Introduction to Deduction Systems:

From Natural Deduction to Sequent Calculus

(and back)

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Reading

- F. Pfenning: Automated Theorem Proving, Course at Carnegie Mellon University. Draft. 1999.
- A.S. Troelstra and H. Schwichtenberg: Basic Proof Theory. Cambridge. 2nd Edition 2000.
- John Byrnes: Proof Search and Normal Forms in Natural Deduction. PhD Thesis. Carnegie Mellon University. 1999.
- . . . many more books on Proof Theory . . .

Natural Deduction: Motivation

Frege, Russel, Hilbert Predicate calculus and type theory as formal basis for mathematics

Gentzen ND as intuitive formulation of predicate calculus; introduction and elimination rules for each logical connective

The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. . . . In contrast I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a **calculus of natural deduction** *(NJ for intuitionist, NK for classical predicate logic).* [Gentzen: Investigations into logical deduction]

Sequent Calculus: Motivation

Gentzen had a pure technical motivation for sequent calculus

- Same theorems as natural deduction
- Prove of the Hauptsatz (all sequent proofs can be found with a simple strategy)
- Corollary: Consistency of formal system(s)

The Hauptsatz says that every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not roundabout. . . .
In order to be able to prove the Hauptsatz in a convenient form, **I had to provide a logical calculus especially for the purpose***. For this the natural calculus proved unsuitable.* [Gentzen: Investigations into logical deduction]

Sequent Calculus: Introduction

Sequent calculus exposes many details of fine structure of proofs in a very clear manner. Therefore it is well suited to serve as a **basic representation formalism** for many automation oriented search procedures

- Backward: tableaux, connection methods, matrix methods, some forms of resolution
- Forward: classical resolution, inverse method

Don't be afraid of the many variants of sequent calculi.

Choose the one that is most suited for you.

Natural Deduction

Natural deduction rules operate on proof trees.

Example:

- Conjunction: $\dfrac{\begin{array}{cc} D_1 & D_2 \\ \mathbf{A} & \mathbf{B} \end{array}}{\mathbf{A} \wedge \mathbf{B}} \wedge I \quad \dfrac{\begin{array}{c} D_1 \\ \mathbf{A} \wedge \mathbf{B} \end{array}}{\mathbf{A}} \wedge E_l \quad \dfrac{\begin{array}{c} D_1 \\ \mathbf{A} \wedge \mathbf{B} \end{array}}{\mathbf{B}} \wedge E_r$

The presentation on the next slides treats the proof tree aspects implicit.

Example:

- Conjunction: $\dfrac{\mathbf{A} \quad \mathbf{B}}{\mathbf{A} \wedge \mathbf{B}} \wedge I \quad \dfrac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{A}} \wedge E_l \quad \dfrac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{B}} \wedge E_r$

Natural Deduction Rules Ia

- Conjunction: $\dfrac{\mathbf{A} \quad \mathbf{B}}{\mathbf{A} \wedge \mathbf{B}} \wedge I \quad \dfrac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{A}} \wedge E_l \quad \dfrac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{B}} \wedge E_r$
- Disjunction:

$$\dfrac{\mathbf{A}}{\mathbf{A} \vee \mathbf{B}} \vee I_l \quad \dfrac{\mathbf{B}}{\mathbf{A} \vee \mathbf{B}} \vee I_r \quad \dfrac{\mathbf{A} \vee \mathbf{B} \quad \begin{array}{c} [\mathbf{A}]_1 \\ \vdots \\ \mathbf{C} \end{array} \quad \begin{array}{c} [\mathbf{B}]_2 \\ \vdots \\ \mathbf{C} \end{array}}{\mathbf{C}} \vee E_r^{1,2}$$

- Implication: $\dfrac{\begin{array}{c} [\mathbf{A}]_1 \\ \vdots \\ \mathbf{B} \end{array}}{\mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow I^1 \quad \dfrac{\mathbf{A} \Rightarrow \mathbf{B} \quad \mathbf{A}}{\mathbf{B}} \Rightarrow E$
- Truth and Falsehood: $\dfrac{}{\top} \top I \quad \dfrac{\bot}{\mathbf{C}} \bot E$

