Finite Model Theory Tutorial

Lecture 1

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Finite Model Theory

In the 1980s, the term *finite model theory* came to be used to describe the study of the expressive power of logics (from first-order to second-order logic and in between), on the class of all finite structures.

The motivation for the study is that problems in computer science (especially in *complexity theory* and *database theory*) are naturally expressed as questions about the expressive power of logics.

And, the structures involved in computation are finite.

Model Theoretic Questions

The kind of questions we are interested in are about the *expressive power* of logics. Given a formula φ, its class of models is the collection of *finite* relational structures $\mathbb{A}$ in which it is true.

$$\mathrm{Mod}(\varphi) = \{\mathbb{A} \mid \mathbb{A} \models \varphi\}$$

What classes of structures are definable in a given logic $\mathcal{L}$?

How do syntactic restrictions on φ relate to semantic restrictions on $\mathrm{Mod}(\varphi)$?

How does the computational complexity of $\mathrm{Mod}(\varphi)$ relate to the syntactic complexity of φ?

Descriptive Complexity

A class of finite structures is definable in existential second-order logic if, and only if, it is decidable in NP.

(Fagin)

A closs of *ordered* finite structures is definable in least fixed-point logic if, and only if, it is decidable in P.

(Immerman; Vardi)

Open Question: Is there a logic that captures P without order?

Can *model-theoretic* methods cast light on questions of computational complexity?

Compactness and Completeness

The *Compactness Theorem* fails if we restrict ourselves to finite structures.

The *Completeness Theorem* also fails:

> **Theorem (Trakhtenbrot 1950)**
>
> The set of finitely valid sentences is not *recursively enumerable*.

Various preservation theorems *(Łoś-Tarski, Lyndon)* fail when restricted to finite structures.

The finitary analogues of *Craig Interpolation Theorem* and the *Beth Definability Theorem* also fail.

Tools for Finite Model Theory

It seems that the class of finite structures is not well-behaved for the study of definability.

What *tools and methods* are available to study the expressive power of logic in the finite?

- *Ehrenfeucht-Fraïssé Games* (reviewed in this lecture);
- *Locality Theorems* (examined in Lecture 2);
- *Complexity* (the topic of Lectures 3 and 4);
- *Asymptotic Combinatorics* (later in this lecture and again in Lecture 5).

Elementary Equivalence

On finite structures, the elementary equivalence relation is trivial:

$\mathbb{A} \equiv \mathbb{B}$ if, and only if, $\mathbb{A} \cong \mathbb{B}$

Given a structure $\mathbb{A}$ with n elements, we construct a sentence

$$\varphi_{\mathbb{A}} = \exists x_1 \ldots \exists x_n \psi \wedge \forall y \bigvee_{1 \leq i \leq n} y = x_i$$

where, $\psi(x_1, \ldots, x_n)$ is the conjunction of all atomic and negated atomic formulas that hold in $\mathbb{A}$.

Theories vs. Sentences

First order logic can make all the distinctions that are there to be made between finite structures.

Any isomorphism closed class of finite structures S can be defined by a *first-order theory*:

$$\{\neg\varphi_{\mathbb{A}} \mid \mathbb{A} \notin S\}.$$

To understand the limits on the expressive power of *first-order sentences*, we need to consider coarser equivalence relations than $\equiv$.

We will also be interested in the expressive power of logics extending first-order logic. This amounts to studying theories satisfying a weaker *axiomatisibality* requirement than *finite axiomatisability*.

Quantifier Rank

The *quantifier rank* of a formula φ, written $\mathrm{qr}(\varphi)$ is defined inductively as follows:

1. if φ is atomic then $\mathrm{qr}(\varphi) = 0$,
2. if $\varphi = \neg\psi$ then $\mathrm{qr}(\varphi) = qr(\psi)$,
3. if $\varphi = \psi_1 \vee \psi_2$ or $\varphi = \psi_1 \wedge \psi_2$ then $\mathrm{qr}(\varphi) = \max(\mathrm{qr}(\psi_1), \mathrm{qr}(\psi_2))$.
4. if $\varphi = \exists x \psi$ or $\varphi = \forall x \psi$ then $\mathrm{qr}(\varphi) = \mathrm{qr}(\psi) + 1$

Note: For the rest of this lecture, we assume that our signature consists only of relation and constant symbols.

