STUDIES IN LOGIC, GRAMMAR AND RHETORIC 5 (18) 2002

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ON PROOFS OF THE CONSISTENCY OF ARITHMETIC

1. The main aim and purpose of Hilbert's programme was to defend the integrity of classical mathematics (refering to the actual infinity) by showing that it is safe and free of any inconsistencies. This problem was formulated by him for the first time in his lecture at the Second International Congress of Mathematicians held in Paris in August 1900 (cf. Hilbert, 1901). Among twenty three problems Hilbert mentioned under number 2 the problem of proving the consistency of axioms of arithmetic (under the name "arithmetic" Hilbert meant number theory and analysis).

Hilbert returned to the problem of justification of mathematics in lectures and papers, especially in the twentieth[1], where he tried to describe and to explain the problem more precisely (in particular the methods allowed to be used) and simultaneously presented the partial solutions obtained by his students.

Hilbert distinguished between the unproblematic, finitistic part of mathematics and the infinitistic part that needed justification. Finitistic mathematics deals with so called real sentences, which are completely meaningful because they refer only to given concrete objects. Infinitistic mathematics on the other hand deals with so called ideal sentences that contain reference to infinite totalities. It should be justified by finitistic methods – only they can give it security (*Sicherheit*). Hilbert proposed to base mathematics on finitistic mathematics via proof theory (*Beweistheorie*). It should be shown that proofs which use ideal elements in order to prove results in the real part of mathematics always yield correct results, more exactly, that (1) finitistic mathematics is conservative over finitistic mathematics with respect to real sentences and (2) the infinitistic

[1] More information on this can be found for example in (Mancosu, 1998).

ISBN 83-89031-54-X ISSN 0860-150X

mathematics is consistent. This should be done by using finitistic methods only.

2. It seems that the first result in this direction was obtained by Wilhelm Ackermann in 1924. In his paper "Begründung des 'tertium non datur' mittels der Hilbertschen Theorie der Widerspruchsfreiheit" (cf. Ackermann, 1924) Ackermann gave a finitistic proof of the consistency of arithmetic of natural numbers without the axiom (scheme) of induction. In fact it was a much weaker system than the usual systems of arithmetic but the paper provided the first attempt to solve the problem of consistency. Add that Ackermann used in (1924) a formalism with Hilbert's ε-functions.

3. Next attempt to solve the second Hilbert's problem was the paper by Janos (later Johann, John) von Neumann "Zur Hilbertschen Beweistheorie" published in 1927. He used another formalism than that in (Ackermann, 1924) and, similarly as Ackermann, proved in fact the consistency of a fragment of arithmetic of natural numbers obtained by putting some restrictions on the induction. It is worth mentioning here that in the introductory section of von Neumann's paper a nice and precise formulation of aims and methods of Hilbert's proof theory was given. It indicated how was at that time the state of affairs and how Hilbert's programme was understood. Therefore we shall quote the appropriate passages.

Von Neumann writes that the essential tasks of proof theory are (cf. von Neumann, 1927, 256–257):

I. First of all one wants to give a proof of the consistency of the classical mathematics. Under 'classical mathematics' one means the mathematics in the sense in which it was understood before the begin of the criticism of set theory. All settheoretic methods essentially belong to it but not the proper abstract set theory. [...]

II. To this end the whole language and proving machinary of the classical mathematics should be formalized in an absolutely strong way. The formalism cannot be too narrow.

III. Then one must prove the consistency of this system, i.e., one should show that certain formulas of the formalism just described can never be "proved".

IV. One should always strongly distinguish here between various types of "proving": between formal ("mathematical") proving in a given formal system and contents ("metamathematical") proving [of statements] about the system. Whereas the former one is an arbitrarily defined logical game (which should to a large extent be analogues to the classical mathematics), the latter is a chain of directly evident contents insights. Hence this "contents proving" must proceed according to the intuitionistic logic of Brouwer and Weyl. Proof theory should so to speak construct classical mathematics on the intuitionistic base and in this way lead the strict intuitionism ad absurdum[2].

