Questions in Recursion Theory

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This is an informal list of some open problems in recursion theory. Solutions and new questions are welcome, as well as corrections to the attributions given below. Please, send any submissions to Ted Slaman at `slaman@math.berkeley.edu`.

1 Turing Degrees

Let $\mathcal{D}$ denote the partial ordering of the Turing degrees. $\mathcal{D}(\leq a)$ denotes the degrees less than or equal to a.

1.1. (Sacks) Suppose that P is a locally countable partially ordered set of cardinality less than or equal to the continuum. Is there an order preserving embedding of P into $\mathcal{D}$?

1.2. (Rogers) Is there a nontrivial automorphism of $\mathcal{D}$? *November 1994: Cooper has announced an affirmative solution.*

1.3. (Slaman–Woodin Conjectures)

   (a) $\mathcal{D}$ is biinterpretable with second order arithmetic. In other words, the relation (on $\vec{p}$ and d) "$\vec{p}$ codes a standard model of first order arithmetic and a real X such that X is of degree d" is definable in $\mathcal{D}$.

   (b) Suppose that $\mathcal{I}$ is an ideal in $\mathcal{D}$ such that there is a 1-generic degree in $\mathcal{I}$. Then $\mathcal{I}$ is biinterpretable with that fragment of second order arithmetic in which the real numbers are just those sets whose Turing degrees belong to $\mathcal{I}$.

   November 1994: Cooper has announced negative solutions to both conjectures.

1.4. (Jockusch) Do there exist distinct degrees a and b such that a and b have isomorphic upper cones in $\mathcal{D}$? *November 1994: Cooper has announced an affirmative solution.*

1.5. (Yates) Does every minimal degree have a strong minimal cover?

1.6. (Chong) Is there a minimal degree which is the base of a cone of minimal covers?

1.7. (a) (Kučera) Suppose that p is the Turing degree of a complete extension of Peano arithmetic and that x is a nonzero degree below p. Does there exist a degree a strictly below p such that $a \vee x = p$?

   (b) (Kučera) Characterize the recursively enumerable degrees w such that there is a p for which p is the Turing degree of a complete extension of Peano arithmetic and $w <_T p <_T 0'$.

1.8. (Marker) Let $I \subset \mathcal{P}(\omega)$ be a countable Turing ideal. We say that d is a *uniform upper bound* for I if there is $D \in d$ such that I is equal to $\{D_n : n \in \omega\}$, where $D_n = \{m : (n, m) \in D\}$.

(a) Does I have a minimal uniform upper bound? That is, is there a uniform upper bound d such that for all $e <_T d$, e is not a uniform upper bound? What if I is a Scott set?

(b) Can a minimal upper bound for I be a uniform upper bound?

1.9. (Fenner) Let a and b be Turing degrees with b (Cohen) generic. If $a \leq b \leq a'$, is it the case that $a' = a \vee 0'$?

Fenner comments: If A and G are sets, G is 2-generic, and $A \leq_{tt} G \leq_T A'$, then $A' \equiv_T A \vee K$ (equivalently, $G \leq_T A \vee K$).

1.10. (Lerman) What is the largest decidable fragment (in terms of quantifier complexity) of the first order theory of the Turing degrees in the language with 0, $\leq$ and $'$? Here, $'$ denotes the Turing jump.

2 Degree Invariant Functions

2.1. (Sacks) Is there a degree invariant solution to Post's problem? That is, is there an e such that for all X, $X <_T W_e^X <_T X'$ and for all X and Y, $X \equiv_T Y$ implies that $W_e^X \equiv_T W_e^Y$?

2.2. (Martin's Conjecture) Assume AD. Let $\mathcal{F}$ be the class of degree invariant functions from 2^ω to 2^ω. Order $\mathcal{F}$ by $f \leq_M g$ if there is an X such that whenever $Y \geq_T X$, $f(Y) \leq_T g(Y)$ (write $f \leq_T g$ a.e.).

(a) If $f \not\geq_M id$ then f is constant a.e.

(b) On those elements of $\mathcal{F}$ which are greater than or equal to id, $\leq_M$ is a prewellordering with successor equal to the Turing jump.

