3 The Model Existence Theorem

Although we don't have compactness or a useful Completeness Theorem, Henkin-style arguments can still be used in some contexts to build models. In this section we describe the general framework and give two applications.

Throughout this section we will assume that $\mathcal{L}$ is a countable language and that C is an infinite set of constant symbols of $\mathcal{L}$ (though perhaps not all of the constant symbols of $\mathcal{L}$). The following idea due to Makkai is the key idea. It tells us exactly what we need to do a Henkin argument.

In the following a *basic term* is either a constant symbol (not necessarily from C) or a term of the form $f(c_1, \ldots, c_n)$ where f is a function symbol of $\mathcal{L}$ and $c_1, \ldots, c_n \in C$. A *closed term* is a term with no variables.

Definition 3.1 A *consistency property* Σ is a collection of countable sets σ of $\mathcal{L}_{\omega_1,\omega}$-sentences with the following properties. Let $\sigma \in \Sigma$

C0) $\emptyset \in \Sigma$ and if $\sigma, \tau \in \Sigma$ with $\sigma \subseteq \tau$ and $\phi \in \tau$, then $\sigma \cup \{\phi\}$;

C1) if $\phi \in \sigma$, then $\neg\phi \notin \sigma$;

C2) if $\neg\phi \in \sigma$, then $\sigma \cup \{\sim \phi\} \in \Sigma$;

C3) if $\bigwedge_{\phi \in X} \phi \in \sigma$, then $\sigma \cup \{\phi\} \in \Sigma$ for all $\phi \in X$;

C4) if $\bigvee_{\phi \in X} \phi \in \sigma$, then $\sigma \cup \{\phi\} \in \Sigma$ for some $\phi \in X$;

C5) if $\forall v \phi(v) \in \sigma$, then $\sigma \cup \{\phi(c)\} \in \Sigma$ for all $c \in C$;

C6) if $\exists v \phi(v) \in \sigma$, then $\sigma \cup \{\phi(c)\} \in \Sigma$ for some $c \in C$;

C7) let t be a basic term and let $c, d \in C$,

a) if $c = d \in \sigma$, then $\sigma \cup \{d = c\} \in \Sigma$;

b) if $c = t, \phi(t) \in \sigma$, then $\sigma \cup \{\phi(c)\} \in \Sigma$;

c) for some $b \in C$, $\sigma \cup \{b = t\} \in \Sigma$.

Lemma 3.2 *Let Σ be a consistency property with $\sigma \in \Sigma$, $c, d, e \in C$.*

a) There is $\sigma \cup \{c = c\} \in \tau$.

b) If $c = d, d = e \in \sigma$, then $\sigma \cup \{c = e\} \in \Sigma$.

c) If $\phi, \phi \rightarrow \psi \in \sigma$, then $\sigma \cup \{\psi\} \in \Sigma$.

Proof a) By C7b) there is $b \in C$ such that $\sigma \cup \{b = c\} \in \Sigma$. By C7a) $\sigma_1 = \sigma \cup \{b = c, c = b\} \in \Sigma$. Let $\phi(v)$ be the formula $v = c$. Since $\phi(b)$ and $c = b$ are in σ_1, by C7b) $\tau = \sigma \cup \{c = c\} \in \Sigma$. By C0) $\sigma \cup \{c = c\} \in \Sigma$.

b) Let $\phi(v)$ be the formula $v = e$. Then $\phi(d) \in \sigma$ and $c = d \in \sigma$. Thus by C7b) $\sigma \cup \{c = e\} \in \Sigma$.

c) Since $\neg\phi \vee \psi \in \sigma$, by C4) either $\sigma \cup \{\neg\phi\} \in \Sigma$ or $\sigma \cup \{\psi\} \in \Sigma$. By C1) the former is impossible.

The next exerecise shows that we really need only verify C1)-C7).

Exercise 3.3 Suppose Σ_0 satisfies C1)-C7). Let

$$\Sigma = \{\emptyset\} \cup \{\sigma_0 \cup \Delta : \sigma_0 \in \Sigma_0 \text{ and } \exists \sigma \in \Sigma\ \Delta \subseteq \sigma, \Delta \text{ finite } \}.$$

Prove that Σ is a consistency property.