Note that von Neumann identifies here finitistic methods with intuitionistic ones. This was then current among members of the Hilbert's school. The distinction between those two notions was to be made explicit a few years later – cf. (Hilbert and Bernays, 1934, pp. 34 and 43) and (Bernays 1934, 1935, 1941).

4. In 1930 Kurt Gödel obtain a result which undermined Hilbert's programme. Gödel proved that any consistent theory extending the arithmetic of natural numbers and based on a recursive set of axioms is incomplete (this result is called today Gödel's First Incompleteness Theorem). This result was announced for the first time by Gödel during a conference in Königsberg in September 1930. It seems that the only participant of the conference in Königsberg who immediately grasped the meaning of Gödel's theorem and understood it was J. von Neumann. After Gödel's talk he had a long discussion with him and asked him about details of the proof. Soon after coming back from the conference to Berlin he wrote a letter to Gödel (on 20th November 1930) in which he announced that he had received a remarkable corollary from Gödel's First Theorem, namely a theorem on the unprovability of the consistency of arithmetic in arithmetic itself. In the meantime Gödel developed his Second Incompleteness Theorem

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[2] I. In erster Linie wird der Nachweis der Widerspruchsfreiheit der klassischen Mathematik angestrebt. Unter „klassischer Mathematik" wird dabei die Mathematik in demjenigen Sinne verstanden, wie sie bis zum Auftreten der Kritiker der Mengenlehre anerkannt war. Alle mengentheoretischen Methoden gehören im wesentlichen zu ihr, nicht aber die eigentliche abstrakte Mengenlehre. [...]

II. Zu diesem Zwecke muß der ganze Aussagen- und Beweisapparat der klassischen Mathematik absolut streng formalisiert werden. Der Formalismus darf keinesfalls zu eng sein.

III. Sodann muß die Widerspruchsfreiheit dieses Systems nachgewiesen werden, d.h. es muß gezeigt werden, daß gewisse Aussagen „Formeln" innerhalb des beschriebenen Formalismus niemals „bewiesen" werden können.

IV. Hierbei muß stets scharf zwischen verschiedenen Arten des „Beweisens" unterschieden werden: Dem formalistischen („mathematischen") Beweisen innerhalb des formalen Systems, und dem inhaltlichen („metamathematischen") Beweisen über das System. Während das erstere ein willkürlich definiertes logisches Spiel ist (das freilich mit der klassischen Mathematik weitgehend analog sein muß), ist das letztere eine Verkettung unmittelbar evidenter inhaltlicher Einsichten. Dieses „inhaltliche Beweisen" muß also ganz im Sinne der Brouwer-Weylschen intuitionistischen Logik verlaufen: Die Beweistheorie soll sozusagen auf intuitionistischer Basis die klassische Mathematik aufbauen und den strikten Intuitionismus so ad absurdum führen.

and included it in his paper "Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme. I" (cf. Gödel, 1931). In this situation von Neumann decided to leave the priority of the discovery to Gödel.

In fact in (Gödel, 1931) one finds only a statement of the theorem on the unprovability of consistency (called today Gödel's Second Incompleteness Theorem) and a remark that it can be proved by formalizing the proof of the first theorem. Gödel promissed also there to publish the full proof in the second part of the paper which would be ready soon. But this second part was never written and Gödel published in fact no proof of his second theorem. Moreover, his remark on the proof was not correct. The first proof of the theorem on the unprovability of consistency appeared in the second volume of Hilbert and Bernay's monograph *Grundlagen der Mathematik* (1939). It has turned out that the way in which the metamathematical sentence "the theory T is consistent" is formalized in the formal language of T is significant here. Hilbert and Bernays formulated certain so called derivability conditions for formulas representing in T the metamathematical notion of provability in T (in fact those conditons require certain internal properties of provability to be formally derivable in T). If those conditions are fulfilled then the second incompleteness theorem holds.

