Reflections

on

Finite Model Theory

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What is finite model theory?

It is the study of logics on classes of finite structures.

- **Logics:**
  - First-order logic FO and various extensions of FO:
    - Fragments of second-order logic SO.
    - Logics with fixed-point operators.
    - Logics with generalized quantifiers.

- **Classes of finite structures:**
  - All finite structures $\mathbf{A} = (A, R_1,\ldots R_m)$ over a fixed vocaculary.
  - All ordered finite structures $\mathbf{A} = (A, <, R_1,\ldots,R_m)$.
  - Restricted classes of finite structures of combinatorial or of algorithmic interest (trees, planar graphs, partial orders, …).

Contrast with traditional focus of logic

- Study of logics on the class of all structures
  - **Gödel's Completeness Theorem**

    Truth in FO on the class of all (finite & infinite) structures

- Study of logics on a fixed infinite structure
  - **Gödel's Incompleteness Theorem**

    Truth in FO on the structure **N** = (N, +, x) of the integers
  - **Tarski's Theorem**

    Truth in FO on the structure **R** = (R, +, x) of the reals.

Brief History

- Late 1940s to 1970:
  - Early scattered results and problems about FO in the finite.
- Early 1970s to present:
  - Steady development of finite model theory in its own right.
  - Extensive interaction with computational complexity, database theory, asymptotic combinatorics, automated verification, constraint satisfaction.
- Finite model theory has had a constant presence in LICS.
  - At least five times the Kleene Award for Best Student Paper has been given for work in finite model theory.

Aims of this Talk

To reflect on finite model by

- Highlighting some of its successes;
- Examining obstacles that were encountered;
- Discussing some open problems that have resisted solution.

This talk is

neither

- a comprehensive survey of finite model theory

nor

- a "personal perspective" on the development of finite model theory.

Early Beginnings: a theorem and two problems.

Theorem: Trakhtenbrot – 1950

First-order finite validities **cannot** be axiomatized:

The set of finitely valid first-order sentences is **not** recursively enumerable.

- "Anti-completeness" theorem
- Sharp contrast with Gödel's Completeness Theorem: first-order validities **can** be axiomatized.

The Spectrum Problem

Definition:

A set S of positive integers is a spectrum if there is a FO-sentence ϕ such that

$$S = \{m: \phi \text{ has a finite model with } m \text{ elements} \}$$

Example: The set of all powers of primes is a spectrum.

The Spectrum Problem

- Scholz – 1952: Characterize all spectra
- Asser – 1955: Are spectra closed under complement?
  Is the complement of a spectrum a spectrum?

Preservation under Substructures

- **Theorem:** Łoś- Tarski – 1948

  If a FO-sentence ψ is preserved under substructures on all (finite and infinite) structures, then there is a universal FO-sentence $\psi*$ that is equivalent to ψ on all structures.

- **Conjecture:** Scott and Suppes – 1958

  The Łoś- Tarski Theorem holds in the finite:

  If a FO-sentence ψ is preserved under substructures on all finite structures, then there is a universal FO-sentence ψ*
  that is equivalent to ψ on all finite structures.

Main Themes in Finite Model Theory

- **Descriptive complexity:**
  computational complexity vs. uniform definability.

- **Expressive power of logics in the finite:**
  What **can** and what **cannot** be expressed in various logics on classes of finite structures.

- **Logic and asymptotic probabilities on finite structures**
  0-1 laws and convergence laws.

- **Classical Model theory in the finite:**
  Do the classical results of model theory hold in the finite?

Notation and Terminology

- σ: a fixed relational vocabulary $\{R_1, \ldots, R_m\}$
- ***C***: a class of finite σ-structures closed under isomorphisms.

- A k-ary query on ***C*** is a mapping Q defined on ***C*** such that
  - If **A** $\in$ ***C***, then Q(**A**) is a k-ary relation on A;
  - Q is invariant under isomorphisms:
    if f: **A** $\rightarrow$ **B** is an isomorphism, then Q(**B**) = f(Q(**A**)).
- **Example:** TRANSITIVE CLOSURE of a graph **G** = (V,E)

- A Boolean query on ***C*** is a mapping Q: ***C*** $\rightarrow \{0, 1\}$
  that is invariant under isomorphisms
- **Example:** CONNECTIVITY, 3-COLORABILITY, …

Complexity vs. Definability

- **Computational complexity** is concerned with the **computational resources** (model of computation, time, space) needed to **compute** queries.
- **Logical definability** is concerned with the **logical resources** (type of quantification, number of variables, operators extending the syntax of first-order logic, …) needed to **express** queries.
- **Descriptive complexity** studies the connections between computational complexity and logical definability.