Natural Deduction Rules IIa

- Negation: $\dfrac{\begin{array}{c} [\mathbf{A}]_1 \\ \vdots \\ \bot \end{array}}{\neg \mathbf{A}} \neg I^1 \quad \dfrac{\neg \mathbf{A} \quad \mathbf{A}}{\bot} \neg E$
- Universal Quantif.: $\dfrac{\{a^*/x\}\mathbf{A}}{\forall x.\, \mathbf{A}} \forall I \quad \dfrac{\forall x.\, \mathbf{A}}{\{t/x\}\mathbf{A}} \forall E$
- Existential Quantif.: $\dfrac{\{t/x\}\mathbf{A}}{\exists x.\, \mathbf{A}} \exists I \quad \dfrac{\exists x.\, \mathbf{A} \quad \begin{array}{c} [\{a^*/x\}\mathbf{A}] \\ \vdots \\ \mathbf{C} \end{array}}{\mathbf{C}} \exists E$

*: parameter a must be new in context

Natural Deduction Rules IIIa

For classical logic choose one of the following

- Excluded Middle $$\frac{}{\mathbf{A} \vee \neg \mathbf{A}}\ XM$$
- Double Negation $$\frac{\neg\neg \mathbf{A}}{\mathbf{A}}\ \neg\neg C$$
- Proof by Contradiction $$\frac{\begin{array}{c}[\neg \mathbf{A}]\\ \vdots \\ \bot\end{array}}{\mathbf{A}}\ \bot_c$$

Natural Deduction

Structural properties

- Exchange — hypotheses order is irrelevant
- Weakening — hypothesis need not be used
- Contraction — hypotheses can be used more than once

Natural Deduction Proofs

$$\cfrac{\cfrac{\cfrac{[\mathbf{A}]_1 \quad [\mathbf{A}]_2}{\mathbf{A} \wedge \mathbf{A}}\ \wedge I}{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})}\ \Rightarrow I^2}{\mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))}\ \Rightarrow I^1$$

$$\cfrac{\cfrac{\cfrac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{B}}\ \wedge E_r \qquad \cfrac{\cfrac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{A}}\ \wedge E_l}{\mathbf{C} \vee \mathbf{A}}\ \vee I_r}{\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})}\ \wedge I}{(\mathbf{A} \wedge \mathbf{B}) \Rightarrow (\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A}))}\ \Rightarrow I^1$$

Natural Deduction with Contexts

Idea: Localizing hypotheses; explicit representation of the available assumptions for each formula occurrence in a ND proof:

$$\Gamma \vdash \mathbf{A}$$

Γ is a multiset of the (uncanceled) assumptions on which formula $\mathbf{A}$ depends. Γ is called context.

Example proof in context notation:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\mathbf{A}_1 \vdash \mathbf{A}} \quad \cfrac{}{\mathbf{A}_2 \vdash \mathbf{A}}}{\mathbf{A}_1, \mathbf{A}_2 \vdash \mathbf{A} \wedge \mathbf{A}}\ \wedge I}{\mathbf{A}_1 \vdash \mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})}\ \Rightarrow I_2}{\vdash \mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))}\ \Rightarrow I_1$$

Natural Deduction with Contexts

Another Idea: Consider sets of assumptions instead of multisets.

$$\Gamma \vdash \mathbf{A}$$

Γ is now a set of (uncanceled) assumptions on which formula $\mathbf{A}$ depends.

Example proof:

$$\dfrac{\dfrac{\dfrac{\overline{\mathbf{A} \vdash \mathbf{A}} \quad \overline{\mathbf{A} \vdash \mathbf{A}}}{\mathbf{A} \vdash \mathbf{A} \wedge \mathbf{A}} \wedge I}{\mathbf{A} \vdash \mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})} \Rightarrow I}{\vdash \mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))} \Rightarrow I$$

Natural Deduction with Contexts

Structural properties to ensure

- Exchange (hypotheses order is irrelevant)

$$\dfrac{\Gamma, \mathbf{B}, \mathbf{A} \vdash \mathbf{C}}{\Gamma, \mathbf{A}, \mathbf{B} \vdash \mathbf{C}}$$

- Weakening (hypothesis need not be used)

$$\dfrac{\Gamma \vdash \mathbf{C}}{\Gamma, \mathbf{A} \vdash \mathbf{C}}$$

- Contraction (hypotheses can be used more than once)

$$\dfrac{\Gamma, \mathbf{A}, \mathbf{A} \vdash \mathbf{C}}{\Gamma, \mathbf{A} \vdash \mathbf{C}}$$

