Gödel's Incompleteness Theorems

Reference Pages

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1 Gödel's incompleteness theorem (weak version)

1.1 Abstract Framework for the Incompleteness Theorems

1. E - set of expressions.
2. $S \subseteq E$ - set of sentences.
3. $N \subseteq E$ - set of numerals.
4. $P \subseteq E$ - set of predicates.
5. A Gödel function: $g : E \to N$, denoted by $g(\psi) = \ulcorner \psi \urcorner$.
6. A function $\Phi : P \times N \to S$, i.e $\Phi(h, n) = h(n)$.
7. $T \subseteq S$ - representing intuitively the set of "true" sentences.

Definition

1. *We say a predicate $h \in P$ T-defines the set $B \subseteq N$ of numerals, if for all $n \in N$, $n \in B \iff h(n) \in T$.*
2. *We say a predicate $h \in P$ T-defines the set $B \subseteq S$ of sentences, if for all $\psi \in S$, $\psi \in B \iff h(\ulcorner \psi \urcorner) \in T$.*
3. *We say a predicate $H \in P$ T-defines the set $B \subseteq P$ of predicates, if for all $h \in P$, $h \in B \iff H(\ulcorner h \urcorner) \in T$.*

Definition(Diagonalization)

1. *Let $B \subseteq S$; The diagonalization function is defined as follows:* $D(B) \stackrel{\text{def}}{=} \{h \in P \mid h(\ulcorner h \urcorner) \in B\}$.
2. *We say that $T \subseteq B$ satisfies the diagonalization condition if when B is T-definable then $D(B)$ is T-definable.*

Proposition:

1. *if T satisfies the diagonalization condition then for every T-definable set of sentences B there is a (Gödel) sentence φ such that* $\varphi \in T \iff \varphi \in B$.

2. *if T satisfies the diagonalization condition then $S \setminus T$ is not T-definable.*

3. (Tarski Theorem - abstract version) *if T satisfies the diagonalization condition and for every T-definable set $B \subseteq S$, $S \setminus B$ is also T-definable then T is not T-definable.*

Theorem: application I (Concrete Tarski)
Let L be a FOL with infintely many closed terms. Let M be a Model for L and T_M the set of true sentences of M. if T_M satisfies the diagonalization condition then T_M is not T_M-definable.

Theorem: application II (Gödel's incompleteness theorem (weak version))
Let L be a FOL with infintely many closed terms. Let M be a Model for L and T_M the set of true sentences of M.
Let $\mathcal{T}$ be a theory such that $M \models \mathcal{T}$.
Let $Pr_{\mathcal{T}}$ denote the set of sentences that are provable in $\mathcal{T}$.
If for some coding we have that:
(i) *T_M satisfies the diagonalization condition;*
(ii) *$Pr_{\mathcal{T}}$ is T_M-definable*
then $T_M \neq Pr_{\mathcal{T}}$. That is, there are true sentences that are not provable in $\mathcal{T}$.

Theorem: Application I: Concrete Tarski's theorem for AE (arithmetic with exponentiation)
Let T_N be the set of AE sentences that are true in N, then T_N is not T_N-definable.

Theorem: Application II for AE: Gödel's incompleteness theorem (weak version) for AE
The language - AE; the model - N; T_N - the set of AE sentences that are true in N. Let $\mathcal{T}$ be PA + the following two more axiom for exponent:
(i) $x^0 = 1$ (ii) $x^{s(n)} = x^n \cdot x$
$PR_{\mathcal{T}}$ is the provable sentences of $\mathcal{T}$.
If for some coding we have that:
(i) *T_M satisfies the diagonalization condition;*
(ii) *$Pr_{\mathcal{T}}$ is T_M-definable*
then $T_M \neq Pr_{\mathcal{T}}$. That is, there are true sentences that are not provable in $\mathcal{T}$.

Application II: Gödel's incompleteness theorem (weak version) for PA.
The same as above, only for PA.

2 Gödel's incompleteness theorem (strong version)

Our goal now is to prove the following:

Theorem: (Gödel's incompleteness theorem (strong version) - application III)
Let L be a FOL with infintely many closed terms.
Let $\mathcal{T}$ be a consistent theory of L.
Let $Pr_{\mathcal{T}}$ denote the set of sentences that are provable in $\mathcal{T}$; Thus, "truth" here is actually "provability".
If for some coding we have that:
(i) *$Pr_{\mathcal{T}}$ satisfies the diagonalization condition ;*
(ii) *$Pr_{\mathcal{T}}$ is $Pr_{\mathcal{T}}$-definable*
then $\mathcal{T}$ is incomplete.