2.3. (Kechris's Conjecture) Consider Borel equivalence relations on Polish spaces with countable equivalence classes. Say that such an equivalence relation U is *universal* if for every other one E there is a Borel function f such that for all x and y,

$$x \sim_E y \iff f(x) \sim_U f(y).$$

Kechris's conjecture states that $\equiv_T$ is universal.

Slaman comments: Kechris's conjecture implies the failure of Martin's Conjecture 2.2. Slaman and Steel have announced that arithmetic equivalence is universal.

3 Turing Degrees of the Recursively Enumerable Sets

Let $\mathcal{R}$ denote the partial ordering of the Turing degrees of the recursively enumerable sets.

3.1. Is there a nontrivial automorphism of $\mathcal{R}$? *November 1994: Cooper has announced an affirmative solution.*

3.2. (Harrington, Slaman–Woodin) Is $\mathcal{R}$ biinterpretable with first order arithmetic? In other words, is the relation (on $\vec{p}$ and d) "$\vec{p}$ codes a standard copy of first order arithmetic and an integer e such that W_e is of degree d" definable in $\mathcal{R}$? *November 1994: Cooper has announced an negative solution.*

An affirmative answer here settles the following questions.

(a) (Slaman) Is there a definable element of $\mathcal{R}$ other than 0 and $0'$?

(b) (Harrington) Is there an element of $\mathcal{R}$ which is not definable? *November 1994: Cooper has announced a positive solution to Harrington's question.*

3.3. (Slaman) Is every nontrivial upper cone an automorphism base in $\mathcal{R}$?

3.4. (a) (Lerman) Is the $\exists\forall$-theory of $\mathcal{R}$ decidable?

(b) (Lerman's Conjecture) Define the nonembedding condition as follows.

NEC: There are $a, b, c, d, p, q \in L$ such that b, c and d are incomparable and $b < a$, $b \cup c = b \cup d = a$, $c \cap d \leq b$, p and q are incomparable, $c \leq p \cap q \leq a$, and p and a are incomparable.

Lerman's Conjecture states that a finite lattice L can be embedded into $\mathcal{R}$ preserving order, meet and join if and only if it does not satisfy **NEC**. (**NEC** implies not embeddable is known.) *August 1995: Lempp and Lerman have announced a counterexample to Lerman's conjecture.*

3.5. (Downey's Conjecture) A finite lattice L can be embedded in all nontrivial intervals (ideals or filters) if and only if it does not contain a *critical triple* of incomparable elements a, b and c such that $a \vee b = a \vee c$ and $b \wedge c \leq a$. *August 1995: Lempp and Lerman have announced a counterexample to Downey's conjecture.*

Downey has now modified the conjecture to read "If a finite lattice has no critical triples and can be embedded into the recursively enumerable degrees at all then it can be embedded into every interval."

3.6. (Slaman) Define $S(\vec{a}, \vec{b}, \vec{c}, d)$ in the language of partially ordered sets by

$$(\forall z)\left[(\exists a \in \vec{a})(z \not\leq a) \text{ or } (\exists c \in \vec{c})(z \leq c) \text{ or } z \vee (\vee_{b \in \vec{b}} b) \geq d\right].$$

(a) Suppose that P is a finite partial order. Suppose that s is a relation on the elements of P such that there is an (upper semilattice) extension Q of P in which s is the intersection P with the set defined by S in Q. Is there an embedding of P into $\mathcal{R}$ such that s is the intersection of the image of P with the set defined by S in $\mathcal{R}$? *June 1995: Slaman has announced a negative solution.*

(b) If the answer to (a) is YES then is the same true when P and $\mathcal{R}$ are considered as partial lattices with 0 and 1 such that P can be embedded into $\mathcal{R}$? *November 1993: Cooper, Slaman and Yi have announced a negative solution to the second part of this question.*

Slaman comments: The analysis of satisfying S in $\mathcal{R}$ is one of the necessary steps in giving a decision procedure for the $\exists\forall$-theory of $\mathcal{R}$.

3.7. (Cooper) Is jump equivalence ($x' = y'$) a definable relation in $\mathcal{R}$ or $\mathcal{D}(\leq 0')$? *May 1995: Cooper has announced that there is an automorphism of $\mathcal{R}$ which moves a low degree to one which is not low and so has announced a negative answer for $\mathcal{R}$. Nies, Slaman, and Shore have announced a positive answer to the analogous question for the double jump.*

3.8. (Cooper) Is $\mathcal{R}(\leq a)$ definable in $\mathcal{D}(\leq a)$ for each recursively enumerable a?