With this proviso, it is easily proved that in a finite vocabulary, for each q, there are (up to logical equivalence) only finitely many sentences φ with $\mathrm{qr}(\varphi) \leq q$.

Finitary Elementary Equivalence

For two structures $\mathbb{A}$ and $\mathbb{B}$, we say $\mathbb{A} \equiv_p \mathbb{B}$ if for any sentence φ with $\mathrm{qr}(\varphi) \leq p$,

$$\mathbb{A} \models \varphi \text{ if, and only if, } \mathbb{B} \models \varphi.$$

Key fact:

a class of structures S is definable by a first order sentence if, and only if, S is closed under the relation $\equiv_p$ for some p.

In a *finite* relational vocabulary, for any structure $\mathbb{A}$ there is a sentence $\theta^p_{\mathbb{A}}$ such that

$$\mathbb{B} \models \theta^p_{\mathbb{A}} \quad \text{if, and only if,} \quad \mathbb{A} \equiv_p \mathbb{B}$$

Partial Isomorphisms

The equivalence relations $\equiv_p$ can be characterised in terms of sequences of partial isomorphisms

(Fraïssé 1954)

or two player games.

(Ehrenfeucht 1961)

$$\mathbb{A} \equiv_p \mathbb{B}$$

if, and only if, there is a sequence

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_p$$

of non-empty sets of partial isomorphisms from $\mathbb{A}$ to $\mathbb{B}$ such that

for each $f \in I_{i+1}, a \in \mathbb{A}$ and $b \in \mathbb{B}$, there are:

- $f' \in I_i$ such that $f \subseteq f'$ and $a \in \mathrm{dom}(f')$
- $f'' \in I_i$ such that $f \subseteq f''$ and $b \in \mathrm{rng}(f'')$

Ehrenfeucht-Fraïssé Game

The p-round Ehrenfeucht game on structures $\mathbb{A}$ and $\mathbb{B}$ proceeds as follows:

- There are two players called Spoiler and Duplicator.
- At the ith round, Spoiler chooses one of the structures (say $\mathbb{B}$) and one of the elements of that structure (say b_i).
- Duplicator must respond with an element of the other structure (say a_i).
- If, after p rounds, the map $a_i \mapsto b_i$ is a partial isomorphism, then Duplicator has won the game, otherwise Spoiler has won.

Theorem (Fraïssé 1954; Ehrenfeucht 1961)

Duplicator has a strategy for winning the p-round Ehrenfeucht game on $\mathbb{A}$ and $\mathbb{B}$ if, and only if, $\mathbb{A} \equiv_p \mathbb{B}$.

Using Games

To show that a class of structures S is not definable in FO, we find, for every p, a pair of structures $\mathbb{A}_p$ and $\mathbb{B}_p$ such that

- $\mathbb{A}_p \in S$, $\mathbb{B}_p \in \overline{S}$; and
- *Duplicator* wins a p round game on $\mathbb{A}_p$ and $\mathbb{B}_p$.

Examples:

- Sets of *even cardinality*;
- *Linear orders* of even cardinality;
- *2-colourable* graphs;
- *connected* graphs.

Undefinability in First-Order Logic

For many of these examples, one can show undefinability by more classical methods.

Let QFLO be the theory consisting of sentences:

- $<$ is a linear order with end points;
- every element except the minimal one has a unique predecessor;
- every element except the maximal one has a unique successor;
- for each n: there are at least n elements.

We can show QFLO is *complete* by showing a countably saturated model.

If φ was a sentence defining evenness, both QFLO $\cup \{\varphi\}$ and QFLO $\cup \{\neg\varphi\}$ would be *consistent*.

Asymptotic Probabilities

Let S be any isomorphism closed class of σ-structures (where σ is a *finite, relational* signature).

Let C_n be the set of all σ-structures whose universe is $\{0, \ldots, n-1\}$.