Descriptive Complexity

Main Finding:

All major computational complexity classes, including P, NP, and PSPACE, can be characterized in terms of definability in various logics on classes of finite structures.

- Reinforces the unity of computation and logic.
- Yields machine-independent characterizations of computational complexity classes.

Descriptive Complexity: Characterizing NP

Theorem: Fagin – 1974

Let ***F*** be the class of all finite σ-structures and let Q be a query on ***F***. Then the following are equivalent:

- Q is is NP.
- Q is definable by an existential second-order formula

$$\exists S_1 \ldots \exists S_k \phi(S_1, \ldots, S_k).$$

In symbols, NP = ESO on ***F***.

Example: 3-COLORABILITY of a graph (V,E) is definable by

$\exists B \exists R \exists G$ ((B,R,G) form a partition of V

$\wedge \forall x \forall y$ (E(x,y) $\rightarrow$ x, y are in different parts)).

Descriptive Complexity: Characterizing NP

Corollary: The following are equivalent:

- NP is closed under complement (i.e., NP = coNP).
- ESO is closed under complement on the class ***G*** of all finite graphs.
- NON 3-COLORABILITY is ESO-definable on ***G***.

Proof:

Fagin's Theorem and NP-completeness of 3-COLORABILITY.

Descriptive Complexity & Spectrum Problem

Theorem: Jones and Selman, Fagin – 1974.
The following are equivalent for a set S of positive integers in binary notation:

- S is a spectrum.
- S is in NEXPTIME.

Corollary: The following are equivalent:

- Spectra are closed under complement.
- NEXPTIME is closed under complement.

Conclusion: Asser's question is equivalent to a major open problem in computational complexity.

Descriptive Complexity: Characterizing P

Theorem: Immerman – 1982, Vardi – 1982

Let ***O*** be the class of all ordered finite σ-structures **A** = $(A, <, R_1,\ldots,R_m)$ and let Q be a query on ***O***. Then the following are equivalent:

- Q is in P.
- Q is definable in least-fixed point logic LFP.

In symbols, P = LFP on ***O***.

Note: LFP = (FO + Least fixed-points of positive FO-formulas)

Example: The TRANSITIVE CLOSURE query is definable by the least fixed point of the FO-formula $E(x,y) \vee \exists z(E(x,z) \wedge T(z,y))$

$$T(x,y) \equiv E(x,y) \vee \exists z(E(x,z) \wedge T(z,y))$$

Descriptive Complexity Results

Two groups of results:

Group I: A complexity class (typically, NP or higher) can be characterized in terms of uniform definability in a logic on the class ***F*** of all finite σ-structures (and, hence, on all subclasses of ***F***).

Group II: A complexity class (typically, P or lower) can be characterized in terms of definability in a logic on the class ***O*** of all ordered finite σ-structures $\mathbf{A} = (A, <, R_1, \ldots, R_m)$.

Note: LFP cannot express counting queries on ***F***
(eg., EVEN CARDINALITY).

The Quest for a Logic for P

Problem: Chandra and Harel – 1982

Is there an effective enumeration of all polynomial-time computable queries on the class ***F*** of all finite σ-structures?

Conjecture: Gurevich – 1988

There is **no** logic that captures P on the class ***F*** of all finite σ-structures.

Note:

If P = NP, then there is logic for P (namely, ESO).

The Quest for a Logic for P

- Has motivated numerous investigations in finite model theory:
  - Systematic study of various extensions of first-order logic, including generalized quantifiers and fixed-point operators.
  - Systematic development of tools to delineate the expressive power of extensions of first-order logic in the finite, such as Ehrenfeucht – Fraïssé games and their variants: Ehrenfeucht – Fraïssé games for ESO, pebble games, and games for logics with generalized quantifiers.
- However,
  Chandra and Harel's Problem and Gurevich's Conjecture remain outstanding open problems in finite model theory.

Restricted Classes of Finite Structures

- Progressive shift of emphasis from the class of all finite structures to restricted classes of finite structures.

- **Theorem:** Let (IFP + C) be the extension of FO with inflationary fixed-points and counting quantifiers.
  - Grohe – 1998

    P = (IFP + C) on the class ***P*** of all planar graphs.
  - Grohe and Mariño – 1999

    P = (IFP + C) on the class ***T***(k) of graphs of treewidth $\leq$ k.

- **Note:** Deeper properties of the restricted classes are used to find an (IFP + C)-definable linear order on structures in the restricted class.

