(u) \text{ is defined and } u \neq e \\ k \text{ if } u = e \\ \text{undefined otherwise} \end{cases}$$

and by Application 3 one can find integers c, c_k $(k \geq 0)$ with $\varphi_c \simeq \varphi_{f(c)}$, $\varphi_{c_k} \simeq \varphi_{f_k(c_k)}$. Hence, $\varphi_c \simeq \varphi_{c_k} \simeq \delta$, but $\varphi_c(c)$ is undefined while $\varphi_{c_k}(c_k) = k$. Thus, at least in the case ψ is the identity function, one can obtain any solution θ of the equation $\psi \circ \theta \simeq \theta$, where θ is a constant function or is completely undefined, simply by varying one's choice of $\ulcorner\delta\urcorner$.

The main virtue of Application 5 is that it, together with Application 3, demonstrates the necessity of having four operations in the statement of Theorem 1. Otherwise, not every "Kleene-like" argument would be a corollary. Actually, Theorem 1 can be further strengthened by not requiring $\circ$, $*$, and $\cdot$ to be total operations, but this seems purely academic.

Many people believe that something akin to self-reference must be inherent in a situation before one can apply Theorem 1. We share this impression but are not sure how to make it precise. The equation $\delta \,\Box f = f * f$ says there is an object δ which is capable of "squaring" all the others (including itself), but we do not consider this just cause for calling δ self-referential. On the other hand, it is an easy corollary of Application 3 that, for some e, the range of φ_e is $\{e\}$, which clearly suggests that e is able to talk about itself.

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