2.1 Safety Relations

Goal: To make $\varphi(x_1, ..., x_n, y_1, ..., y_k)$ *safe* for $x_1, ..., x_n$, when for all k numerals $n_1, ..., n_k$, the question $\varphi(x_1, ..., x_n, n_1, ..., n_k)$ can be computed *effectively*: there is a finite number of n-tuples, and there is an effective way to find them. Therefore we have,

Definition: *A $\succ$ saftey relation between a set of formulas and sets of variables is a relation that satisfy the following conditions:*

1. $A \succ X,\ \ Z \subseteq X \implies A \succ Z$.
2. $x \notin Fv(t) \implies t = x \succ \{x\}$ and $x = t \succ \{x\}$.
3. $A \succ \emptyset \implies \neg A \succ \emptyset$.
4. $A \succ X,\ B \succ X \implies A \vee B \succ X$ [1]
5. $A \succ X,\ \ B \succ Z,\ \ Z \cap Fv(A) = \emptyset \implies A \wedge B \succ X \cup Z$ and $B \wedge A \succ X \cup Z$
6. $A \succ X,\ \ y \in X \implies \exists y.A \succ X \setminus \{y\}$.
7. $A \equiv B,\ \ A \succ X \implies B \succ X$.

Definition: *If t is a term and $X \subseteq Fv(t)$ then we say that $t \succ X$ if $t = z \succ X$ when $z \notin Fv(t)$* .
Remark: $t \succ \emptyset$ for all t.

2.2 Implementation of Safety Relations

Definition: $A(\bar{x}, \bar{z})\ {}_N\!\succ\ \bar{x}$ *if for all $\bar{n} \in N^k$ the set $\{\bar{x} \mid A(\bar{x}, \bar{n})\}$ is finite.*

proposition: ${}_N\!\succ$ is a safety relation.

[1] Notice that both A and B are safe in respect to X, since if for example, $x \leq y \succ y$ and $z \leq w \succ w$ then its not the case that $x \leq y \vee z \leq w \succ \{y, w\}$, because all x's are valid whenwe fix the w, for instance.

2.2.1 Safety relations in Arithmetic

Definition:

1. Bounded Safety: *We define the* ${}_b\succ$ *safety relation as follows:*

   (i) $x \leq y \ {}_b\succ x$

   (ii) By induction, all the other conditions (1-7) of the safety relations hold.

   Remark: Actually, it is sufficient to say that ${}_b\succ$ is a safety relation such that $x \leq y \ {}_b\succ x$. Since, if ${}_b\succ$ is a safety relation then all other conditions of the definition of safety relation hold.

2. Polynomial safety, ${}_p\succ$:

   (i) $s(x) \ {}_p\succ x$

   (ii) $x + y \ {}_p\succ \{x, y\}$

   (iii) $s(x) \cdot s(y) = z \ {}_p\succ \{x, y\}$

3. Exponential safety, ${}_E\succ$:

   (i) $x^y = z \ {}_E\succ z$

   (ii) $s(s(x))^y = z \ {}_E\succ \{x, y\}$

All of the above are effective safety relations in respect to N. That is, if $\varphi(x, y) \succ \{x\}$, then given $y \in N$, we can effectively find a finite set of x's that satisfy φ.

Definition:

1. φ *is safe if* $\varphi \succ Fv(\varphi)$.

2. φ *is effective if* $\varphi \succ \emptyset$.

2.3 r.e. and $\sum_1$

Definition: *Let* $\succ$ *be a safety relation. A formula* φ *is said to be in* $\sum_1$ *if it is of the form:* $\exists x_1, ...x_k.\varphi$, *where* $\varphi \succ \emptyset$.

Remarks:
(1) $\sum_1$ formulas are also called *semi-effective* formulas.
(2) We shall usualy treat $\sum_1$ formulas as formulas of the form $\exists x_1, ...x_k.\psi$, where ψ is ${}_p\succ$ (that is, ψ is in a language of N.)

Definition: *r.e. or* $\sum$ *formulae are defined as follows:*
(i) Every ${}_b\succ$ *effective or* ${}_p\succ$ *effective formula is r.e. formula .*
(ii) If A *and* B *are r.e. formulae then so is* $A \vee B$ *and* $A \wedge B$.
(iii) If A *is a r.e. formula then so is* $\exists x.A$.
(iv) IF $A \ {}_b\succ \bar{x}$ *or* $A \ {}_p\succ \bar{x}$ *and* B *is r.e., then* $\forall \bar{x}(A \rightarrow B)$ *is r.e. .*

Proposition: *Every r.e. formula is equivalent to a* $\sum_1$ *formula over* N.