Reflecting on Descriptive Complexity

Early Optimism:

- Descriptive complexity results reduce the separation of complexity classes to the separation of logics in the finite.
- Combinatorial games (Ehrenfeucht – Fraïssé games and their variants) provide a sound and complete method for delineating the expressive power of logics in the finite.
- Use logic to resolve open problems in computational complexity.

Example: Recall that the following are equivalent:

- NP is **not** closed under complement (i.e., NP $\neq$ coNP).
- NON 3-COLORABILITY is **not** ESO-definable on ***G***.

Reflecting on Descriptive Complexity

Reality: The implementation of this approach is confronted with seemingly insurmountable combinatorial obstacles.

- Combinatorial games have been successfully used to analyze the expressive power of **monadic** ESO

  $\exists S_1 \ldots \exists S_k \phi(S_1, \ldots, S_k)$, where the S_i's are unary symbols.
- The expressive power of **binary** ESO is poorly understood.

Problem: Fagin – 1990

Prove or disprove that there is a query Q on graphs such that

- Q is ESO-definable.
- Q is **not** definable in binary ESO with a single existentially quantified binary symbol

  $\exists S\ \phi(S)$, where S is a binary relation symbol.

Reflecting on Descriptive Complexity

Reality:

- The expressive power of FO on the class ***F*** of all finite structures is **well** understood.
- The expressive power of FO on classes of ordered finite structures **A** = (A, <, $R_1,\ldots,R_m$) is **poorly** understood.

The Ordered Conjecture: K … and Vardi – 1992

If ***C*** is a class of ordered finite structures of arbitrarily large cardinalities, then FO ≠ LFP on ***C*** (i.e., FO ≠ P on ***C***).

Note: Either way of resolving the Ordered Conjecture has complexity-theoretic implications.

Main Themes in Finite Model Theory

- ✓ **Descriptive complexity:**
  computational complexity vs. uniform definability.

- ✓ **Expressive power of logics in the finite:**
  What **can** and what **cannot** be expressed in various logics on classes of finite structures.

- **Logic and asymptotic probabilities on finite structures**
  0-1 laws and convergence laws.

- **Classical Model theory in the finite:**
  Do the classical results of model theory hold in the finite?

Logic and Asymptotic Probabilities

- **Notation:**
  - Q: Boolean query on the class ***F*** of all finite structures
  - $\mathbf{F}_n$: Class of finite structures of cardinality n
  - μ_n: Probability measure on $\boldsymbol{F}_n$, $n \geq 1$
  - $\mu_n(Q)$ = Probability of Q on $\boldsymbol{F}_n$ with respect to μ_n, $n \geq 1$.

- **Definition:** Asymptotic probability of query Q

  $\mu(Q) = \lim \mu_{n \to \infty}(Q)$ (provided the limit exists)

- **Examples:** For the uniform measure μ on finite graphs **G**:
  - μ(**G** contains a triangle) = 1.
  - μ(**G** is connected) = 1.
  - μ(**G** is 3-colorable) = 0.
  - μ(**G** has even cardinality) does not exist.

0-1 Laws and Convergence Laws

Question: Is there a connection between the definability of a query Q in some logic L and its asymptotic probability?

Definition: Let L be a logic

- The 0-1 law holds for L w.r.t. to a measure μ_n, $n \geq 1$, if

$$\mu(Q) = 0 \quad \text{or} \quad \mu(Q) = 1,$$

  for every L-definable Boolean query Q.

- The convergence law holds for L w.r.t. to a measure μ_n, $n \geq 1$, if $\mu(Q)$ exists, for every L-definable Boolean query Q.

0-1 Law for First-Order Logic

Theorem: Glebskii et al. – 1969, Fagin – 1972
The 0-1 law holds for FO w.r.t. to the uniform measure.

Transfer Theorem: Fagin – 1972
There is a unique countable graph **R** such that for every FO-sentence ψ, we have that

$$\mu(\psi) = 1 \text{ if and only if } \mathbf{R} \models \psi.$$

Note:

- **R** is Rado's graph: the unique countable, homogeneous, and universal graph.
- **R** is characterized by a set of first-order extension axioms.

Decision Problem for 0-1 Law

Problem: Given a FO-sentence ψ, tell whether
$\mu(\psi) = 0$ or $\mu(\psi) = 1$.

Note:

- By the Transfer Theorem, this is equivalent to deciding first-order truth on **R**.
- Fagin's proof shows it is a decidable problem.

