WHAT'S SO SPECIAL ABOUT KRUSKAL'S THEOREM AND THE ORDINAL Γ_0? A SURVEY OF SOME RESULTS IN PROOF THEORY

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Abstract: This paper consists primarily of a survey of results of Harvey Friedman about some proof theoretic aspects of various forms of Kruskal's tree theorem, and in particular the connection with the ordinal Γ_0. We also include a fairly extensive treatment of normal functions on the countable ordinals, and we give a glimpse of Veblen hierarchies, some subsystems of second-order logic, slow-growing and fast-growing hierarchies including Girard's result, and Goodstein sequences. The central theme of this paper is a powerful theorem due to Kruskal, the "tree theorem", as well as a "finite miniaturization" of Kruskal's theorem due to Harvey Friedman. These versions of Kruskal's theorem are remarkable from a proof-theoretic point of view because they are *not* provable in relatively strong logical systems. They are examples of so-called "natural independence phenomena", which are considered by most logicians as more natural than the metamathematical incompleteness results first discovered by Gödel. Kruskal's tree theorem also plays a fundamental role in computer science, because it is one of the main tools for showing that certain orderings on trees are well founded. These orderings play a crucial role in proving the termination of systems of rewrite rules and the correctness of Knuth-Bendix completion procedures. There is also a close connection between a certain infinite countable ordinal called Γ_0 and Kruskal's theorem. Previous definitions of the function involved in this connection are known to be incorrect, in that, the function is not monotonic. We offer a repaired definition of this function, and explore briefly the consequences of its existence.

1 Introduction

This paper consists primarily of a survey of results of Harvey Friedman [47] about some proof theoretic aspects of various forms of Kruskal's tree theorem [28], and in particular the connection with the ordinal Γ_0. Initially, our intention was to restrict ourselves to Kruskal's tree theorem and Γ_0. However, as we were trying to make this paper as self contained as possible, we found that it was necessary to include a fairly extensive treatment of normal functions on the countable ordinals. Thus, we also give a glimpse of Veblen hierarchies, some subsystems of second-order logic, slow-growing and fast-growing hierarchies including Girard's result, and Goodstein sequences.

The central theme of this paper is a powerful theorem due to Kruskal, the "tree theorem", as well as a "finite miniaturization" of Kruskal's theorem due to Harvey Friedman. These versions of Kruskal's theorem are remarkable from a proof-theoretic point of view because they are *not* provable in relatively strong logical systems. They are examples of so-called "natural independence phenomena", which are considered by most logicians as more natural than the metamathematical incompleteness results first discovered by Gödel.

Kruskal's tree theorem also plays a fundamental role in computer science, because it is one of the main tools for showing that certain orderings on trees are well founded. These orderings play a crucial role in proving the termination of systems of rewrite rules and the correctness of Knuth-Bendix completion procedures [27].

There is also a close connection between a certain infinite countable ordinal called Γ_0 (Feferman [13], Schütte [46]) and Kruskal's theorem. This connection lies in the fact that there is a close relationship between the embedding relation $\preceq$ on the set T of finite trees (see definition 4.11) and the well-ordering $\leq$ on the set $\mathcal{O}(\Gamma_0)$ of all ordinals $< \Gamma_0$. Indeed, it is possible to define a function $h : T \to \mathcal{O}(\Gamma_0)$ such that h is (1). surjective, and (2). preserves order, that is, if $s \preceq t$, then $h(s) \leq h(t)$. Previous definitions of this function are known to be incorrect, in that, the function is not monotonic. We offer a repaired definition of this function, and explore briefly the consequences of its existence.

We believe that there is a definite value in bringing together a variety of topics revolving around a common theme, in this case, ordinal notations and their use in mathematical logic. We are hoping that our survey will help in making some beautiful but seemingly rather arcane tools and techniques known to more researchers in logic and theoretical computer science.

The paper is organized as follows. Section 2 contains all the definitions about preorders, well-founded orderings, and well-quasi orders (WQO's), needed in the rest of the paper. Higman's theorem for WQO's on strings is presented in section 3. Several versions of Kruskal's tree theorem are presented in section 4. Section 5 is devoted to several versions of the finite miniaturization of Kruskal's theorem due to Harvey Friedman. Section 6 is a fairly lengthy presentation of basic facts about the countable ordinals, normal functions, and Γ_0. Most of this material is taken from Schütte [46], and we can only claim to have presented it our own way, and hopefully made it more accessible. Section 7 gives a glimpse at Veblen hierarchies. A constructive system of notations for Γ_0 is presented in section 8. The connection between Kruskal's tree theorem and Γ_0 due to Friedman is presented in section 9. A brief discussion of some relevant subsystems of second-order arithmetic occurs in section 10. An introduction to the theory of term orderings is presented in section 11, including the recursive path ordering and the lexicographic path ordering. A glimpse at slow-growing and fast-growing hierarchies is given in section 12. Finally, constructive proofs of Higman's lemma are briefly discussed in section 13.

2 Well Quasi-Orders (WQO's)

We let $\mathbf{N}$ denote the set $\{0, 1, 2, \ldots\}$ of natural numbers, and $\mathbf{N}_+$ denote the set $\{1, 2, \ldots\}$ of positive natural numbers. Given any $n \in \mathbf{N}_+$, we let $[n]$ denote the finite set $\{1, 2, \ldots, n\}$, and we let $[0] = \emptyset$. Given a set S, a *finite sequence* u over S, or *string* over S, is a function $u : [n] \to S$, for some $n \in \mathbf{N}$. The integer n is called the *length* of u and is denoted by $|u|$. The special sequence with domain $\emptyset$ is called the *empty sequence*, or *empty string*, and will be denoted by e. Strings can be *concatenated* in the usual way: Given two strings $u : [m] \to S$ and $v : [n] \to S$, their *concatenation* denoted by $u.v$ or uv, is the string $uv : [m+n] \to S$ such that, $uv(i) = u(i)$ if $1 \leq i \leq m$, and $uv(i) = v(i-m)$ if $m+1 \leq i \leq m+n$. Clearly, concatenation is associative and e is an identity element. Occasionally, a finite sequence u of length n will be denoted as $\langle u_1, \ldots, u_n \rangle$ (denoting $u(i)$ as u_i), or as $u_1 \ldots u_n$. Strings of length 1 are identified with elements of S. The set of all strings over S is denoted as S^*.