We define $\mu_n(S)$ as:

$$\mu_n(S) = \frac{|S \cap C_n|}{|C_n|}$$

The asymptotic probability, $\mu(S)$, of S is defined as

$$\mu(S) = \lim_{n\to\infty} \mu_n(S)$$

if this limit exists.

Asymptotic Probabilities

Many interesting properties (of graphs, for instance) have asymptotic probability either 0 or 1.

- $\mu(\text{connectivity}) = 1$
- $\mu(3\text{-colourability}) = 0$
- $\mu(\text{planarity}) = 0$
- $\mu(\text{Hamiltonicity}) = 1$
- $\mu(\text{rigidity}) = 1$
- $\mu(k\text{-clique}) = 1$ for fixed k

By contrast, $\mu(\text{even number of nodes})$ is not defined and $\mu(\text{even number of edges}) = 1/2$.

0–1 Law

Theorem (Glebskĭi *et al.* 1969; Fagin 1974)
For every first order sentence in a relational signature φ, $\mu(\mathrm{Mod}(\varphi))$ is defined and is either 0 or 1.

This provides a very general result on the limits of first order definability.

Compare the results (recall *La Roche tutorial* to the effect that *even cardinality* is not definable in first-order logic

in the language of *equality*;

in the language of *linear order*.

Extension Axioms

Given a *relational* signature σ,

> an atomic type $\tau(x_1, \ldots, x_k)$ is the conjunction of a maximally consistent set of atomic and negated atomic formulas.

Let $\tau(x_1, \ldots, x_k)$ and $\tau'(x_1, \ldots, x_{k+1})$ be two atomic types such that τ' is consistent with τ.

The τ, τ'-extension axiom is the sentence:

$$\forall x_1 \ldots \forall x_k \exists x_{k+1} (\tau \rightarrow \tau').$$

Asymptotic Probability of Extension Axioms

Fact:

For any extension axiom $\eta_{\tau,\tau'}$, $\mu(\mathrm{Mod}(\eta_{\tau,\tau'})) = 1$.

Proof Idea:

- Given a σ-structure $\mathbb{A}$ of size n, and a k-tuple a in $\mathbb{A}$ satisfying τ, there is a probability $\sim \frac{1}{\alpha^n}$ that there is no extension of a satisfying τ'.
- There are roughly $\sim \frac{n^k}{\beta}$ tuples in $\mathbb{A}$ satisfying τ.
- The expected number of counterexamples to η in $\mathbb{A}$ is $\sim \gamma \frac{n^k}{\alpha^n}$.
- The probability that $\mathbb{A}$ satisfies η goes to 1 as n grows.

where α, β and γ are constants.

Finite Collections of Extension Axioms

Note:

If Δ is a finite set of extension axioms, then

$$\mu(\mathrm{Mod}(\Delta)) = 1.$$

For any finite set of classes $S_1, \ldots, S_m$, each of asymptotic probability 1,

$$\mu(S_1 \cap \ldots \cap S_m) = 1.$$

The Gaifman Theory

Let Γ be the set of all extension axioms for a fixed signature σ.

By compactness, Γ has a model (albeit an infinite one).

Moreover, Γ is *complete*.

To prove this, we show If $\mathbb{A} \models \Gamma$ and $\mathbb{B} \models \Gamma$, then, for every p,

$$\mathbb{A} \equiv_p \mathbb{B}.$$

The extension axioms gurantee that Duplicator has a response to any move by Spoiler.

Thus, any two models of Γ are elementarily equivalent.

Proof of 0–1 Law

Let φ be any σ-sentence.

By completeness of Γ, either

$$\Gamma \models \varphi \quad \text{or} \quad \Gamma \models \neg \varphi.$$

By *compactness*, in the first case, there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models \varphi$.

Since $\mu(\mathrm{Mod}(\Delta)) = 1$, it follows that $\mu(\mathrm{Mod}(\varphi)) = 1$.

Similarly, in the second case, $\mu(\mathrm{Mod}(\neg\varphi)) = 1$, and therefore $\mu(\mathrm{Mod}(\varphi)) = 0$.