Theorem: Grandjean – 1983
The decision problem for the 0-1 law for FO is PSPACE-complete.

FO Truth vs. FO Almost Sure Truth

Everywhere true (valid)
Somewhere true & Somewhere false
Everywhere false (contradiction)

Almost surely true
Almost surely false

- **First-Order Truth**
  Testing if a FO-sentence is **true** on all finite graphs is an **undecidable** problem.

- **Almost Sure First-Order Truth**
  Testing if a FO-sentence is **almost surely true** on all finite graphs is a **decidable** problem; in fact, it is PSPACE-complete.

Three Directions of Research on 0-1 Laws

- 0-1 laws for extensions of FO w.r.t. the uniform measure.
- 0-1 laws for FO on restricted classes of finite structures
- 0-1 laws on graphs under variable probability measures.

0-1 Laws for Fragments of ESO

Fact:

- The convergence law fails for ESO
  - EVEN CARDINALITY is ESO-definable.

- Many natural NP-complete problems have probability 0 or 1:
  - 3-COLORABILITY
  - HAMILTONIAN PATH
  - SATISFIABILITY
  - KERNEL
  - …

Question: Do 0-1 laws hold for fragments of ESO?

0-1 Laws for Fragments of ESO

Idea:

Pursue 0-1 laws for fragments of ESO obtained by restricting the quantifier pattern in the FO-part $\phi(\mathbf{S})$ of ESO-sentences $\exists\, \mathbf{S}\ \phi(\mathbf{S})$.

Guiding Principle: Skolem Normal Form for ESO:

$$\exists\, \mathbf{S}\ \exists\, \mathbf{x}\ \forall\, \mathbf{y}\ \exists\, \mathbf{z}\ \theta(\mathbf{S}, \mathbf{x}, \mathbf{y}, \mathbf{z}),$$

where $\mathbf{S}$ is a tuple of SO-variables, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are tuples of FO-variables, and $\theta(\mathbf{S}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ is a quantifier-free formula.

Thus, it suffices to consider first-order prefix classes that are subclasses of $\exists^* \forall^* \exists^*$.

0-1 Laws for Fragments of ESO

Theorem: K … and Vardi – 1987

- For every ESO($\exists^*\forall^*$)-sentence ψ, we have that $\mu(\psi) = 1$ if and only if $\mathbf{R} \models \psi$.
- The 0-1 law holds for ESO($\exists^*\forall^*$).

Theorem: K … and Vardi – 1988

- For every ESO($\exists^*\forall\exists^*$)-sentence ψ, we have that $\mu(\psi) = 1$ if and only if $\mathbf{R} \models \psi$.
- The 0-1 law holds for ESO($\exists^*\forall\exists^*$).

Theorem: Pacholski and Szwast – 1991

The convergence law fails for ESO($\forall\forall\exists$).

0-1 Laws for Fragments of ESO

ESO Fragment	**0-1 Law**	**Decision Problem**
ESO($\exists^*\forall^*$)	**Yes**	NEXPTIME-complete
ESO($\exists^*\forall\exists^*$)	**Yes**	NEXPTIME-complete
ESO($\forall\forall\exists$)	**No**	Undecidable

Classification Theorem:

The Bernays-Schönfinkel Class $\exists^*\forall^*\exists^*$ and the Ackermann Class $\exists^*\forall\exists^*$ are the **only** prefix classes Ψ of FO such that the 0-1 law holds for the corresponding fragment ESO(Ψ) of ESO.

0-1 Laws for Fragments of ESO

Note:
The Bernays-Schönfinkel Class $\exists^*\forall^*\exists^*$ and the Ackermann Class $\exists^*\forall\exists^*$ are the **only** prefix classes of FO (with equality) for which the satisfiability problem is decidable.

Theorem: Gödel – 1932
The satisfiability problem for the prefix class $\forall\forall\exists$ without equality is decidable.

Theorem: Le Bars – 1998
The convergence law fails for ESO($\forall\forall\exists$) without equality.

Reflecting on 0-1 Laws

On the positive side:

- 0-1 laws are new phenomena that are meaningful only in the context of finite structures.
- Finiteness is a feature, not a limitation.
- The study of 0-1 laws gave rise to an extensive interaction between finite model theory and asymptotic combinatorics (genuine two-way interaction; e.g., 0-1 laws for restricted classes of finite structures: partial orders, clique-free graphs).

Reflecting on 0-1 Laws

On the negative side:

- The study of 0-1 laws had less interaction with and impact on computer science than other areas of FMT.