Natural Deduction Rules Ib

- Hypotheses: $\overline{\Gamma, \mathbf{A}, \Delta \vdash \mathbf{A}}$
- Conjunction:

$$\dfrac{\Gamma \vdash \mathbf{A} \quad \Gamma \vdash \mathbf{B}}{\Gamma \vdash \mathbf{A} \wedge \mathbf{B}} \wedge I \qquad \dfrac{\Gamma \vdash \mathbf{A} \wedge \mathbf{B}}{\Gamma \vdash \mathbf{A}} \wedge E_l \qquad \dfrac{\Gamma \vdash \mathbf{A} \wedge \mathbf{B}}{\Gamma \vdash \mathbf{B}} \wedge E_r$$

- Disjunction: $\dfrac{\Gamma \vdash \mathbf{A}}{\Gamma \vdash \mathbf{A} \vee \mathbf{B}} \vee I_l \qquad \dfrac{\Gamma \vdash \mathbf{B}}{\Gamma \vdash \mathbf{A} \vee \mathbf{B}} \vee I_r$

$$\dfrac{\Gamma \vdash \mathbf{A} \vee \mathbf{B} \quad \Gamma, \mathbf{A} \vdash \mathbf{C} \quad \Gamma, \mathbf{B} \vdash \mathbf{C}}{\Gamma \vdash \mathbf{C}} \vee E_r$$

- Implication: $\dfrac{\Gamma, \mathbf{A} \vdash \mathbf{B}}{\Gamma \vdash \mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow I \qquad \dfrac{\Gamma \vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Gamma \vdash \mathbf{A}}{\Gamma \vdash \mathbf{B}} \Rightarrow E$

Natural Deduction Rules IIb

- Truth and Falsehood: $\overline{\Gamma \vdash \top} \top I \qquad \dfrac{\Gamma \vdash \bot}{\Gamma \vdash \mathbf{C}} \bot E$
- Negation: $\dfrac{\Gamma, \mathbf{A} \vdash \bot}{\Gamma \vdash \neg \mathbf{A}} \neg I \qquad \dfrac{\Gamma \vdash \neg \mathbf{A} \quad \Gamma \vdash \mathbf{A}}{\Gamma \vdash \bot} \neg E$
- Universal Quantif.: $\dfrac{\Gamma \vdash \{a^*/x\}\mathbf{A}}{\Gamma \vdash \forall x.\ \mathbf{A}} \forall I \qquad \dfrac{\Gamma \vdash \forall x.\ \mathbf{A}}{\Gamma \vdash \{t/x\}\mathbf{A}} \forall E$
- Existential Quantif.:

$$\dfrac{\Gamma \vdash \{t/x\}\mathbf{A}}{\Gamma \vdash \exists x.\ \mathbf{A}} \exists I \qquad \dfrac{\Gamma \vdash \exists x.\ \mathbf{A} \quad \Gamma, \{a^*/x\}\mathbf{A} \vdash \mathbf{C}}{\Gamma \vdash \mathbf{C}} \exists E$$

*: parameter a must be new in context

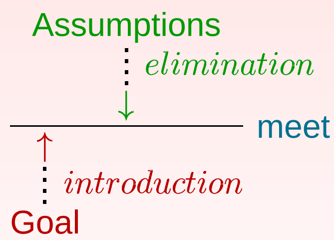
Natural Deduction Rules IIIb

For classical logic add:

- Proof by Contradiction: $$\frac{\Gamma, \neg\mathbf{A} \vdash \bot}{\Gamma \vdash \mathbf{A}}\ \bot_c$$

Intercalation

Idea (Prawitz, Sieg & Scheines, Byrnes & Sieg):
Detour free proofs: strictly use introduction rules bottom up (from proposed theorem to hypothesis) and elimination rules top down (from assumptions to proposed theorem). When they meet in the middle we have found a proof in normal form.