An *infinite sequence* is a function $s : \mathbf{N}_+ \to S$. An infinite sequence s is also denoted by $(s_i)_{i \geq 1}$, or by $\langle s_1, s_2, \ldots, s_i, \ldots \rangle$. Given an infinite sequence $s = (s_i)_{i \geq 1}$, an *infinite subsequence* of s is any infinite sequence $s' = (s'_j)_{j \geq 1}$ such that there is a strictly monotonic function[1] $f : \mathbf{N}_+ \to \mathbf{N}_+$, and $s'_i = s_{f(i)}$ for all $i > 0$. An infinite subsequence s' of s associated with the function f is also denoted as $s' = (s_{f(i)})_{i \geq 1}$.

We now review preorders and well-foundedness.

Definition 2.1 Given a set A, a binary relation $\preceq \subseteq A \times A$ on the set A is a *preorder*

[1] A function $f : \mathbf{N}_+ \to \mathbf{N}_+$ is *strictly monotonic* (or *increasing*) iff for all $i, j > 0$, $i < j$ implies that $f(i) < f(j)$.

(or *quasi-order*) iff it is reflexive and transitive. A preorder that is also antisymmetric is called a *partial order*. A preorder is *total* iff for every $x, y \in A$, either $x \preceq y$ or $y \preceq x$. The relation $\succeq$ is defined such that $x \succeq y$ iff $y \preceq x$, the relation $\prec$ such that

$$x \prec y \quad \text{iff} \quad x \preceq y \quad \text{and} \quad y \not\preceq x,$$

the relation $\succ$ such that $x \succ y$ iff $y \prec x$, and the equivalence relation $\approx$ such that

$$x \approx y \quad \text{iff} \quad x \preceq y \quad \text{and} \quad y \preceq x.$$

We say that x and y are *incomparable* iff $x \not\preceq y$ and $y \not\preceq x$, and this is also denoted by $x \mid y$.

Given two preorders $\preceq_1$ and $\preceq_2$ on a set A, $\preceq_2$ is an *extension* of $\preceq_1$ iff $\preceq_1 \subseteq \preceq_2$. In this case, we also say that $\preceq_1$ is a *restriction* of $\preceq_2$.

Definition 2.2 Given a preorder $\preceq$ over a set A, an infinite sequence $(x_i)_{i\geq 1}$ is an *infinite decreasing chain* iff $x_i \succ x_{i+1}$ for all $i \geq 1$. An infinite sequence $(x_i)_{i\geq 1}$ is an *infinite antichain* iff $x_i \mid x_j$ for all i, j, $1 \leq i < j$. We say that $\preceq$ is *well-founded* and that $\succ$ is *Noetherian* iff there are no infinite decreasing chains w.r.t. $\succ$.

We now turn to the fundamental concept of a well quasi-order. This concept goes back at least to Janet [23], whose paper appeared in 1920, as recently noted by Pierre Lescanne [31]. Irving Kaplanski also told me that this concept is defined and used in his Ph.D thesis [25] (1941). The concept was further investigated by Higman [22], Kruskal [28], and Nash-Williams [36], among the forerunners.

Definition 2.3 Given a preorder $\preceq$ over a set A, an infinite sequence $(a_i)_{i\geq 1}$ of elements in A is termed *good* iff there exist positive integers i, j such that $i < j$ and $a_i \preceq a_j$, and otherwise, it is termed a *bad* sequence. A preorder $\preceq$ is a *well quasi-order*, abbreviated as *wqo*, iff every infinite sequence of elements of A is good.

Among the various characterizations of *wqo*'s, the following ones are particularly useful.

Lemma 2.4 Given a preorder $\preceq$ on a set A, the following conditions are equivalent:

1. Every infinite sequence is good (w.r.t. $\preceq$).
2. There are no infinite decreasing chains and no infinite antichains (w.r.t. $\preceq$).
3. Every preorder extending $\preceq$ (including $\preceq$ itself) is well-founded.

Proof. (1) $\Longrightarrow$ (2). Suppose that $(x_i)_{i\geq 1}$ is an infinite sequence over A such that $x_i \succ x_{i+1}$ for all $i \geq 1$. Hence, for every $i \geq 1$,

$$x_{i+1} \preceq x_i, \quad \text{and} \quad x_i \not\preceq x_{i+1}. \tag{$*$}$$

Since $\preceq$ satisfies (1), there exist integers $i, j > 0$ such that $i < j$ and $x_i \preceq x_j$. If $j = i+1$, this contradicts ($*$). If $j > (i+1)$, by transitivity of $\preceq$, since $x_{j-1} \preceq \ldots \preceq x_{i+1} \preceq x_i \preceq x_j$, we have $x_{j-1} \preceq x_j$, contradicting ($*$). Hence there are no infinite decreasing sequences, that is, $\preceq$ is well-founded. Also, it is clear that the existence of an infinite antichain would contradict (1).

(2) $\Longrightarrow$ (3). We argue by contradiction. Let $\preceq'$ be any preorder extending $\preceq$ and assume that $\preceq'$ is not well-founded. Then, there is some strictly decreasing chain

$$x_1 \succ' x_2 \succ' \cdots \succ' x_i \succ' x_{i+1} \succ' \cdots.$$

Either infinitely many elements in this sequence are related under $\preceq$, in which case we have an infinite decreasing chain w.r.t. $\preceq$, contradicting (2), or infinitely many elements in this sequence are incomparable under $\preceq$, in which case we have an infinite antichain, again contradicting (2).

(3) $\Longrightarrow$ (1). If (1) fails, then there is some infinite sequence $s = (x_i)_{i\geq 1}$ such that $x_i \not\preceq x_j$ for all i, j, $1 \leq i < j$. But then, we can extend $\preceq$ to a preorder $\preceq'$ such that s becomes an infinite decreasing chain in $\preceq'$, contradicting (3). $\square$

It is interesting to observe that the property of being a *wqo* is substantially stronger that being well-founded. Indeed, it is not true in general that any preorder extending a given well-founded preorder is well-founded. However, by (3) of lemma 2.4, this property characterizes a *wqo*. Every preorder on a finite set (including the equality relation) is a *wqo*, and by (3) of lemma 2.4, every partial ordering that is total and well-founded is a *wqo* (such orderings are called *well-orderings*).

The following lemma turns out to be the key to the proof of Kruskal's theorem. It is implicit in Nash-Williams [36], lemma 1, page 833.

Lemma 2.5 Given a preorder $\preceq$ on a set A, the following are equivalent:

(1) $\preceq$ is a *wqo* on A.

(2) Every infinite sequence $s = (s_i)_{i\geq 1}$ over A contains some infinite subsequence $s' = (s_{f(i)})_{i\geq 1}$ such that $s_{f(i)} \preceq s_{f(i+1)}$ for all $i > 0$.