3.9. (Lerman) If $a' = b'$, are the degrees REA in a isomorphic (elementarily equivalent) to those REA in b?

3.10. (Li) Given the language $L \subseteq \{0, 1, \vee, \wedge\}$ and a property P in L. A *neighborhood* of a is an interval $[c, b]$ with $c < a < b$. Property P is an *isolated* property of the recursively enumerable degree a if there is a neighborhood of a such that a is the unique element of the interval which satisfies P. Call a *isolable* if there is a property P of a which is isolated.

(a) Is there an isolated property in $\mathcal{R}$? *Cholak, Downey, Nies, and Walk have announced a positive solution. The property is that of maximal contiguity. See Cholak, Downey, and Walk, JSL, 2002.*

(b) Characterize the isolable degrees.

(c) Is there an a which is not isolable?

3.11. (Li–Yang) Call an interval $[c, a]$ in $\mathcal{R}$ a *capping interval* if there is a recursively enumerable degree b such that

- $b > c$,
- for any recursively enumerable degree x, $x \wedge b = c$ if and only if $x \in [c, a]$.

Call b a witness to $[c, a]$'s being a capping interval.

$[c, a]$ is a *maximal capping interval* if there is a b such that

- $b > c$,
- $a \wedge b = c$,
- for any recursively enumerable degrees x and y, if $x > a$ and $y > c$ then $x \wedge y \neq c$.

$[c, a]$ is a *principal capping interval* if there is a recursively enumerable degree b such that

- b is a witness to $[c, a]$'s being a capping interval
- for any recursively enumerable degree x, if x is a witness to $[c, a]$'s being a capping interval then $x \in (c, b]$.

(a) Are the capping intervals dense in $\mathcal{R}$? *October, 1995: Ambos-Spies has announced an affirmative solution.*

(b) Are the maximal capping intervals upwards dense in $\mathcal{R}$? *October, 1995: Ambos-Spies has announced an affirmative solution.*

3.12. (Li) Call a recursively enumerable degree a a *center* of $\mathcal{R}$ if for every recursively enumerable degree x other than $0'$, there are recursively enumerable degrees c and b such that

- $c < b$
- $c < a \vee x$,
- $(a \vee x) \wedge b = c$.

A pair a and b of recursively enumerable degrees is a *cupping–cappping pair* if both a and b are intermediate degrees and for every recursively enumerable degree x, one of the following conditions holds.

- $0' \leq a \vee x$.
- $b \wedge x$ exists.

(a) Is there a center in $\mathcal{R}$?

(b) Is there a cupping–capping pair in $\mathcal{R}$? *October, 1995: Ambos-Spies has announced a negative solution.*

(c) Is there a pair of d.r.e. degrees which form a cupping–capping pair in $\mathcal{R}$? *August, 1995 Li and Yi have announced a positive solution, as a corollary to their solution to Question 3.15.*

3.13. (Downey)

(a) Can each non-*low* recursively enumerable set be split into a pair of recursively enumerable sets, at least one of which is not *low*?

(b) Is there a *high* recursively enumerable set that cannot be split into a pair of recursively enumerable sets both of which are nonrecursive and at least one of which is *high*? *Shore has announced that there is a high r.e. degree that can only be split into* low_2 *degrees.*

3.14. (Li) A recursively enumerable degree a is *undirectedly cappable* if $0 < a < 0'$ and for every recursively enumerable degree x, if $x \not\leq a$ then there is a recursively enumerable degree b such that $B \leq x$, $b \neq 0$ and $a \wedge b = 0$.

(a) (Conjecture) There is an undirectedly cappable degree in $\mathcal{R}$. *October, 1995: Ambos-Spies and Yi independently pointed out this conjecture is refuted by a theorem of Seetapun.*

(b) Define *undirectedly cuppable* dually as above. Is there an undirectedly cuppable degree? *October, 1995: Ambos-Spies and Yi independently pointed out this question is directly related to a question of Lachlan on major subdegrees.*

3.15. (Li) Call a pair a and b of d.r.e. degrees a *focal pair* for $\mathcal{R}$ if a and b are incomparable and for every recursively enumerable degree x, if $x \neq 0$ then either $a \vee x \geq 0'$ or $b \vee x \geq 0'$.