Definition(Varinat of Church's Thesis):

1. *A relation is semi-effective iff it is definable by a P-semi-effective formula.*

2. *A relation R is effective if both R and $\neg R$ is P-semi-effective (semi-effective).*

Definition:

1. *We say a relation $R \in N^k$ is defined in N by a formula $\psi(x_1, ..., x_k)$ when* $\bar{x} \in R \iff N \models \psi(\bar{x})$.

2. *A relation R over N is r.e. iff R is definable in N by a r.e. formula ψ iff there is a $\sum_1$ formula ϕ such that $N \models \psi \leftrightarrow \phi$.*

3. *A theory T is axiomatic if the set of its axioms is r.e.*

4. *We say a relation R over N is decidable or recursive if both R and $\bar{R}$ is r.e.*

Proposition: *P-semi-effective is equivalent to E-semi-effective.*
proof. To be completed.

Proposition:

1. *If a theory T is axiomatic then the set of all its theorems is r.e.*

2. *If a theory T is* exponentially *safe, i.e. for all its axioms A, A $_E\!\succ$ $Fv(A)$, and thus E-effective* [2] *, then the syntax predicates 1-11 are all* r.e. *and* exponentially safe*. 12 is not anymore effective.*

3. *If T is not $_E\!\succ$ safe but rather semi-effective, that is, in $\sum_1$, then 12 is also semi-effective (since, $\sum_1$ is closed under $\exists$.)*

2.4 Numeral Accurate Theories

Definition($_T\!\succ$):
Let T be a consistent theory that satisfies these conditions:
(i) If $k \neq n$ then $T \vdash \bar{k} \neq \bar{n}$.
(ii) If $t(\bar{y})$ is a term then for every $\bar{n}$ there is a k such that $T \vdash t(\bar{n}) = k$.

Then $_T\!\succ$ is defined as follows:
$\varphi(\bar{x}, \bar{y})$ $_T\!\succ \bar{x}$ if for all $\bar{k}$ there exists a finite set A such that:

$$T \vdash \varphi(\bar{x}, \bar{k}) \leftrightarrow \bar{x} \in A$$

Definition(BA):
(i) *A numeral accurate consistent theory that satisfies both (i) and (ii) conditions for $_T\!\succ$ is a theory in which the following conditions hold for every n, k and m (BA):*

1. If $n \neq k$ then $T \vdash \bar{n} \neq \bar{k}$

2. If $n + k = m$ then $T \vdash \bar{n} + \bar{k} = \bar{m}$

[2]Notice that if a formula A is safe for some $\succ$ then it is also effective. But the opposite is not allways true.

3. If $n \cdot k = m$ then $T \vdash \bar{n} \cdot \bar{k} = \bar{m}$

(ii) *T is accurate with respect to a formula φ if for every closed instance φ' of φ we have:*

$$N \models \varphi' \iff T \vdash \varphi'$$
$$N \not\models \varphi' \iff T \vdash \neg\varphi'$$

Note: from now on T is a numeral accurate theory.

Definition(B.N.): *PA without induction scheme is a numeral accurate and finite theory (i.e. includes BA).*

Definition: *T respects a safety relation $\succ \subseteq {}_N\!\succ$ when*

1. ${}_T\!\succ \; \supseteq \succ$
2. *T is accurate with respect to every formula φ that is $\succ$-effective (i.e. $\varphi \succ \emptyset$).*

Proposition: *T respects a safety relation $\succ \subseteq {}_N\!\succ$ that is defined by a standard induction on the basic rules (2) if for every basic rule of the form $\varphi \succ X$:*
(i) *$\varphi \;{}_T\!\succ X$.*
(ii) *T is accurate with respect to φ.*

Proposition: (RR^-)
Let RR^- be an infinite theory containing BA and all formulae of the form

$$T \vdash \forall x \leq k \longleftrightarrow (x = 0) \vee (x = 1) \vee ... \vee (x = k).$$

Then a consistent theory T respects b-safety iff it includes RR^- (i.e. it prooves all axioms of RR^-.)

Definition(Q): *The theory Q is obtained from B.N. by adding the axiom:*

$$\forall x(x = 0 \vee \exists y.x = s(y))$$

(The $\leq$ is not in the language of Q and is defined by the $+$ and $=$ signs.)

2.5 $\sum_1 -$ *consistency*

Proposition: If T is a consistent extension of RR^- and φ is a true $\sum_1$ sentence then $T \vdash \varphi$.