Proof. It is clear that (2) implies (1). Next, assume that $\preceq$ is a *wqo*. We say that a member s_i of a sequence s is *terminal* iff there is no $j > i$ such that $s_i \preceq s_j$. We claim that the number of terminal elements in the sequence s is finite. Otherwise, the infinite sequence t of terminal elements in s is a bad sequence (because if the sequence t was good, then we would have $s_h \preceq s_k$ for two terminal elements in s, contradicting the fact that s_h is terminal), and this contradicts the fact that $\preceq$ is a *wqo*. Hence, there is some $N > 0$ such that s_i is not terminal for every $i \geq N$. We can define a strictly monotonic function f inductively as follows. Let $f(1) = N$, and for any $i \geq 1$, let $f(i+1)$ be the least integer such that $s_{f(i)} \preceq s_{f(i+1)}$ and $f(i+1) > f(i)$ (since every element $s_{f(i)}$ is not terminal by the choice of N and the definition of f, such an element exists). The infinite subsequence $s' = (s_{f(i)})_{i \geq 1}$ has the property stated in (2). $\square$

As a corollary of lemma 2.5, we obtain another result of Nash-Williams [36]. Given two preorders $\langle \preceq_1, A_1 \rangle$ and $\langle \preceq_2, A_2 \rangle$, the cartesian product $A_1 \times A_2$ is equipped with the preorder $\preceq$ defined such that $(a_1, a_2) \preceq (a'_1, a'_2)$ iff $a_1 \preceq_1 a'_1$ and $a_2 \preceq_2 a'_2$.

Lemma 2.6 If $\preceq_1$ and $\preceq_2$ are *wqo*, then $\preceq$ is a *wqo* on $A_1 \times A_2$.

Proof. Consider any infinite sequence s in $A_1 \times A_2$. This sequence is formed of pairs $(s'_i, s''_i) \in A_1 \times A_2$, and defines an infinite sequence $s' = (s'_i)_{i \geq 1}$ over A_1 and an infinite sequence $s'' = (s''_i)_{i \geq 1}$ over A_2. By lemma 2.5, since $\preceq_1$ is a *wqo*, there is some infinite subsequence $t' = (s'_{f(i)})_{i \geq 1}$ of s' such that $s'_{f(i)} \preceq_1 s'_{f(i+1)}$ for all $i > 0$. Since $\preceq_2$ is also a *wqo* and $t'' = (s''_{f(i)})_{i \geq 1}$ is an infinite sequence over A_2, there exist some i, j such that $f(i) < f(j)$ and $s''_{f(i)} \preceq_2 s''_{f(j)}$. Then, we have $(s'_{f(i)}, s''_{f(i)}) \preceq (s'_{f(j)}, s''_{f(j)})$, which shows that the sequence s is good, and that $\preceq$ is a *wqo*. $\square$

In turn, lemma 2.6 yields an interesting result due to Dickson [12], published in 1913!

Lemma 2.7 Let n be any integer such that $n > 1$. Given any infinite sequence $(s_i)_{i \geq 1}$ of n-tuples of natural numbers, there exist positive integers i, j such that $i < j$ and $s_i \preceq_n s_j$, where $\preceq_n$ is the partial order on n-tuples of natural numbers induced by the natural ordering $\leq$ on $\mathbf{N}$.

Proof. The proof follows immediately by observing that $\leq$ is a *wqo* on $\mathbf{N}$ and that lemma 2.6 extends to any $n > 1$ by a trivial induction. $\square$

Next, given a *wqo* $\preceq$ on a set A, we shall extend $\preceq$ to the set of strings A^*, and prove what is known as *Higman's theorem* [22].

3 WQO's On Strings, Higman's Theorem

Our presentation of Higman's theorem is inspired by Nash-Williams's proof of a similar theorem ([36], lemma 2, page 834), and is also very similar to the proof given by Steve Simpson ([47], lemma 1.6, page 92). Nash-Williams's proof is not entirely transparent, and Simpson's proof appeals to Ramsey's theorem. Using lemma 2.5, it is possible to simplify the proof. A proof along this line has also been given by Jean Jacques Levy in some unpublished notes [33] that came mysteriously in my possession.

Definition 3.1 Let $\sqsubseteq$ be a preorder on a set A. We define the preorder $\ll$ (*string embedding*) on A^* as follows: $e \ll u$ for each $u \in A^*$, and, for any two strings $u = u_1u_2 \ldots u_m$ and $v = v_1u_2 \ldots v_n$, $1 \le m \le n$,

$$u_1u_2 \ldots u_m \ll v_1v_2 \ldots v_n$$

iff there exist integers $j_1, \ldots, j_m$ such that $1 \le j_1 < j_2 < \ldots < j_{m-1} < j_m \le n$ and

$$u_1 \sqsubseteq v_{j_1}, \ \ldots, \ u_m \sqsubseteq v_{j_m}.$$

It is easy to show that $\ll$ is a preorder, and we leave as an exercise to show that $\ll$ is a partial order if $\sqsubseteq$ is a partial order. It is also easy to check that $\ll$ is the least preorder on A^* satisfying the following two properties:

(1) (deletion property) $uv \ll uav$, for all $u, v \in A^*$ and $a \in A$;

(2) (monotonicity) $uav \ll ubv$ whenever $a \sqsubseteq b$, for all $u, v \in A^*$ and $a, b \in A$.

Theorem 3.2 (Higman) If $\sqsubseteq$ is a *wqo* on A, then $\ll$ is a *wqo* on A^*.

Proof. Assume that $\ll$ is not a *wqo* on A^*. Then, there is at least one bad sequence from A^*. Following Nash-Williams, we define a *minimal bad sequence* t inductively as follows. Let t_1 be a string of minimal length starting a bad sequence. If $t_1, \ldots, t_n$ have been defined, let t_{n+1} be a string of minimal length such that there is a bad sequence whose first n elements are $t_1, \ldots, t_n$. Note that we must have $|t_i| \ge 1$ for all $i \ge 1$, since otherwise the sequence t is not bad (since $e \ll u$ for each $u \in A^*$). Since $|t_i| \ge 1$ for all $i \ge 1$, let

$$t_i = a_i s_i,$$

where $a_i \in A$ is the leftmost symbol in t_i. The elements a_i define an infinite sequence $a = (a_i)_{i \ge 1}$ in A, and the s_i define an infinite sequence $s = (s_i)_{i \ge 1}$ in A^*. Since $\sqsubseteq$ is a *wqo* on A, by lemma 2.5, there is an infinite subsequence $a' = (a_{f(i)})_{i \ge 1}$ of a such that

$a_{f(i)} \sqsubseteq a_{f(i+1)}$ for all $i > 0$. We claim that the infinite subsequence $s' = (s_{f(i)})_{i \geq 1}$ of s is good. Otherwise, if $s' = (s_{f(i)})_{i \geq 1}$ is bad, there are two cases.