$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots\\ \mathbf{A} & \mathbf{B}\end{matrix}}{\mathbf{A} \wedge \mathbf{B}}\ \wedge I}{\begin{matrix}\mathbf{A}\\ \vdots\end{matrix}}\ \wedge E_l$$

Intercalating Natural Deductions

New annotations:

- $\mathbf{A}\uparrow$: $\mathbf{A}$ is obtained by an introduction derivation
- $\mathbf{A}\downarrow$: $\mathbf{A}$ is extracted from a hypothesis by an elimination derivation

Example:

$$\frac{\Gamma, \mathbf{A} \vDash_{\text{ic}} \mathbf{B}\uparrow}{\Gamma \vDash_{\text{ic}} \mathbf{A} \Rightarrow \mathbf{B}\uparrow}\ \Rightarrow I \qquad \frac{\Gamma \vDash_{\text{ic}} \mathbf{A} \Rightarrow \mathbf{B}\downarrow \quad \Gamma \vDash_{\text{ic}} \mathbf{A}\downarrow}{\Gamma \vDash_{\text{ic}} \mathbf{B}\downarrow}\ \Rightarrow E$$

ND Intercalation Rules I

- Hypotheses: $$\overline{\Gamma, \mathbf{A}, \Delta \vDash_{\text{ic}} \mathbf{A}\downarrow}$$
- Conjunction:

$$\frac{\Gamma \vDash_{\text{ic}} \mathbf{A}\uparrow \quad \Gamma \vDash_{\text{ic}} \mathbf{B}\uparrow}{\Gamma \vDash_{\text{ic}} \mathbf{A} \wedge \mathbf{B}\uparrow}\ \wedge I \qquad \frac{\Gamma \vDash_{\text{ic}} \mathbf{A} \wedge \mathbf{B}\downarrow}{\Gamma \vDash_{\text{ic}} \mathbf{A}\downarrow}\ \wedge E_l \qquad \frac{\Gamma \vDash_{\text{ic}} \mathbf{A} \wedge \mathbf{B}\downarrow}{\Gamma \vDash_{\text{ic}} \mathbf{B}\downarrow}\ \wedge E_r$$

- Disjunction: $$\frac{\Gamma \vDash_{\text{ic}} \mathbf{A}\uparrow}{\Gamma \vDash_{\text{ic}} \mathbf{A} \vee \mathbf{B}\uparrow}\ \vee I_l \qquad \frac{\Gamma \vDash_{\text{ic}} \mathbf{B}\uparrow}{\Gamma \vDash_{\text{ic}} \mathbf{A} \vee \mathbf{B}\uparrow}\ \vee I_r$$

$$\frac{\Gamma \vDash_{\text{ic}} \mathbf{A} \vee \mathbf{B}\downarrow \quad \Gamma, \mathbf{A} \vDash_{\text{ic}} \mathbf{C}\uparrow \quad \Gamma, \mathbf{B} \vDash_{\text{ic}} \mathbf{C}\uparrow}{\Gamma \vdash \mathbf{C}\uparrow}\ \vee E_r$$

- Implication:

$$\frac{\Gamma, \mathbf{A} \vDash_{\text{ic}} \mathbf{B}\uparrow}{\Gamma \vDash_{\text{ic}} \mathbf{A} \Rightarrow \mathbf{B}\uparrow}\ \Rightarrow I \qquad \frac{\Gamma \vDash_{\text{ic}} \mathbf{A} \Rightarrow \mathbf{B}\downarrow \quad \Gamma \vDash_{\text{ic}} \mathbf{A}\downarrow}{\Gamma \vDash_{\text{ic}} \mathbf{B}\downarrow}\ \Rightarrow E$$

ND Intercalation Rules II

- Truth and Falsehood: $\dfrac{}{\Gamma \vdash_{\mathrm{ic}} \top \uparrow}\ \top I \qquad \dfrac{\Gamma \vdash_{\mathrm{ic}} \bot \downarrow}{\Gamma \vdash_{\mathrm{ic}} \mathbf{C} \uparrow}\ \bot E$
- Negation: $\dfrac{\Gamma, \mathbf{A} \vdash_{\mathrm{ic}} \bot \uparrow}{\Gamma \vdash_{\mathrm{ic}} \neg\mathbf{A} \uparrow}\ \neg I \qquad \dfrac{\Gamma \vdash_{\mathrm{ic}} \neg\mathbf{A} \downarrow \quad \Gamma \vdash_{\mathrm{ic}} \mathbf{A} \uparrow}{\Gamma \vdash_{\mathrm{ic}} \bot \uparrow}\ \neg E$
- Universal Quantif.:

