Presented by: Margareta Ackerman

Kruskal's Tree Theorem

Given an infinite set of trees, one of the trees in the set is topologically contained in another tree in the set.

Quasi Order

$(S, \leq)$ where S is a set and $\leq$ is a relation, such that

(1) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)

(2) $a \leq a$ $\forall a \in S$ (reflexivity)

Antichain

A set $\{a_1, a_2, a_3, ..\} \subseteq S$ of incomparable elements.

Example

Consider the set $\mathbb{Z}$x$\mathbb{Z}$. Let $(a_i, b_i) \leq (a_j, b_j)$ iff $a_i \leq a_j$ and $b_i \leq b_j$.

The set contains the antichain $(1, -1), (2, -2), (3, -3), ...$

Bad Sequence

A sequence $(a_1, a_2, a_3, ...)$ such that $a_i \not\leq a_j$ whenever $i < j$

ex. In $(\mathbb{Z}, \leq)$, the sequence -1, -2, -3, ...

Well Quasi Order

A quasi order $(S, \leq)$ such that there is no infinite bad sequence in S.
i.e. For every infinite sequence $(a_1, a_2, a_3, ...)$ $\exists$ $i < j$ such that $a_i \leq a_j$

Lemma 1

Given a quasi order $(S, \leq)$ containing an infinite sequence $(a_1, a_2, a_3, ...)$, either
(1) $(a_1, a_2, a_3, ...)$ contains an infinite bad sequence,
or
(2) $(a_1, a_2, a_3, ...)$ contains an infinite increasing sequence (need not be strict)

proof

- Let N be the subsequence of all elements $a_i \in (a_1, a_2, a_3, ...)$ such that $a_i \not\leq a_j \ \forall j > i$.
- Assume that $(a_1, a_2, a_3, ...)$ does not contain an infinite bad chain. Then N is finite.
- Let a_k be the last element in N. Then a_{k+1} is a start of an infinite increasing sequence.

Example

Consider the quasi order $(\mathbb{Z}_+ \mathrm{x} \mathbb{Z}_+, \leq)$. Is it well quasi ordered?

Yes.

• Consider a sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), ...$
Since $a_1 \in \mathbb{Z}_+$, $a_1, a_2, a_3, ...$ cannot contain an infinite bad sequence.

• Let $a_{\alpha_1}, a_{\alpha_2}, a_{\alpha_3}, ...$ be an infinite increasing subsequence.

• By a similar argument, $b_{\alpha_1}, b_{\alpha_2}, b_{\alpha_3}, ...$ must contain an infinite increasing subsequence $b_{\beta_1}, b_{\beta_2}, b_{\beta_3}, ...$

• Therefore, $(a_{\beta_1}, b_{\beta_1}), (a_{\beta_2}, b_{\beta_2}), (a_{\beta_3}, b_{\beta_3}), ...$ is an infinite increasing sequence.

Therefore, $(\mathbb{Z}_+ \mathrm{x} \mathbb{Z}_+, \leq)$ is well quasi ordered.

Higman's Theorem

Given the quasi order $(S, \leq)$, define $(L(S), \leq)$ where $L(S)$ is the set of all lists in S, and

$$[s_1, s_2, s_3] \leq [t_1, t_2, t_3, t_4, t_5]$$

If $(S, \leq)$ is a well quasi order, then $(L(S), \leq)$ is a well quasi order.

Theorem

Let $(T_1, T_2, T_3, ...)$ be an infinite sequence of **rooted** cubic trees. Then T_i is a topological minor of T_j for some $i < j$.

i.e. Rooted cubic trees are well quasi ordered under taking topological minors.

Proof

Suppose $A = (T_1, T_2, T_3, ..)$ is an infinite bad sequence of cubic trees chosen such that $(|T_1|, |T_2|, |T_3|, ...)$ is lexicographically minimal.

Define T_i^L and T_i^R.

Claim
$(\{T_i^L : i \in [1,\infty)\}, \leq)$ and $(\{T_i^R : i \in [1,\infty)\}, \leq)$ are well quasi ordered.

subproof
• Assume $A^L = (T^L_{\alpha_1}, T^L_{\alpha_2}, T^L_{\alpha_3}, ...)$ is a bad sequence.
We may assume that $\alpha_1 < \alpha_2 < \alpha_3 < ...$

• Let $B = (T_1, T_2, T_3, ..., T_{\alpha_1 - 1}, T^L_{\alpha_1}, T^L_{\alpha_2}, T^L_{\alpha_3}, ...)$. Since $|T^L_{\alpha_1}| < |T_{\alpha_1}|$, B is lexicographically smaller than A. Therefore, B is not a bad sequence.

• Since A and A^L are bad sequences, $T_i \leq T^L_j$ for some i. But $T^L_j \leq T_j$. Then $T_i \leq T_j$ for $i < j$. A contradiction.

By the same argument, $(\{T_i^R : i \in [1,\infty)\}, \leq)$ is well quasi ordered.

Therefore, $\exists$ a sequence $(T_{\beta_1}, T_{\beta_2}, T_{\beta_3}, ...)$ such that

(1) $T^L_{\beta_1} \leq T^L_{\beta_2} \leq T^L_{\beta_3} \leq ...$, and

(2) $T^R_{\beta_1} \leq T^R_{\beta_2} \leq T^R_{\beta_3} \leq ...$

Thus, $T_{\beta_1} \leq T_{\beta_2}$.

Q.E.D.

Kruskal's Tree Theorem

Rooted trees are well quasi ordered under topological minors.

Proof

For each tree T_i, define $T_i^1, T_i^2, T_i^3, ..., T_i^{n_i}$

- Let $S = \cup_{i=0}(\cup_{j=0}^{n_i} T_i^j)$

- By the construction of a minimal lexicographical bad sequence, we show that S is w.q.o.

- By Higman's theorem, $L(S)$ is well quasi ordered.

- In particular, $[T_1^1, T_1^2, ...T_1^{n_1}], [T_2^1, T_2^2, ...T_2^{n_2}], ...$

contains an infinite increasing sequence.

$Q.E.D.$

Generalizations

- Graphs are not well quasi ordered under topological minors.

- Graphs are well quasi ordered under minors (Robertson & Seymour).