Case 1: $f(1) = 1$. Then, the infinite sequence $s' = (s_{f(i)})_{i \geq 1}$ is a bad sequence with $|s_1| < |t_1|$, contradicting the minimality of t.

Case 2: $f(1) > 1$. Then, the infinite sequence

$$t' = \langle t_1, \ldots, t_{f(1)-1}, s_{f(1)}, s_{f(2)}, \ldots, s_{f(j)}, \ldots \rangle$$

is also bad, because $t_k = a_k s_k$ for all $k \geq 1$ and $t_i \ll s_{f(j)}$ implies that $t_i \ll t_{f(j)}$ by the definition of $\ll$. But $|s_{f(1)}| < |t_{f(1)}|$, and this contradicts the minimality of t.

Since the sequence $s' = (s_{f(i)})_{i \geq 1}$ is good, there are some positive integers i, j such that $f(i) < f(j)$ and $s_{f(i)} \ll s_{f(j)}$. Since the infinite sequence $a' = (a_{f(i)})_{i \geq 1}$ was chosen such that $a_{f(i)} \sqsubseteq a_{f(i+1)}$ for all $i > 0$, by the definition of $\ll$, we have

$$a_{f(i)} s_{f(i)} \ll a_{f(j)} s_{f(j)},$$

that is, $t_{f(i)} \ll t_{f(j)}$ (since $t_k = a_k s_k$ for all $k \geq 1$). But this shows that the sequence t is good, contradicting the initial assumption that t is bad. $\square$

A theorem similar to theorem 3.2 applying to finite subsets of A can be shown. Following Nash-Williams [36], let $\mathcal{F}(S)$ denote the set of all finite subsets of S. Given any two subsets A, B of S, a function $f : A \to B$ is *non-descending* if $a \sqsubseteq f(a)$ for every $a \in A$. The set $\mathcal{F}(S)$ is equipped with the preorder $\ll$ defined as follows: $\emptyset \ll A$ for every $A \in \mathcal{F}(S)$, and for any two nonempty subsets $A, B \in \mathcal{F}(S)$, $A \ll B$ iff there is an injective non-descending function $f : A \to B$. The proof of theorem 3.2 can be trivially modified to obtain the following.

Theorem 3.3 (Nash-Williams) If $\sqsubseteq$ is a *wqo* on A, then $\ll$ is a *wqo* on $\mathcal{F}(A)$.

We now turn to trees.

4 WQO's On Trees, Kruskal's Tree Theorem

First, we review the definition of trees in terms of tree domains.

Definition 4.1 A *tree domain* D is a nonempty subset of strings in $\mathbf{N}_+^*$ satisfying the conditions:

(1) For all $u, v \in \mathbf{N}_+^*$, if $uv \in D$ then $u \in D$.

(2) For all $u \in \mathbf{N}_+^*$, for every $i \in \mathbf{N}_+$, if $ui \in D$ then, for every j, $1 \leq j \leq i$, $uj \in D$.

The elements of D are called *tree addresses* or *nodes*. We now consider labeled trees.

Definition 4.2 Given any set Σ of labels, a Σ-*tree (or term)* is any function $t : D \to \Sigma$, where D is a tree domain denoted by $dom(t)$.

Hence, a labeled tree is defined by a tree domain D and a labeling function t with domain D and range Σ. The tree address e is called the *root* of t, and its label $t(e)$ is denoted as $root(t)$. A tree is *finite* iff its domain is finite. In the rest of this paper, only finite trees will be considered. The set of all finite Σ-trees is denoted as T_Σ.

Definition 4.3 Given a (finite) tree t, the number of tree addresses in $dom(t)$ is denoted by $|t|$. The depth of a tree t is defined as $depth(t) = max(\{|u| \mid u \in dom(t)\})$. The number of immediate successors of the root of a tree is denoted by $rank(t)$, and it is defined formally as the number of elements in the set $\{i \mid i \in \mathbf{N}_+ \text{ and } i \in dom(t)\}$. Given a tree t and some tree address $u \in dom(t)$, the *subtree of* t *rooted at* u is the tree t/u whose domain is the set $\{v \mid uv \in dom(t)\}$ and such that $t/u(v) = t(uv)$ for all v in $dom(t/u)$.

A tree t such that $rank(t) = 0$ is a one-node tree, and if $root(t) = f$, t will also be denoted by f. Given any $k \geq 1$ trees $t_1, \ldots, t_k$ and any element $f \in \Sigma$, the tree $t = f(t_1, \ldots, t_k)$ is the tree whose domain is the set

$$\{e\} \cup \bigcup_{i=1}^{i=k} \{iu \mid u \in dom(t_i)\},$$

and whose labeling function is defined such that $t(e) = f$ and $t(iu) = t_i(u)$, for $u \in dom(t_i)$, $1 \leq i \leq k$. It is well known that every finite tree t is either a one-node tree, or can be written uniquely as $t = f(t/1, \ldots, t/k)$, where $f = root(e)$, and $k = rank(t)$. It is also convenient to introduce the following abbreviations. Let $\sqsubseteq$ be a binary relation on trees. Then

$$s \sqsubseteq f(\ldots, s, \ldots)$$

is an abbreviation for $s \sqsubseteq f(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n)$,

$$f(\ldots) \sqsubseteq f(\ldots, s, \ldots)$$

is an abbreviation for $f(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \sqsubseteq f(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n)$,

$$f(\ldots, s, \ldots) \sqsubseteq g(\ldots, t, \ldots)$$

is an abbreviation for $f(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) \sqsubseteq g(s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_n)$, for some trees s, t, $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$, $1 \leq i \leq n$. When $n = 1$, these are understood as $s \sqsubseteq f(s)$, $f \sqsubseteq f(s)$, and $f(s) \sqsubseteq g(t)$.

4.1 Kruskal's Theorem, Version 1

Assuming that Σ is preordered by $\sqsubseteq$, we define a preorder $\preceq$ on Σ-trees extending $\sqsubseteq$ in the following way.