Theorem 3.4 (Model Existence Theorem) *If Σ is a consistency property and $\sigma \in \Sigma$, there is a countable $\mathcal{M} \models \sigma$.*

Proof Let $\phi_0, \phi_1, \ldots$ list all $\mathcal{F}$-sentences. We assume that each sentence is listed infinitely often. Let $t_0, t_1, \ldots,$ list all basic terms.

Using the fact that Σ is a consistency property, we build

$$\sigma = \sigma_0 \subseteq \sigma_1 \subset \ldots$$

such that each $\sigma_i \in \Sigma$ and

A) if $\sigma_n \cup \{\phi_n\} \in \Sigma$, then $\phi_n \in \sigma_{n+1}$, in this case:

a) if ϕ_n is $\bigvee_{\phi \in X} \phi$, then $\phi \in \sigma_{n+1}$ for some $\phi \in X$;

b) if ϕ_n is $\exists v \phi(v)$, then $\phi(c) \in \sigma_{n+1}$ for some $c \in C$;

B) $c = t_n \in \sigma_{n+1}$ for some $c \in C$.

Let $\Gamma = \bigcup_{n=1}^{\infty} \sigma_n$. We will build a model of Γ.

For $c, d \in C$ we say $c \sim d$ if $c = d \in \Gamma$.

Claim $\sim$ is an equivalence relation

Let $c \in C$. The sentence $c = c$ is ϕ_n for some n. By Lemma 3.2, and A) $c = c \in \sigma_{n+1} \subseteq \Gamma$. Thus $c \sim c$.

If $c = d \in \Gamma$, then we can find n such that $c = d \in \sigma_n$ and $d = c$ is ϕ_n. Then by C7) and condition A) $d = c \in \Gamma$.

If $c = d, d = e \in \Gamma$, choose n such that $c = d, d = e \in \sigma_n$ and $c = e$ is ϕ_n. Then by Lemmma 3.2 and A) $c = e \in \Gamma$.

Let $[c]$ denote the $\sim$-class of c. Let $M = \{[c] : c \in C\}$. We make M into an $\mathcal{L}$-structure.

Let f be an n-ary function symbol of $\mathcal{L}$ and let $c_1, \ldots, c_n \in C$. By ii) there is $d \in C$ such that $d = f(c_1, \ldots, c_n) \in \Gamma$. Suppose $d_1 = f(c_1, \ldots, c_n)$ is also in Γ, using C7b) and A) we see that $d = d_1 \in \Gamma$ and $d \sim d_1$. Also, note that if $c_0 = f(c_1, \ldots, c_n) \in \Gamma$, and $d_i \sim c_i$ for $i = 0, \ldots, n$ then,repeatedly using C7b), $d_0 = f(d_1, \ldots, d_n) \in \Gamma$. Thus we can define $f^{\mathcal{M}} : M^n \to M$ by

$$f([c_1], \ldots, [c_n]) = [d] \Leftrightarrow d = f(c_1, \ldots, c_n) \in \Gamma.$$

For $c_1, \ldots, c_n$ and R an n-ary relation symbol of $\mathcal{L}$, we say

$$R^{\mathcal{M}}([c_1], \ldots, [c_n]) \Leftrightarrow R(c_1, \ldots, c_n) \in \Gamma.$$

Again, this does not depend on the choice of c_i.

Claim If $\phi \in \Gamma$, then $\mathcal{M} \models \phi$. We need an annoying analysis of terms.

We prove this by induction on complexity.[5]

i) ϕ is $s = t$, where s and t are closed terms

We build a sequence of formulas $s_1 = t_1, \ldots, s_m = t_m$ where:

- $s_1 = s$ and $t_1 = t$;
- $s_i^{\mathcal{M}} = s_{i+1}^{\mathcal{M}}$ and $t_i^{\mathcal{M}} = t_{i+1}^{\mathcal{M}}$;

[5]We need to be slightly careful how we define "complexity". For example, $\exists v \psi(v)$ is more complex than any $\psi(c)$ and $\neg \bigvee_{\psi \in X} \psi$ is more complicated than $\neg \psi$ for any $\psi \in X$.