$$\frac{\Gamma \vdash_{\mathrm{ic}} \{a^*/x\}\mathbf{A} \uparrow}{\Gamma \vdash_{\mathrm{ic}} \forall x.\ \mathbf{A} \uparrow}\ \forall I \qquad \frac{\Gamma \vdash_{\mathrm{ic}} \forall x.\ \mathbf{A} \downarrow}{\Gamma \vdash_{\mathrm{ic}} \{t/x\}\mathbf{A} \downarrow}\ \forall E$$

- Existential Quantif.:

$$\frac{\Gamma \vdash_{\mathrm{ic}} \{t/x\}\mathbf{A} \uparrow}{\Gamma \vdash_{\mathrm{ic}} \exists x.\ \mathbf{A} \uparrow}\ \exists I \qquad \frac{\Gamma \vdash_{\mathrm{ic}} \exists x.\ \mathbf{A} \downarrow \quad \Gamma, \{a^*/x\}\mathbf{A} \vdash_{\mathrm{ic}} \mathbf{C} \uparrow}{\Gamma \vdash \mathbf{C} \uparrow}\ \exists E$$

*: parameter a must be new in context

ND Intercalation Rules III

For classical logic add:

- Proof by Contradiction: $\dfrac{\Gamma, \neg\mathbf{A} \vdash_{\mathrm{ic}} \bot \uparrow}{\Gamma \vdash_{\mathrm{ic}} \mathbf{A} \uparrow}\ \bot_c$

Intercalation and ND

Normal form proofs

Assumptions

⋮ *elimination*

$\downarrow$

——— meet

$\uparrow$

⋮ *introduction*

Goal

guaranteed by $\dfrac{\Gamma \vdash_{\mathrm{ic}} \mathbf{A} \downarrow}{\Gamma \vdash_{\mathrm{ic}} \mathbf{A} \uparrow}$ meet

. . . proofs without detour . . .

To model all ND proofs add $\dfrac{\Gamma \vdash_{\mathrm{ic}} \mathbf{A} \uparrow}{\Gamma \vdash_{\mathrm{ic}} \mathbf{A} \downarrow}$ roundabout

Example Proofs

In normal form

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{M}\wedge\mathbf{Q} \downarrow}\ \wedge E_r}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q} \downarrow}\ \text{meet}}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q} \uparrow}\ \vee I_l}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q}\vee\mathbf{S} \uparrow}\ \Rightarrow I}{\vdash_{\mathrm{ic}} (\mathbf{M}\wedge\mathbf{Q}) \Rightarrow (\mathbf{Q}\vee\mathbf{S}) \uparrow}$$

With detour

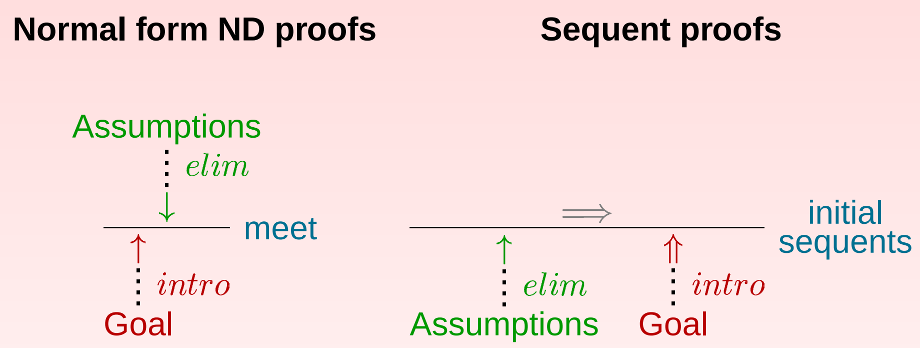
$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}\vdots & \vdots\\ \mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q} \uparrow & \mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{M} \uparrow\end{matrix}}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q}\wedge\mathbf{M} \uparrow}\ \wedge I}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q}\wedge\mathbf{M} \downarrow}\ \text{roundabout}}{\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q} \downarrow}\ \wedge E_l}{\begin{matrix}\mathbf{M}\wedge\mathbf{Q} \vdash_{\mathrm{ic}} \mathbf{Q} \uparrow\\ \vdots\end{matrix}}\ \text{meet}$$

Soundness and Completeness

Let $\vdash^{\pm}_{ic}$ denote the intercalation calculus with rule roundabout and $\vdash_{ic}$ the calculus without this rule.