Definition 4.4 Assume that $\sqsubseteq$ is a preorder on Σ. The preorder $\preceq$ on T_Σ (*homeomorphic embedding*) is defined inductively as follows: Either

(1) $f \preceq g(t_1, \ldots, t_n)$ iff $f \sqsubseteq g$; or

(2) $s \preceq g(\ldots, t, \ldots)$ iff $s \preceq t$; or

(3) $f(s_1, \ldots, s_m) \preceq g(t_1, \ldots, t_n)$ iff $f \sqsubseteq g$, and there exist some integers $j_1, \ldots, j_m$ such that $1 \leq j_1 < j_2 < \ldots < j_{m-1} < j_m \leq n$, $1 \leq m \leq n$, and

$$s_1 \preceq t_{j_1}, \ \ldots, \ s_m \preceq t_{j_m}.$$

Note that (1) can be viewed as the special case of (3) for which $m = 0$, and $n = 0$ is possible. It is easy to show that $\preceq$ is a preorder. One can also show that $\preceq$ is a partial order if $\sqsubseteq$ is a partial order. This can be shown by observing that $s \preceq t$ implies that $depth(s) \leq depth(t)$. Hence, if $s \preceq t$ and $t \preceq s$, we have $depth(s) = depth(t)$ and $rank(s) = rank(t)$ (since only case (1) or (3) can apply). Then, we can show that $s = t$ by induction on the depth of trees.

It is also easy to show that the preorder $\preceq$ can be defined as the least preorder satisfying the following properties:

(1) $s \preceq f(\ldots, s, \ldots)$;

(2) $f(\ldots) \preceq f(\ldots, s, \ldots)$;

(3) $f(\ldots, s, \ldots) \preceq g(\ldots, t, \ldots)$ whenever $f \sqsubseteq g$ and $s \preceq t$.

We now prove a version of Kruskal's theorem [28].

Theorem 4.5 (Kruskal's tree theorem) If $\sqsubseteq$ is a *wqo* on Σ, then $\preceq$ is a *wqo* on T_Σ.

Proof. Assume that $\preceq$ is not a *wqo* on T_Σ. As in the proof of theorem 3.2, we define a minimal bad sequence t of elements of T_Σ satisfying the following properties:

(i) $|t_1| \leq |t'_1|$ for all bad sequences t';

(ii) $|t_{n+1}| \leq |t'_{n+1}|$ for all bad sequences t' such that $t'_i = t_i$, $1 \leq i \leq n$.

We claim that $|t_i| \geq 2$ for all but finitely many $i \geq 1$. Otherwise, the sequence of one-node trees in t must be infinite, and since $\sqsubseteq$ is a *wqo*, by clause (1) of the definition of $\preceq$, there are $i, j > 0$ such that $i < j$ and $t_i \preceq t_j$, contradicting the fact that t is bad.

Let $s = (s_i)_{i \geq 1}$ be the infinite subsequence of t consisting of all trees having at least two nodes, and let $f = (f_i)_{i \geq 1}$ be the infinite sequence over Σ defined such that $f_i = root(s_i)$ for every $i \geq 1$. Since $\sqsubseteq$ is a *wqo* over Σ, by lemma 2.5, there is some infinite subsequence $f' = (f_{\varphi(i)})_{i \geq 1}$ of f such that $f_{\varphi(i)} \sqsubseteq f_{\varphi(i+1)}$ for all $i \geq 1$. Let

$$\mathcal{D} = \{s_{\varphi(i)}/j \mid i \geq 1,\ 1 \leq j \leq rank(s_{\varphi(i)})\}.$$

We claim that $\preceq$ is a *wqo* on $\mathcal{D}$. Otherwise, let $r = \langle r_1, r_2, \ldots, r_j, \ldots \rangle$ be a bad sequence in $\mathcal{D}$. Because r is bad, it contains a bad subsequence $r' = \langle r'_1, r'_2, \ldots, r'_j, \ldots \rangle$ with the following property: if $i < j$, then r'_i is a subtree of a tree t_p and r'_j is a subtree of a tree t_q such that $p < q$. Indeed, every t_i only has finitely many subtrees, and r being bad must contain an infinite number of distinct trees. Thus, we consider a bad sequence r with the additional property that if $i < j$, then r_i is a subtree of a tree t_p and r_j is a subtree of a tree t_q such that $p < q$. Let n be the index of the first tree in the sequence t such that $t_n/j = r_1$ for some j. If $n = 1$, since $|r_1| < |t_1|$ and the sequence r is bad, this contradicts the fact that t is a minimal bad sequence. If $n > 1$, then the sequence

$$\langle t_1, t_2, \ldots, t_{n-1}, r_1, r_2, \ldots, r_j, \ldots \rangle$$

is bad, since by clause (ii) of the definition of $\preceq$, for any k s.t. $1 \leq k \leq n-1$, $t_k \preceq r_j$ would imply that $t_k \preceq t_h$ for some t_h and l such that $r_j = t_h/l$ and $k < h$, since each r_i is a subtree of some t_p such that $n - 1 < p$. But since $|r_1| < |t_n|$, this contradicts the fact that t is a minimal bad sequence. Hence, $\mathcal{D}$ is a *wqo*.

By Higman's theorem (theorem 3.2), the string embedding relation $\ll$ extending the preorder $\preceq$ on $\mathcal{D}$ is a *wqo* on $\mathcal{D}^*$. Hence, considering the infinite sequence over $\mathcal{D}^*$

$$\langle \langle s_{\varphi(1)}/1, s_{\varphi(1)}/2, \ldots, s_{\varphi(1)}/rank(s_{\varphi(1)}) \rangle, \ldots, \langle s_{\varphi(j)}/1, s_{\varphi(j)}/2, \ldots, s_{\varphi(j)}/rank(s_{\varphi(j)}) \rangle, \ldots \rangle,$$

there exist some $i, j > 0$ such that, letting $m = rank(s_{\varphi(i)})$ and $n = rank(s_{\varphi(j)})$,

$$\langle s_{\varphi(i)}/1, s_{\varphi(i)}/2, \ldots, s_{\varphi(i)}/m \rangle \ll \langle s_{\varphi(j)}/1, s_{\varphi(j)}/2, \ldots, s_{\varphi(j)}/n \rangle,$$

that is, there are some positive integers $j_1 < j_2 < \ldots < j_{m-1} < j_m \leq n$ such that

$$s_{\varphi(i)}/1 \preceq s_{\varphi(j)}/j_1,\ \ldots,\ s_{\varphi(i)}/m \preceq s_{\varphi(j)}/j_m.$$

Since we also have $f_{\varphi(i)} \sqsubseteq f_{\varphi(j)}$, by clause (3) of the definition of $\preceq$, we have $s_{\varphi(i)} \preceq s_{\varphi(j)}$. But s is a subsequence of t, and this contradicts the fact that t is bad. Hence, $\preceq$ is a *wqo* on T_Σ. $\square$

The above proof is basically due to Nash-Williams.