- $s_i = t_i \in \Gamma$;
- s_m and t_m are constants in C.

If we accomplish i)-iv) then $[s_m] = [t_m]$ and, by induction, $s^{\mathcal{M}} = t^{\mathcal{M}}$.

Given the formula $s_i = t_i \in \Gamma$ unless $s_i, t_i \in C$, we can find a basic subterm $\tau \notin C$ of one of them, say s_i, by B) there is $c \in C$ such that $c = \tau \in \Gamma$. Obtain s_{i+1} from s_i by substituting c for τ and let $t_{i+1} = t_i$. An easy induction shows that $s_i^{\mathcal{M}} = s_{i+1}^{\mathcal{M}}$. By C7) and A) $s_{i+1} = t_{i+1} \in \Gamma$. $s_{i+1} = t_{i+1}$.

Thus $\mathcal{M} \models t_1 = t_2$.

ii) ϕ is $R(t_1, \ldots, t_n)$

Exercise 3.5 Prove that $\mathcal{M} \models R(t_1, \ldots, t_n)$. [Hint: Similar to case i).]

iii) ϕ is $\bigwedge_{\psi \in X} \phi$

By C3) and A), $\phi \in \Gamma$ for each $\phi \in X$. By induction $\mathcal{M} \models \psi$ for all $\phi \in X$. Thus $\mathcal{M} \models \psi$.

iv) ϕ is $\bigvee_{\psi \in X} \psi$

By C4) and A), there is $\psi \in X$ such that $\psi \in \Gamma$. By induction $\mathcal{M} \models \psi$. Thus $\mathcal{M} \models \phi$.

v) ϕ is $\forall v\ \psi(v)$

By C5) and A), $\psi(c) \in \Gamma$ for all $c \in C$. By induction $\mathcal{M} \models \psi(c)$ for all $c \in C$. Since every element of M is named by a constant, $\mathcal{M} \models \phi$.

vi) ϕ is $\exists v\ \psi(v)$

By A) there is $c \in C$ such that $\psi(c) \in \Gamma$. By induction, $\mathcal{M} \models \psi(c)$. Thus $\mathcal{M} \models \phi$.

vii) ϕ is $\neg\psi$

By C2) and A), $\sim \psi \in \Gamma$. This now breaks into cases depending on ψ.

a) ψ is $s = t$ where s and t are closed terms

As in the proof of i) above we can find a sequence of formulas

$$s_1 \neq t_1, \ldots, s_m \neq t_m$$

where each of the formulas is in Γ, $s_i^{\mathcal{M}} = s_{i+1}^{\mathcal{M}}$, $t_i^{\mathcal{M}} = t_{i+1}^{\mathcal{M}}$, and $s_m, t_m \in C$. Thus $\mathcal{M} \models \neg\psi$.

b) ψ is $R(t_1, \ldots, t_m)$

Exercise 3.6 Prove $\mathcal{M} \models \neg R(t_1, \ldots, t_m)$.

c) ψ is $\neg\theta$

Then $\theta \in \Gamma$. My induction $\mathcal{M} \models \theta$ and $\mathcal{M} \models \phi$.

d) ψ is $\bigwedge_{\theta \in X} \theta$

Then $\bigvee_{\theta \in X} \neg\theta \in \Gamma$ and $\neg\theta \in \Gamma$ for some $\theta \in X$. By induction, $\mathcal{M} \models \neg\theta$. Thus $\mathcal{M} \models \phi$.

e) ψ is $\bigvee_{\theta \in X} \theta$

Then $\bigwedge_{\theta \in X} \neg\theta \in \Gamma$ and $\neg\theta \in \Gamma$ for all $\theta \in X$. By induction $\mathcal{M} \models \neg\theta$ for all $\theta \in X$. Thus $\mathcal{M} \models \phi$.

f) ψ is $\exists v\ \theta(v)$

Then $\forall v \neg\theta(v) \in \Gamma$ and $\theta(c) \in \Gamma$ for all $c \in C$. By induction, $\mathcal{M} \models \neg\theta(c)$ for all $c \in C$. Thus $\mathcal{M} \models \neg\phi$.

g) ψ is $\forall v\ \theta(v)$.