Hilbert-Bernay's conditions were not elegant. A useful and elegant form of them was given by M. H. Löb in 1954 (cf. Löb, 1955). It was also shown that there exist formal translations of the sentence "T is consistent" which are provable in T and for which the second incompleteness theorem fails. Examples of such formulas were given by J. B. Rosser and A. Mostowski[3].

Those results weakened in a sense (the metamathematical and philosophical meaning of) Gödel's Second Incompleteness Theorem. In fact this theorem does not say simply that Peano arithmetic, if consistent, cannot prove its own consistency (and similarly for any consistent extension of it). It turns out that the way in which the metamathematical property of consistency is expressed in the language of the considered theory plays here the crucial role. The crude numerical adequacy in the sense of strong representability is not enough here – one needs in fact that the formal representation "reflects" the very structure of the notion of provability (cf. Feferman, 1960). Nevertheless Gödel's theorem indicated certain limitations of formalized systems and showed that certain corrections in Hilbert's programme are necessary.

[3] For technical as well as philosophical and historical information on Gödel's theorems see, e.g., (Murawski, 1999).

In spite of those new circumstances Hilbert defended the very idea of his programme. In the Preface to the first volume of *Grundlagen der Mathematik* he wrote:

> [...] the occasionally held opinion that from the results of Gödel follows the non-executability of my Proof Theory, is shown to be erroneous. This result shows indeed only that for more advanced consistency proofs one must use the finite standpoint in a deeper way than is necessary for the consideration of elementary formalisms[4].

5. Through von Neumann about Gödel's incompleteness theorems learned (in November 1930) Jacques Herbrand. He found them to be of great interest. They also stimulated him to reflect on the nature of intuitionistic proofs and of schemes for the recursive definition of functions. In a letter to Gödel of 7th April 1931 Herbrand suggested the idea of extending the schemes for the recursive definition of functions. His remarks inspired Gödel to formulate the notion of general recursive function (in the lectures he gave at Princeton in 1934 – cf. Gödel, 1934).

From the point of view of the present paper however more important is Herbrand's paper "Sur la non-contradiction de l'arithmétique" published in 1931 already after the Gödel's "Über formal unentscheidbare Sätze...". Herbrand probably started to write his paper before Gödel's paper reached him (the manuscript sent for publication to the *Journal für reine und angewandte Mathematik* was dated "Göttingen, 14 July 1931"; it was sent just before Herbrand left for a vacation trip in the Alps, and was received on 27 July 1931 – on that day Herbrand was killed in a fall). Nevertheless, he had opportunity to examine Gödel's results (in particular his second theorem) and in the last section of his paper he was dealing with them.

Herbrand's paper presents a proof of the consistency of a fragment of arithmetic of natural numbers. It was certainly intended to be a contribution to the realization of Hilbert's programme. The fragment considered by Herbrand is arithmetic with induction for formulas containing no bounded variables and induction for formulas containing bounded variables but

[4] [...] die zeitweilig aufgekommene Meinung, aus gewissen neueren Ergebnissen von Gödel folge die Undurchführbarkeit meiner Beweistheorie, als irrtümlich erwiesen ist. Jenes Ergebnis zeigt in der Tat auch nur, daß man für die weitergehenden Widerspruchsfreiheitsbeweise den finiten Standpunkt in einer schärferen Weise ausnutzen muß, als dieses bei der Betrachtung der elementaren Formalismen erforderlich ist.

containing no function symbols except eventually the successor function. The proof uses Herbrand's fundamental theorem[5] (section 1 consists of a very clear presentation of this theorem).

It is worth noting here that Herbrand, similarly as von Neumann (see above), uses the name "intuitionistic" to describe methods which are allowed in the metamathematics, hence finitistic methods. This identification was then current in Hilbert's school.