4.2 Kruskal's Theorem, Version 2

Another version of Kruskal's theorem that assumes a given preorder on T_Σ (and not just Σ) can also be proved. This version (found in J.J. Levy's unpublished notes [33]) can be used to show that certain orderings on trees are well-founded.

Definition 4.6 Assume that $\sqsubseteq$ is a preorder on T_Σ. The preorder $\preceq$ on T_Σ is defined inductively as follows: Either

(1) $f \preceq g(t_1, \ldots, t_n)$ iff $f \sqsubseteq g(t_1, \ldots, t_n)$; or

(2) $s \preceq g(\ldots, t, \ldots)$ iff $s \preceq t$; or

(3) $s = f(s_1, \ldots, s_m) \preceq g(t_1, \ldots, t_n) = t$ iff $s \sqsubseteq t$, and there exist some integers $j_1, \ldots, j_m$ such that $1 \leq j_1 < j_2 < \ldots < j_{m-1} < j_m \leq n$, $1 \leq m \leq n$, and

$$s_1 \preceq t_{j_1},\ \ldots,\ s_m \preceq t_{j_m}.$$

It is easy to show that $\preceq$ is a preorder. It can also be shown that $\preceq$ is a partial order if $\sqsubseteq$ is a partial order. Again, (1) can be viewed as the special case of (3) for which $m = 0$ and, $n = 0$ is possible. It is also easy to see that $\preceq$ can be defined as the least preorder satisfying the following properties:

(1) $s \preceq f(\ldots, s, \ldots)$;

(2) $s = f(s_1, \ldots, s_m) \preceq g(t_1, \ldots, t_n) = t$ whenever $s \sqsubseteq t$ and there exist some integers $j_1, \ldots, j_m$ such that $1 \leq j_1 < j_2 < \ldots < j_{m-1} < j_m \leq n$, $1 \leq m \leq n$, and

$$s_1 \preceq t_{j_1},\ \ldots,\ s_m \preceq t_{j_m}.$$

We can now prove another version of Kruskal's theorem.

Theorem 4.7 (J.J. Levy) If $\sqsubseteq$ is a *wqo* on T_Σ, then $\preceq$ is a *wqo* on T_Σ.

Proof. Assume that $\preceq$ is not a *wqo* on T_Σ. As in the proof of theorem 4.5, we find a minimal bad sequence t of elements of T_Σ.

Since $\sqsubseteq$ is a *wqo*, there is some infinite subsequence $t' = (t_{\psi(i)})_{i \geq 1}$ of t such that $t_{\psi(i)} \sqsubseteq t_{\psi(i+1)}$ for all $i \geq 1$. We claim that $|t_{\psi(i)}| \geq 2$ for all but finitely many $i \geq 1$. Otherwise, the sequence of one-node trees in t' must be infinite, and since $\sqsubseteq$ is a *wqo*, by clause (1) of the definition of $\preceq$, there are $i, j > 0$ such that $\psi(i) < \psi(j)$ and $t_{\psi(i)} \preceq t_{\psi(j)}$, contradicting the fact that t is bad.

Let $s = (t'_{\eta(i)})_{i\geq 1}$ be the infinite subsequence of t' consisting of all trees having at least two nodes. Since s is a subsequence of t' and t' is a subsequence of t, s is a subsequence of t of the form $s = (t_{\varphi(i)})_{i\geq 1}$ for some strictly monotonic function φ. Let

$$\mathcal{D} = \{t_{\varphi(i)}/j \mid i \geq 1,\ 1 \leq j \leq rank(t_{\varphi(i)})\}.$$

As in the proof of theorem 4.5, we can show that $\preceq$ is a *wqo* on $\mathcal{D}$.

By Higman's theorem (theorem 3.2), the string embedding relation $\ll$ extending the preorder $\preceq$ on $\mathcal{D}$ is a *wqo* on $\mathcal{D}^*$. Hence, considering the infinite sequence over $\mathcal{D}^*$

$$\langle\langle t_{\varphi(1)}/1, t_{\varphi(1)}/2, \ldots, t_{\varphi(1)}/rank(t_{\varphi(1)})\rangle, \ldots, \langle t_{\varphi(j)}/1, t_{\varphi(j)}/2, \ldots, t_{\varphi(j)}/rank(t_{\varphi(j)})\rangle, \ldots\rangle,$$

there exist some $i, j > 0$ such that, letting $m = rank(t_{\varphi(i)})$ and $n = rank(t_{\varphi(j)})$,

$$\langle t_{\varphi(i)}/1, t_{\varphi(i)}/2, \ldots, t_{\varphi(i)}/m\rangle \ll \langle t_{\varphi(j)}/1, t_{\varphi(j)}/2, \ldots, t_{\varphi(j)}/n\rangle,$$

that is, there are some positive integers $j_1 < j_2 < \ldots < j_{m-1} < j_m \leq n$ such that

$$t_{\varphi(i)}/1 \preceq t_{\varphi(j)}/j_1,\ \ldots,\ t_{\varphi(i)}/m \preceq t_{\varphi(j)}/j_m.$$

Since we also have $t_{\varphi(i)} \sqsubseteq t_{\varphi(j)}$ (because $s = (t_{\varphi(i)})_{i\geq 1}$ is also a subsequence of $t' = (t_{\psi(i)})_{i\geq 1}$ and $t_{\psi(i)} \sqsubseteq t_{\psi(i+1)}$ for all $i \geq 1$), by clause (3) of the definition of $\preceq$, we have $t_{\varphi(i)} \preceq t_{\varphi(j)}$. But this contradicts the fact that t is bad. Hence, $\preceq$ is a *wqo* on T_Σ. $\square$

This second version of Kruskal's theorem (theorem 4.7) actually implies the first version (theorem 4.5). Indeed, if $\sqsubseteq$ is a preorder on Σ, we can extend it to a preorder on T_Σ by requiring that $s \sqsubseteq t$ iff $root(s) \sqsubseteq root(t)$. It is easy to check that with this definition of $\sqsubseteq$, definition 4.6 reduces to 4.4, and that theorem 4.7 is indeed theorem 4.5.

Kruskal's theorem has been generalized in a number of ways. Among these generalizations, we mention some versions using unavoidable sets of trees due to Puel [43, 44], and a version using well rewrite orderings due to Lescanne [30].