Then $\exists v \neg\theta(v) \in \Gamma$ and $\theta(c) \in \Gamma$ for some $c \in C$. But then $\mathcal{M} \models \neg\theta(c)$ and $\mathcal{M} \models \neg\phi$.

This completes the induction. Thus $\mathcal{M} \models \Gamma$.

The next exercise gives a useful extension of the Model Existence Theorem.

Exercise 3.7 [Extended Model Existence Theorem] Suppose Σ is a consistency property and T is a countable set of sentences such that for all $\sigma \in \Sigma$ and $\phi \in T$, $\sigma \cup \{\phi\} \in \Sigma$. Then $\sigma \cup T$ has a model for all $\sigma \in \Sigma$. [Hint: Consider

$$\Sigma_1 = \{\sigma \cup T : \sigma \in \Sigma\}.$$

The Interpolation Theorem

We give two applications of the Model Existence Theorem. Both results were first proved by Lopez-Escobar by different means. The first is the $\mathcal{L}_{\omega_1,\omega}$ version of Craig's Interpolation Theorem.

Theorem 3.8 *Suppose ϕ_1 and ϕ_2 are $\mathcal{L}_{\omega_1,\omega}$-sentences with $\phi_1 \models \phi_2$. There is an $\mathcal{L}_{\omega_1,\omega}$-sentence θ such that $\phi_1 \models \theta$, $\theta \models \phi_2$ and every relation, function and constant symbol occurring in θ occurs in both ϕ_1 and ϕ_2.*

Proof Let C be a countably infinite collection of new constant symbols. Let $\mathcal{F}_i$ be a countable fragment containing ϕ_i, where every relation and function symbol and every constant symbol not in C occurs in ϕ_i and every formula contains only finitely many constants from C. Let $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$.

Let Σ be the collection of finite $\sigma = \sigma_1 \cup \sigma_2$ where σ_i is a set of $\mathcal{F}_i$-sentences and if ψ_1, ψ_2 are $\mathcal{F}$-sentences such that $\sigma_1 \models \psi_1$ and $\sigma_2 \models \psi_2$, then $\psi_1 \wedge \psi_2$ is satisfiable.

Claim Σ is a consistency property.

We verify a couple of properties and leave the rest as an exercise.

C3) Suppose $\bigwedge_{\psi \in X} \psi \in \sigma \in \Sigma$, where $\sigma = \sigma_1 \cup \sigma_2$ as above. Suppose $\bigwedge_{\psi \in X} \psi \in \sigma_1$. Let $\sigma_1' = \sigma_1 \cup \{\psi\}$. We claim that $\sigma_1' \cup \sigma_2 \in \Sigma$. Suppose $\sigma_1' \models \theta_1$ and $\sigma_2 \models \theta_2$. Then $\sigma_1 \models \theta_1$. Hence $\theta_1 \wedge \theta_2$ is satisfiable. This is similar if $\bigwedge_{\psi \in X} \psi \in \sigma_2$.

C4) Suppose $\bigvee_{\psi \in X} \psi \in \sigma_1$. Let $\sigma_{1,\psi} = \sigma_1 \cup \{\psi\}$. We claim that some $\sigma_{1,\psi} \cup \sigma_2 \in \Sigma$. Suppose not. Then for each ψ there are $\theta_{1,\psi}$, $\theta_{2,\psi} \in \mathcal{F}$ such that $\sigma_{1,\psi} \models \theta_{1,\psi}$, $\sigma_2 \models \theta_{2,\psi}$ and $\theta_{1,\psi} \wedge \theta_{2,\psi}$ is unsatisfiable. Then $\theta_{1,\psi} \models \neg\theta_{2,\psi}$. Since

$$\sigma_1 \models \bigvee_{\psi \in X} \psi,$$

$$\sigma_1 \models \bigvee_{\psi \in X} \theta_{1,\psi}.$$

But

$$\sigma_2 \models \bigwedge_{\psi \in X} \theta_{2,\psi}$$

and

$$\bigvee_{\psi \in X} \theta_{1,\psi} \models \neg \bigwedge_{\psi \in X} \theta_{2,\psi}$$

contradicting that $\sigma \in \Sigma$.