  N. Immerman – 1999: 0-1 laws are "inimical to computation".

- There was early speculation that the analysis of the asymptotic properties of logically definable queries may be useful in the average-case analysis of algorithms.

  This early optimism and expectation remains largely unrealized.

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- ✓ **Descriptive complexity:**
  computational complexity vs. uniform definability.

- ✓ **Expressive power of logics in the finite:**
  What **can** and what **cannot** be expressed in various logics on classes of finite structures.

- ✓ **Logic and asymptotic probabilities on finite structures**
  0-1 laws and convergence laws.

- **Classical Model theory in the finite:**
  Do the classical results of model theory hold in the finite?

Classical Model Theory in the Finite

- The Skolem-Löwenheim Theorem is meaningless in the finite.
- The Compactness Theorem fails in the finite.
- The Craig Interpolation Theorem fails in the finite:
  the EVEN CARDINALITY query is not FO-definable.
- **Conjecture:** Scott and Suppes – 1958
  The Łoś- Tarski Theorem holds in the finite:
  If a FO-sentence ψ is preserved under substructures on all finite structures, then there is a universal FO-sentence ψ^* that is equivalent to ψ on all finite structures

Classical Model Theory in the Finite

Theorem: Tait – 1959

The Łoś- Tarski Theorem fails in the finite.

(rediscovered by Gurevich and Shelah in the 1980s)

Theorem: Ajtai and Gurevich – 1987

Lyndon's Positivity Theorem fails in the finite:

There is a FO-sentence $\psi(S)$ that is monotone in S on all finite structures, but is not equivalent to any positive-in-S FO-sentence on all finite structures.

Question: Do any of the classical results of model theory survive the passage to the finite?

Classical Model Theory in the Finite

Preservation-under-Homomorphisms Theorem:
If a FO-sentence ψ is preserved under homomorphisms on all structures, then there is an existential positive FO-sentence ψ^* that is equivalent to ψ on all structures.

Problem: Does the preservation-under-homomorphisms theorem hold in the finite?
Suppose that a FO-sentence ψ is preserved under homomorphisms on all finite structures. Is there a FO-sentence ψ^* that is equivalent to ψ on all finite structures?

This problem had remained open for a long time …

Classical Model Theory in the Finite

Theorem: Rossman – 2005

If a FO-sentence ψ is preserved under homomorphisms on all finite structures, then there is an existential positive FO-sentence ψ^* that is equivalent to ψ on all finite structures.

- So, finally, we have a positive result about classical model theory in the finite.
- And there is more …

Model Theory of Restricted Classes

Theorem: Atserias, Dawar, K … – 2004

- Let ***T***(k) be the class of graphs of treewidth at most k.
  If a FO-sentence is preserved under homomorphisms on ***T***(k), then it is equivalent to some existential-positive FO-sentence on ***T***(k).

- If a FO-sentence is preserved under homomorphims on all planar graphs, then it is equivalent to some existential-positive FO-sentence on all planar graphs.

Note: Preservation theorems do **not** relativize to subclasses.

Model Theory of Restricted Classes

Theorem: Atserias, Dawar, Grohe – 2005

- Let ***T***(k) be the class of graphs of treewidth at most k.

  If a FO-sentence is preserved under substructures on ***T***(k), then it is equivalent to some universal FO-sentence on ***T***(k).

- There is a FO-sentence that is preserved under substructures on all planar graphs, but it is **not** equivalent to any universal FO-sentence on all planar graphs.

Abstract Model Theory in the Finite

Theorem: Lindström – 1969

First-order logic is a maximal logic possessing both the Compactness Theorem and the Skolem-Löwenheim Theorem.

Problem: K … and Väänänen - 1992

- Is there a Lindström-type characterization of first-order logic on finite structures?
- Is there a Lindström-type characterization of least fixed-point logic on finite structures?

Concluding Remarks

Many topics were not covered in this talk:

- Finite-variable logics and analysis of k-types.
- Logics with generalized quantifiers.
- Interaction with modal logics, connections with the μ-calculus and automated verication.
- Applications to database theory and to constraint databases.
- Interaction with constraint satisfaction.

Concluding Remarks

- Finite model theory has come a long way from a collection of early sporadic results to a mature research area.

- There have been numerous successes, but also frustrations:
  - Lack of progress on resolving open problems in complexity.
  - Limited impact of 0-1 laws on other areas of CS.

- On the positive side,
  - Shift of focus on restricted classes of structures is bearing fruit.
  - Growing connections with constraint satisfaction.

- One can only hope that the next 30 years of finite model theory will be at least as fruitful as the past 30.