4.3 WQO's and Well-Founded Preorders

This second version of Kruskal's theorem also has the following applications. Recall that from lemma 2.4 a *wqo* is well-founded. The following proposition is very useful to prove that orderings on trees are well-founded.

Proposition 4.8 Let $\ll$ be a preorder on T_Σ and let $\leq$ be another preorder on T_Σ such that:

(1) If $f \ll g(t_1, \ldots, t_n)$, then $f \leq g(t_1, \ldots, t_n)$;

(2) $s \leq f(\ldots, s, \ldots)$;

(3) If $f(s_1, \ldots, s_m) \ll g(t_1, \ldots, t_n)$, and $s_1 \leq t_{j_1}, \ldots, s_m \leq t_{j_m}$ for some $j_1, \ldots, j_m$ such that $1 \leq j_1 < \ldots < j_m \leq n$, then $f(s_1, \ldots, s_m) \leq g(t_1, \ldots, t_n)$.

If $\ll$ is a *wqo*, then $\leq$ is a *wqo*.

Proof. Let $\preceq$ be the preorder associated with $\ll$ as in definition 4.6. Then, an easy induction shows that the conditions of the proposition imply that $\preceq \subseteq \leq$. By theorem 4.7, since $\ll$ is a *wqo*, $\preceq$ is also a *wqo*, which implies that $\leq$ is a *wqo*. By lemma 2.4, $\leq$ is well-founded. $\square$

The following proposition also gives a sufficient condition for a preorder on trees to be well-founded.

Proposition 4.9 Assume Σ is finite, and let $\leq$ be a preorder on T_Σ satisfying the following conditions:

(1) $s \leq f(\ldots, s, \ldots)$;

(2) $s \leq t$ implies that $f(\ldots, s, \ldots) \leq f(\ldots, t, \ldots)$;

(3) $f(\ldots) \leq f(\ldots, s, \ldots)$.

Then, $\leq$ is well-founded.

Proof. Let $\ll$ be the preorder on T_Σ defined such that $s \ll t$ iff $root(s) = root(t)$. Since Σ is finite, $\ll$ is a *wqo*. Since it is clear that $\ll$ and $\leq$ satisfy the conditions of proposition 4.8, $\leq$ is well-founded. $\square$

Proposition 4.8 can be used to show that certain orderings on trees are well-founded. These orderings play a crucial role in proving the termination of systems of rewrite rules and the correctness of Knuth-Bendix completion procedures. An introduction to the theory of these orderings will be presented in section 11, and for more details, the reader is referred to the comprehensive survey by Dershowitz [7] and to Dershowitz's fundamental paper [8].

It is natural to ask whether there is an analogue to Kruskal's theorem with respect to well-founded preorders instead of *wqo*. Indeed, it is possible to prove such a theorem, using Kruskal's theorem.

Theorem 4.10 If $\sqsubseteq$ is a well-founded preorder on T_Σ, then $\preceq$ is well-founded on T_Σ.

Proof. The proof is implicit in Levy [33], Dershowitz [8], and Lescanne [29]. Unfortunately, one cannot directly apply theorem 4.7, since $\sqsubseteq$ is not necessarily a *wqo*. However, there is a way around this problem. We use the fact that every well-founded preorder $\sqsubseteq$ can be extended to a total well-founded preorder $\leq$. This fact can be proved rather simply using Zorn's lemma. The point is that $\leq$ being total and well-founded is also a *wqo*. Now, we can apply theorem 4.7 since $\leq$ is a *wqo* on T_Σ, and so $\preceq_\leq$ is a *wqo* on T_Σ, and thus it is well-founded. Finally, we note that $\preceq_\leq$ contains $\preceq$, which proves that $\preceq$ is well-founded. $\square$

Exercise: Find a proof of theorem 4.10 that does not use Zorn's lemma nor Kruskal's theorem.

4.4 Kruskal's Theorem, A Special Version

Kruskal's tree theorem is a very powerful theorem, and we state more interesting consequences. We consider the case where Σ is a finite set of symbols.

Definition 4.11 The preorder $\preceq$ on T_Σ is defined inductively as follows: Either

(1) $f \preceq f(t_1, \ldots, t_n)$, for every $f \in \Sigma$; or

(2) $s \preceq f(\ldots, t, \ldots)$ iff $s \preceq t$; or

(3) $f(s_1, \ldots, s_m) \preceq f(t_1, \ldots, t_n)$ iff $1 \leq m \leq n$, and there exist some integers $j_1, \ldots, j_m$ such that $1 \leq j_1 < j_2 < \ldots < j_{m-1} < j_m \leq n$ and

$$s_1 \preceq t_{j_1}, \ \ldots, \ s_m \preceq t_{j_m}.$$

Again, (1) can be viewed as the special case of (3) in which $m = 0$. For example,

$$f(f(h, h), h(a, b)) \preceq h(\mathbf{f}(g(\mathbf{f}(\mathbf{h}(b), a, \mathbf{h}(b))), g(a), h(\mathbf{h}(\mathbf{a}, \mathbf{b}, c)))).$$

It is also easy to show that the preorder $\preceq$ can be defined as the least preorder satisfying the following properties:

(1) $s \preceq f(\ldots, s, \ldots)$;

(2) $f(\ldots) \preceq f(\ldots, s, \ldots)$;

(3) $f(\ldots, s, \ldots) \preceq f(\ldots, t, \ldots)$ whenever $s \preceq t$.

Kruskal's theorem implies the following result.

Theorem 4.12 Given a finite alphabet Σ, $\preceq$ is a *wqo* on T_Σ.

Proof. Since any preorder on a finite set is a *wqo*, the identity relation on Σ is a *wqo*. But then, it is trivial to verify that the preorder $\preceq$ of definition 4.11 is obtained by specializing $\sqsubseteq$ to the identity relation in definition 4.4. Hence, the theorem is direct a consequence of theorem 4.5. $\square$

In particular, when Σ consists of a single symbol, we have the well-known version of Kruskal's theorem on unlabeled trees [28], except that in Kruskal's paper, the notion of embedding is defined as a certain kind of function between tree domains. We find it more convenient to define the preorder $\preceq$ inductively, as in definition 4.4. For the sake of completeness, we present the alternate definition used by Simpson [47].

4.5 Tree Domains And Embeddings: An Alternate Definition

First, given a partial order $\leq$ on a set A, given any nonempty subset S of A, we say that $\leq$ is a *total order* on S iff for all $x, y \in S$, either $x \leq y$, or $y \leq x$. We also say that S is a *chain* (under $\leq$).