Exercise 3.9 Finish the proof that Σ is a consistency property.

We now finish the proof of the Interpolation Theorem. Since $\phi_1 \models \phi_2$, by the Model Existence Theorem, $\{\phi_1, \neg\phi_2\} \not\in \Sigma$. Thus there are θ_1 and $\theta_2 \in \mathcal{F}$ such that $\phi_1 \models \theta_1$, $\neg\phi_2 \models \theta_2$ and $\theta_1 \wedge \theta_2$ is unsatisfiable.

Thus

$$\phi_1 \models \theta_1, \theta_1 \models \neg\theta_2, \text{ and } \neg\theta_2 \models \phi_2.$$

It follows that

$$\phi_1 \models \theta_1 \text{ and } \theta_1 \models \phi_2.$$

We would be done except θ_1 may contain constants from C. Let $\theta_1 = \psi(\overline{c})$, where $\psi(\overline{v})$ is a $\mathcal{F}$-formula with no constants from C. Then

$$\phi_1 \models \forall \overline{v}\ \psi(\overline{v}) \text{ and } \exists \overline{v}\ \psi(\overline{v}) \models \phi_2.$$

Thus we can take

$$\forall \overline{v}\ \psi(\overline{v})$$

as the interpolant.

Definition 3.10 We say that K is a $PC_{\omega_1,\omega}$-class of $\mathcal{L}$-structures, $\mathcal{L}^*$ an expansion of $\mathcal{L}$ and $\phi \in \mathcal{L}^*_{\omega_1,\omega}$ such that K is the collection of $\mathcal{L}$-reducts of models of ϕ.

Exercise 3.11 Show that the following classes are $PC_{\omega_1,\omega}$.

a) incomplete dense linear orders;

b) dense linear orderings where $|(a,b)| = |(c,d)|$ for all $a < b$ and $c < d$;

c) bipartite graphs;

d) free groups and free abelian groups;

e) groups with a proper subgroup of finite index;

f) ordered fields with a bounded real closed subfield;

Give other natural examples

Exercise 3.12 Suppose K_1 and K_2 are disjoint $PC_{\omega_1,\omega}$-classes of $\mathcal{L}$-structures. Prove there is $\phi \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M} \models \phi$ for $\mathcal{M} \in K_1$ and $\mathcal{M} \models \neg\phi$ for $\mathcal{M} \in K_2$.

Exercise 3.13 † For this exercise we consider only countable structures with universe $\mathbb{N}$. Suppose $\mathcal{L}$ is a countable language, let X be the set of all $\mathcal{L}$-structures with universe $\mathbb{N}$. Topologize X so that the set of models satisfying any atomic formula is clopen.

We say that a subset of X is *invariant* if it is closed under isomorphism.

a) Prove that $\{\mathcal{M} : \mathcal{M} \models \phi\}$ is an invariant Borel set.

b) Prove that any $PC_{\omega_1,\omega}$-class of models is an invariant $\mathbf{\Sigma}^1_1$-set.

c) (Scott)* Prove that every invariant $\mathbf{\Sigma}^1_1$-set is $PC_{\omega_1,\omega}$.

d) Prove that every invariant Borel set is $\{\mathcal{M} : \mathcal{M} \models \phi\}$ for some $\phi \in \mathcal{L}_{\omega_1,\omega}$.