- Theorem 1 (Soundness):
  If $\Gamma \vdash^{\pm}_{ic} \mathbf{A} \uparrow$ then $\Gamma \vdash \mathbf{A}$.
- Theorem 2 (Completeness):
  If $\Gamma \vdash \mathbf{A}$ then $\Gamma \vdash^{\pm}_{ic} \mathbf{A} \uparrow$.
- Is normal form proof search also complete?:
  If $\Gamma \vdash^{\pm}_{ic} \mathbf{A} \uparrow$ then $\Gamma \vdash_{ic} \mathbf{A} \uparrow$?
  We will investigate this question within the **sequent calculus**.

From ND to Sequent Calculus



Sequents pair $<\Gamma, \Delta>$ of finite lists, multisets, or sets of formulas; notation: $\Gamma \Longrightarrow \Delta$

Intuitive: a kind of implication, Δ "follows from" Γ

Sequent Calculus Rules I

- Initial Sequents: $\dfrac{}{\Gamma, \mathbf{A} \Longrightarrow \Delta, \mathbf{A}}\; init$ ($\mathbf{A}$ atomic)
- Conjunction:
  $$\frac{\Gamma, \mathbf{A}, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \wedge \mathbf{B} \Longrightarrow \Delta}\,\wedge L \qquad \frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \quad \Gamma \Longrightarrow \Delta, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \wedge \mathbf{B}}\,\wedge R$$
- Implication
  $$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \Rightarrow \mathbf{B} \Longrightarrow \Delta}\,\Rightarrow L \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \Rightarrow \mathbf{B}}\,\Rightarrow R$$
- Truth and Falsehood $\dfrac{}{\Gamma, \bot \Longrightarrow \Delta}\,\bot L \qquad \dfrac{}{\Gamma \Longrightarrow \Delta, \top}\,\top R$

Sequent Calculus Rules II

- Negation: $\dfrac{\Gamma \Longrightarrow \Delta, \mathbf{A}}{\Gamma, \neg\mathbf{A} \Longrightarrow \Delta}\,\neg L \qquad \dfrac{\Gamma, \mathbf{A} \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\mathbf{A}}\,\neg R$
- Disjunction:
  $$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A}, \mathbf{B}}{\Gamma \Longrightarrow \Delta, \mathbf{A} \vee \mathbf{B}}\,\vee R \qquad \frac{\Gamma, \mathbf{A} \Longrightarrow \Delta \quad \Gamma, \mathbf{B} \Longrightarrow \Delta}{\Gamma, \mathbf{A} \vee \mathbf{B} \Longrightarrow \Delta}\,\vee L$$
- Universal Quantification:
  $$\frac{\Gamma, \forall x\text{.}\ \mathbf{A}, \{t/x\}\mathbf{A} \Longrightarrow \Delta}{\Gamma, \forall x\text{.}\ \mathbf{A} \Longrightarrow \Delta}\,\forall L \qquad \frac{\Gamma \Longrightarrow \Delta, \{a/x\}\mathbf{A}}{\Gamma \Longrightarrow \Delta, \forall x\text{.}\ \mathbf{A}}\,\forall R$$
- Existential Quantification:
  $$\frac{\Gamma, \{a/x\}\mathbf{A} \Longrightarrow \Delta}{\Gamma, \exists x\text{.}\ \mathbf{A} \Longrightarrow \Delta}\,\forall L \qquad \frac{\Gamma \Longrightarrow \Delta, \exists x\text{.}\ \mathbf{A}, \{t/x\}\mathbf{A}}{\Gamma \Longrightarrow \Delta, \exists x\text{.}\ \mathbf{A}}\,\forall R$$