Definition($\sum_1 -$ *consistency*):
A theory T is $\sum_1 -$ consistent if for every $\sum_1$ formula $\varphi = \exists\bar{x}.\psi(\bar{x})$, i.e. such that $\psi(\bar{x})$ is p-effective:

$$T \vdash \varphi \implies \exists\bar{n} \in N. \;\; T \vdash \psi(\bar{n}).$$

Proposition: If T is a $\sum_1$-consistent extension of RR^- and φ is a $\sum_1$ sentence then $T \vdash \varphi$ iff φ is a true sentence.

Note: from now on T is an axiomatic, $\sum_1$-consistent extension of RR^-.

2.6 Definability of Relations and Functions

Definition:

1. *We say a relation $P \subseteq N^k$ is enumerable in T by a formula $\varphi(\bar{x})$ if for all $\bar{n} \in N$:*

$$T \vdash \varphi\{\bar{n}/\bar{x}\} \iff \bar{n} \in P.$$

2. *We say a relation $P \subseteq N^k$ is binumerable in T by a formula $\varphi(\bar{x})$ if for all $\bar{n} \in N$:*
   $\varphi(\bar{x})$ enumerates P in T;
   $\neg\varphi(\bar{x})$ enumerates $\bar{P}$ in T.

propositions of proof of simple diagonalization theorem. (lexture 9)

Proposition: *If an r.e. relation P is ('semantically') defined by φ in N, then for every T, a $\sum_1$-consistent extension of RR^-, φ enumerates P in T.*

Corollary: *If a $\sum_1$-consistent extension of RR^-, T, is axiomatic then Pr_T is enumerable in T.*

Definition: *We say a function f is representable in a theory T by a formula φ if:*

1. *φ enumerates f in T.*
2. *for all $\bar{n}$ we have:*
   (i) $T \vdash \exists y.\varphi(\bar{n}, y)$
   (ii) $T \vdash \varphi(\bar{n}, y_1) \wedge \varphi(\bar{n}, y_2) \longrightarrow y_1 \neq y_2$.

Proposition: *Let T be a consistent and axiomatic extension of RR^-, then the diagonalization function $d(n) = \ulcorner E_n(\ulcorner E_n \urcorner) \urcorner$ is representable in T.*

3 Results: Gödel's incompleteness theorem (strong version)

Theorem: ((**Simple**) **Diagonalization Theorem**)
If $\varphi(x)$ is a formula, with x as its single free variable, then there exists a Gödel sentence E_n for φ such that $RR^- \vdash E_n \longleftrightarrow \varphi(\ulcorner E_n \urcorner)$, where E_n is a sentence with n as its Gödel number.

Reminder : The two conditions for Gödel's incompleteness theorem, *strong* variant:
(i) Pr_T is enumerable in T.
(ii) Diagonalization condition holds in T, according to the diagonalization theorem.

Theorem: (Tarski on truth defintions)
Let ψ be a truth defintion for T in T such that for every sentence A:

$$T \vdash A \longleftrightarrow \psi(\ulcorner A \urcorner).$$

If T is a consistent extension of RR^- then T has no truth defintion in T.

Theorem: (Gödel's incompleteness theorem) *Let T be an axiomatic and consistent extension of RR^-, then:*

1. *There exists a true Π_1 sentence, φ, such that $T \nvdash \varphi$.*
2. *If T is $\sum_1$-consistent then also $T \nvdash \neg\varphi$ and thus T is incomplete.*
3. *Moreover, T in (2) is ω-incomplete; that is, there exists a sentence $\forall x.A(x)$ such that $T \nvdash \forall x.A(x)$and for all $n \in N \quad T \vdash A\{n/x\}$.*

4 Church's and Gödel-Rosser's theorems

Proposition: *The following propositions are equivalent with respect to a relation $R \subseteq N^k$:*
(i) *R is r.e.*
(ii) *R is enumerable in some axiomatic theory T.*
(iii) *R is enumerable in every axiomatic $\sum_1$-consistent extension of RR^-.*

Definition(RR): *RR is the formal system obtained from RR^- by adding for every $n \in N$ the axiom:*

$$x \leq n \quad \vee \quad n \leq x$$

Proposition: *A relation R is decidable iff it is binumerable in some (any) axiomatic consistent extension of RR.*

Theorem: (Church) *Every consistent extension of RR is incomplete.*

Theorem: (Gödel - Rosser) *Let T be an axiomatic and consistent extension of RR, then T is incomplete.*