The Undefinability of Well-Ordering

Let $\mathcal{L} = \{<, \ldots\}$

Theorem 3.14 *Suppose ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and for all $\alpha < \omega_1$ there is $\mathcal{M} \models \phi$ where $(\alpha, <)$ embeds into $<^{\mathcal{M}}$. Then there is $\mathcal{N} \models \phi$ where $(\mathbb{Q}, <)$ embeds into $<^{\mathcal{N}}$.*

Corollary 3.15 *If ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and $<^{\mathcal{M}}$ is well-ordered for all $\mathcal{M} \models \phi$, then there is $\alpha < \omega_1$ such that $<^{\mathcal{M}}$ has order type at most α for all $\mathcal{M} \models \phi$.*

Proof Let $\mathcal{L}$ be our original language. Form $\mathcal{L}$ by adding new constants C and $D = \{d_q : q \in \mathbb{Q}\}$. Let $\sigma \in \Sigma$ be of the form $\sigma_0 \cup \{\phi\} \cup \{d_q < d_r : q < r\}$ where σ is a finite set of $\mathcal{L}_{\omega_1,\omega}$ sentences using only finitely many constants from $C \cup D$. We write $\sigma_0(\overline{c}, d_{i_1}, \ldots, d_{i_m})$ where $i_1 < \ldots < i_m$ to stress the role of the extra constants used. Then $\sigma \in \Sigma$ if and only if for all $\alpha < \omega_1$ there is an $\mathcal{L}$-structure $\mathcal{M}$ where
(*)

$$\mathcal{M} \models \phi \wedge \exists \overline{x}\ \sigma_0(\overline{x}, b_1 \ldots, b_m)$$

where there is a subset A of $\mathcal{M}$ well ordered by $<^{\mathcal{M}}$, $\overline{b} \in A$ and

$$\alpha \leq b_1, b_1 + \alpha \leq b_2, \ldots, b_{m-1} + \alpha \leq b_m.$$

In particular, taking $\sigma_0 = \emptyset$, we see that $\{\phi\} \cup \{d_q < d_r : q < r\} \in \Sigma$. Thus, once we prove that Σ is a consistency property, we will have a model where $<$ contains a densely ordered subset.

Thus we need only show that Σ is a consistency property. We do several of the non-routine claims and leave the rest of the verification as an exercise.

C4) Suppose $\bigvee_{\psi \in X} \psi \in \sigma$. Then for each α there is $\psi_\alpha \in X$ and $\mathcal{M}$ such that (*) holds for ψ_α. There is $\psi \in X$ such that $\psi = \psi_\alpha$ for uncountably many α. Note that if ψ works for α it works for all $\beta < \alpha$. Thus $\sigma \cup \{\psi\} \in \Sigma$

C7c) Suppose $t = d_r$ and σ_0 uses $d_{i_1}, \ldots, d_{i_m}$ where $i_0 < \ldots < i_m$. Suppose $i_s < r < i_{s+1}$. Let c be element of C not yet used. We claim that $\sigma \cup \{c = d_r\} \in \Sigma$.

Let $\alpha < \omega_1$. Pick $\beta > \alpha + \alpha$. By (*) there is

$$\mathcal{M} \models \phi \wedge \exists \overline{x}\ s_0(\overline{x}, \overline{b})$$

where

$$\beta \le b_1, b_1 + \beta \le b_2, \ldots, b_{m-1} + \beta \le b_m.$$

Let $b = b_s + \alpha$. Then $b_s + \alpha \le b$ and $b + \alpha \le b_{s+1}$ as desired.

Exercise 3.16 Complete the proof that Σ is a consistency property.

We give one more application of the Model Existence Theorem.

Exercise 3.17 [Omitting Types Theorem] Let $\mathcal{F}$ be a countable fragment and let T be a satisfiable set of $\mathcal{F}$-sentences. Suppose $X_n(v_1, \ldots, v_{m_n})$ is a set of $\mathcal{F}$-formulas with free variables from $v_1, \ldots, v_n$ such that for all n and all $\psi(v_1, \ldots, v_{m_n}) \in \mathcal{F}$ such that

$$T \cup \{\exists \overline{x}\ \psi(\overline{x})\}$$

is satisfiable, then there is $\phi \in X_n$ such that

$$T \cup \{\exists \overline{x}\ (\psi(\overline{x}) \wedge \phi(\overline{x}))\}$$

is satisfiable. Prove that

$$T \cup \{\forall \overline{x} \bigvee_{\psi \in X_n} \psi(\overline{x}) : n = 1, 2, \ldots\}$$

is satisfiable.

Why is this called the Omitting Types Theorem?