Example Proof

$$\dfrac{\dfrac{\overline{\mathbf{A},\mathbf{B} \Longrightarrow \mathbf{B}}\ init}{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{B}}\ \wedge L \qquad \dfrac{\dfrac{\dfrac{\overline{\mathbf{A},\mathbf{B} \Longrightarrow \mathbf{C},\mathbf{A}}\ init}{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{C},\mathbf{A}}\ \wedge L}{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{C}\vee\mathbf{A}}\ \vee R}{\dfrac{\mathbf{A}\wedge\mathbf{B} \Longrightarrow \mathbf{B}\wedge(\mathbf{C}\vee\mathbf{A})}{\Longrightarrow (\mathbf{A}\wedge\mathbf{B}) \Rightarrow \mathbf{B}\wedge(\mathbf{C}\vee\mathbf{A})}\ \Rightarrow R}\ \wedge R$$

Sequent Calculus: Cut-rule

To map natural deductions (in $\vdash$ and $\models^{\pm}_{\text{ic}}$) to sequent calculus derivations we add: called **cut-rule**:

$$\frac{\Gamma \Longrightarrow \Delta, \mathbf{A} \qquad \Gamma, \mathbf{A} \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta}\ Cut$$

The question whether normal form proof search ($\models_{\text{ic}}$) is complete corresponds to the question whether the cut-rule can be eliminated (is *admissible*) in sequent calculus.

Sequent Calculus

Let $\Longrightarrow^{+}$ denote the sequent calculus with cut-rule and $\Longrightarrow$ the sequent calculus without the cut-rule.

Theorem 3 (Soundness)

(a) If $\Gamma \Longrightarrow \mathrm{C}$ then $\Gamma \models_{\text{ic}} \mathrm{C} \uparrow$.

(b) If $\Gamma \Longrightarrow^{+} \mathrm{C}$ then $\Gamma \models^{\pm}_{\text{ic}} \mathrm{C} \uparrow$.

Theorem 4 (Completeness)

If $\Gamma \models^{\pm}_{\text{ic}} \mathrm{C} \uparrow$ then $\Gamma \Longrightarrow^{+} \mathrm{C}$.

Gentzen's Hauptsatz

Theorem 5 (Cut-Elimination): Cut-elimination holds for the sequent calculus. In other words: The cut rule is *admissible* in the sequent calculus.

If $\Gamma \Longrightarrow^{+} \mathrm{C}$ then $\Gamma \Longrightarrow \mathrm{C}$

Proof non-trivial; main means: nested inductions and case distinctions over rule applications

This result qualifies the sequent calculus as suitable for automating proof search.

Applications of Cut-Elimination

Theorem (Normalization for ND):

$$\text{If } \Gamma \vdash C \text{ then } \Gamma \models_{ic} C \uparrow .$$

Proof sketch:

Assume $\Gamma \vdash C$.

Then $\Gamma \models^{+}_{ic} C \uparrow$ by completeness of $\models^{+}_{ic}$.

Then $\Gamma \Longrightarrow^{+} C$ by completeness of $\Longrightarrow^{+}$.

Then $\Gamma \Longrightarrow C$ by cut-elimination.

Then $\Gamma \models_{ic} C \uparrow$ by soundness of $\Longrightarrow$.

What have we done?

Natural Deduction	Intercalation	Sequent Calculus
$\vdash$ (with detours) $\longrightarrow$	$\longrightarrow$ $\models^{+}_{ic}$ (with roundabout) $\longrightarrow$	$\longrightarrow$ $\Longrightarrow^{+}$ (with cut) $\downarrow$
$\longleftarrow$ $\vdash$ (without detours)	$\longleftarrow$ $\models_{ic}$ (without roundabout) $\longleftarrow$	$\longleftarrow$ $\Longrightarrow$ (without cut) $\longleftarrow$

Applications of Cut-Elimination

Theorem (Consistency of ND): There is no natural deduction derivation $\vdash \bot$.

Proof sketch:

Assume there is a proof of $\vdash \bot$.

Then $\Longrightarrow^{+} \bot$ by completeness of $\Longrightarrow^{+}$ and $\models^{+}_{ic}$.

But $\Longrightarrow^{+} \bot$ cannot be the conclusion of any sequent rule.

Contradiction.

Summary

We have illustrated the connection of

- natural deduction and sequent calculus
- normal form natural deductions and cut-free sequent calculus.

Fact: Sequent calculus often employed as meta-theory for specialized proof search calculi and strategies.

Question: Can these calculi and strategies be transformed to natural deduction proof search?