Definition 4.13 A finite tree domain is a nonempty set D together with a partial order $\leq$ satisfying the following properties:

(1) D has a least element $\perp$ (with respect to $\leq$).

(2) For every $x \in D$, the set $anc(x) = \{y \in D \mid y \leq x\}$ of ancestors of x is a chain under $\leq$.

Clearly $\perp$ corresponds to the root of the tree, and for every $x \in D$, the set $anc(x) = \{y \in D \mid y \leq x\}$ is the set of nodes in the unique path from the root to x. The main difference between definition 4.1 and definition 4.13 is that independent nodes of a tree domain as defined in definition 4.13 are *unordered*, and, in particular, the immediate successors of a node are unordered.

Given any two elements $x, y \in D$, the greatest element of the set $anc(x) \cap anc(y)$ is the greatest lower bound of x and y, and it is denoted as $x \wedge y$. It is the "lowest" common ancestor of x and y. A (labeled) tree is defined as in definition 4.2, but using definition 4.13 for that of a tree domain. The notion of an *embedding* (or *homeomorphic embedding*) is then defined as follows. Let Σ be a set with some preorder $\sqsubseteq$.

Definition 4.14 Given any two trees t_1 and t_2 with tree domains $\langle D_1, \leq_1\rangle$ and $\langle D_2, \leq_2\rangle$, an *embedding* h from t_1 to t_2 is an injective function $h : \langle D_1, \leq_1\rangle \to \langle D_2, \leq_2\rangle$ such that:

(1) $h(x \wedge y) = h(x) \wedge h(y)$, for all $x, y \in D_1$.

(2) $t_1(x) \sqsubseteq t_2(h(x))$, for every $x \in D_1$.

It is easily shown that h is monotonic (choose x, y such that $x \leq_1 y$). One can verify that when the immediate successors of a node are ordered, definition 4.4 is equivalent to definition 4.14.

Next, we shall consider an extremely interesting version of Kruskal's theorem due to Harvey Friedman. A complete presentation of this theorem and its ramifications is given by Simpson [47].

5 Friedman's Finite Miniaturization of Kruskal's Theorem

Friedman's version of Kruskal's theorem, which has been called a finite miniaturization of Kruskal's theorem, is remarkable from a proof-theoretic point of view because it is *not* provable in relatively strong logical systems. Actually, Kruskal's original theorem is also not provable in relatively strong logical systems, but Kruskal's version is a second-order statement (a Π^1_1 statement, meaning that it is of the form $\forall X A$, where X is a second-order variable ranging over infinite sequences and A is first-order), whereas Friedman's version is a first-order statement (a Π^0_2 statement, meaning that it is of the form $\forall x \exists y A$, where A only contains bounded first-order quantifiers).

From now on, we assume that Σ is a finite alphabet, and we consider the embedding preorder of definition 4.11.

Theorem 5.1 (Friedman) Let Σ be a finite set. For every integer $k \geq 1$, there exists some integer $n \geq 2$ so large that, for any finite sequence $\langle t_1, \ldots, t_n \rangle$ of trees in T_Σ with $|t_m| \leq k(m+1)$ for all m, $1 \leq m \leq n$, there exist some integers i, j such that $1 \leq i < j \leq n$ and $t_i \preceq t_j$.

Proof. Following the hint given by Simpson [47], we give a proof using theorem 4.12 and König's lemma. Assume that the theorem fails. Let us say that a finite sequence $\langle t_1, \ldots, t_n \rangle$ such that $|t_m| \leq k(m+1)$ for all m, $1 \leq m \leq n$, is *good* iff there exist some integers i, j such that $1 \leq i < j \leq n$ and $t_i \preceq t_j$, and otherwise, that it is *bad*. Then, there is some $k \geq 1$ such that for all $n > 1$, there is some bad sequence $\langle t_1, \ldots, t_n \rangle$ (and $|t_m| \leq k(m+1)$ for all m, $1 \leq m \leq n$). Observe that any initial subsequence $\langle t_1, \ldots, t_j \rangle$, $j < n$, of a bad sequence is also bad. Furthermore, the size restriction ($|t_m| \leq k(m+1)$ for all m, $1 \leq m \leq n$) and the fact that Σ is finite implies that there are only finitely many bad sequences of length n. Hence, the set of finite bad sequences can be arranged into an infinite tree $\mathcal{T}$ as follows: the root of $\mathcal{T}$ is the empty sequence, and every finite bad sequence t is connected to the root by the unique path consisting of all the initial subsequences of t. From our previous remark, this infinite tree is finite-branching. By König's lemma, this tree contains an infinite path s. But since all finite initial subsequences of s are bad, s itself is bad, and this contradicts theorem 4.12. $\square$

A stronger version of the previous theorem also due to Friedman holds.

Theorem 5.2 (Friedman) Let Σ be a finite set. For every integer $k \geq 2$, there exists some integer $n \geq 2$ so large that, for any finite sequence $\langle t_1, \ldots, t_n \rangle$ of trees in T_Σ with $|t_m| \leq m$ for all m, $1 \leq m \leq n$, there exist some integers $i_1, \ldots, i_k$ such that $1 \leq i_1 < \ldots < i_k \leq n$ and $t_{i_1} \preceq \ldots \preceq t_{i_k}$.

Proof. The proof is very similar to that of theorem 5.1, but lemma 2.5 also needs to be used at the end. $\square$

Note that theorems 5.1 and 5.2 are both of the form $\forall k \exists n A(k, n)$, where $A(k, n)$ only contains bounded quantifiers, that is, they are Π^0_2 statements. Hence, each statement defines a function Fr, where $Fr(k)$ is the least integer n such that $\forall k \exists n A(k, n)$ holds.

One may ask how quickly this function grows. Is it exponential, super exponential, or worse? Well, this function grows extremely fast. It grows faster than Ackermann's function, and, even though it is recursive, it is not *provably total recursive* in fairly strong logical theories, including Peano's arithmetic. We will consider briefly hierarchies of fast-growing functions in section 12. For more details, we refer the reader to Cichon and Wainer [4], Wainer [54], and to Smoryński's articles [50,51].

The other remarkable property of the two previous theorems is that neither is provable in fairly strong logical theories (ATR_0, see section 10). The technical reason is that it is possible to define a function mapping finite trees to (rather large) countable ordinals, and this function is order preserving (between the embedding relation $\preceq$ on trees and the ordering relation on ordinals). This is true in particular for the ordinal Γ_0 (see Schütte [46], chapters 13, 14). For further details, see the articles by Simpson and Smoryński in [21]. We shall present the connection with Γ_0 in sections 9 and 10.