The key trick of Herbrand's proof of the consistency of the indicated fragment of aritmetic is the elimination of the induction axiom scheme through the introduction of functions. The definition conditions for those functions are such that, for every set of arguments, a well-determined number can be proved in a finitary way to be the value of the function. It should be noted that those functions are (general) recursive functions. This is in fact the first appearance of the notion of a general recursive function as opposed to primitive recursive (cf. Gödel's definition of general recursive functions from 1934 "suggested by Herbrand" – see Gödel, 1934, p. 26).

As indicated above, in the last section of his paper (1931) Herbrand considered the problem of connections between his result and Gödel's theorem on the unprovability of consistency. He explains very clearly why the latter does not hold for the fragment of arithmetic he considers. The reason is that the metamathematical description of the system cannot be projected into the system itself (because the system is too weak).

6. First proof of the consistency of the arithmetic of natural numbers was given by Gerhard Gentzen in the paper "Die Widerspruchsfreiheit der reinen Zahlentheorie" (1936) (cf. also his paper "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie" from 1938). According to Gödel's Second Incompleteness Theorem a proof of the consistency of the full arithmetic of natural numbers should use means stronger than those available in the arithmetic itself (modulo the restrictions concerning the way of expressing in the formal language the property of consistency). Indeed the analysis of Gentzen's proof shows that it is just in the concept of a reduction process applied by Gentzen in (1936) that the transgression of the methods formalizable in the formal system under

[5] This theorem contains a reduction (in a certain sense) of predicate logic to propositional logic, more exactly it shows that a formula is derivable in the axiomatic system of quantification logic if and only if its negation has a truth-functionally inconsistent expansion. Herbrand intended to prove this theorem by finitistic means. The theorem was contained in Chapter 4 of his doctoral dissertation presented to the Sorbonne in 1930 and published in the same year – cf. Herbrand, 1930.

consideration comes about. By assigning ordinals to the derivations one sees that the transfinite induction up to ε_0 suffices for the proof[6].

It is worth noting here that the first version of Gentzen's consistency proof was submitted in 1935 but was withdrawn after criticism directed against the means used in the proof which were considered to be too strong. Gentzen took care of the criticism and modified his original proof before it was published (the modified proof was published in the paper (1936)). Fortunately the text of the original proof was preserved in galley proof. It became publically known because of the paper by Bernays (1970) and was recently published in the name of Gentzen (cf. Gentzen, 1974). Bernays remarks in (1970) that Gentzen's original proof was certainly easier to follow than the first published proof and at least as easy to follow as the second Gentzen consistency proof from (1938).

7. Gentzen's proof was apparently accepted by Hilbert and Bernays in the second volume of *Grundlagen der Mathematik* (1939). Indeed, in the Preface Bernays wrote there (p. VII):

> In any case one can say on the basis of Gentzen's proof that the short-lived failure of proof theory was caused solely by the whimsicality of the methodological demand put on it[7].

In the same Preface it was also announced that W. Ackermann is working on extending his earlier consistency proof (published in 1927) along the lines indicated by Gentzen, i.e., by applying the transfinite induction. Indeed, in 1940 appeared Ackermann's paper "Zur Widerspruchsfreiheit der Zahlentheorie" in which the consistency of the full arithmetic of natural numbers was proved by using methods from his paper (1927) and the transfinite induction.

Since then other proofs along Gentzen's lines have been published. One should mention here among others papers by Lorenzen (1951), Schütte (1951, 1960) and Hlodovskii (1959).

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[6] The countable ordinal ε_0 is defined as the smallest ordinal ε such that $\omega^\varepsilon = \varepsilon$ or as the limit of the sequence ω, ω^ω, ω^{ω^ω},...

[7] Jedenfalls kann schon auf Grund des Gentzenschen Beweises die Auffassung vertreten werden, daß das zeitweilige Fiasko der Beweistheorie lediglich durch eine Überspannung der methodischen Anforderung verschuldet war, die man an die Theorie gestellt hat.

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