Conjecture: There is a focal pair for $\mathcal{R}$. *August 1995: Li and Yi have announced a confirmation of this conjecture.*

3.16. (Jockusch's Conjecture) For every nonzero recursively enumerable degree x there is a *low* recursively enumerable degree y such that $x \vee y$ is not *low*.

3.17. (Lerman) Is there any nontrivial (definable) ideal $\mathcal{I}$ in $\mathcal{R}$ whose complement is a strong filter, other than the ideal of cappable sets?

3.18. (Communicated by Lempp) For any recursively enumerable degrees $a \not\leq_T b$, must there be a cappable recursively enumerable degree c with $c \leq_T a$ and $c \not\leq_T b$. *1994: Li has announced a negative solution.*

3.19. (Communicated by Lempp) Consider the quotient partial order of the recursively enumerable degrees modulo the cappable degrees. Is it dense? Does it satisfy Shoenfield's conjecture?

Lempp comments: Schwarz showed downward density.

1995: Yi has announced an example which shows that the recursively enumerable degrees modulo the cappable degrees do not satisfy Shoenfield's conjecture.

3.20. (Downey and Shore) Suppose that L is a finite lattice and that there is a lattice embedding of L into $\mathcal{R}$.

(a) Is there a lattice embedding of L into $\mathcal{R}$ which also preserves 0.

(b) If $w \in \mathcal{R}$ is not low_2, is there a lattice embedding of L into the recursively enumerable degrees below w?

(c) Suppose there is a lattice embedding of L into $\mathcal{R}$ which preserves 0. If $h \in \mathcal{R}$ is *high*, then is there a 0-preserving lattice embedding of L into the recursively enumerable degrees below h.

4 The Lattice of Recursively Enumerable Sets

Let $\mathcal{E}$ denote the lattice of recursively enumerable sets and let $\mathcal{E}^*$ denote its quotient by the finite sets.

4.1. For which $n > 1$, are the jump classes H_n or $\overline{L_n}$ invariant under every automorphism of $\mathcal{E}^*$?

4.2. (Slaman–Woodin Conjecture) The index set

$$\left\{ \langle e_0, e_1 \rangle : \begin{array}{l} \text{There is an automorphism of } \mathcal{E}^* \\ \text{taking } W_{e_0} \text{ to } W_{e_1}. \end{array} \right\}$$

is Σ^1_1-complete.

This conjecture implies affirmative solutions to the following questions, which are of independent interest.

(a) Are there two recursively enumerable sets A and B in the same orbit of $\mathcal{E}^*$ for which there is no Δ^0_3 automorphism carrying A to B? *August 1995: Cholak and Downey have announced a positive solution to 4.2a.*

(b) Are there two recursively enumerable sets A and B which realize the same type in $\mathcal{E}^*$ but are not in the same orbit of $\mathcal{E}^*$?

Building on the work of Cholak and Downey, Harrington as announced a proof of the conjecture.

4.3. (Remmel) A recursively enumerable set A is called *speedable* if the set $\{e : W_e \cap \overline{A} \neq \emptyset\}$ is not recursive in $\emptyset'$. Can every speedable set be split into a pair of speedable sets?

4.4. (Slaman) Is every arithmetic predicate on the recursively enumerable sets which is invariant under the action of the automorphism group of $\mathcal{E}^*$ also first order definable in $\mathcal{E}^*$? *1994: Harrington and Nies have announced a negative solution. More elementary counterexamples have been announced by Nies.*

4.5. (Harrington–Soare) Is it the case that for every orbit $\mathcal{O}$ in $\mathcal{E}^*$ and every recursively enumerable set D, if $\mathcal{O}$ has an incomplete element and D is not recursive, then there is a W in $\mathcal{O}$ such that $W \not\geq_T D$?

4.6. (Herrmann) What is the least n such that the Σ_n theory of $\mathcal{E}^*$ is not decidable?

4.7. (Herrmann) Let $\mathcal{O}$ be an orbit in $\mathcal{E}$. Conjecture: the following conditions are equivalent.

(a) $\mathcal{O}$ is Σ^0_3-complete.

(b) $\mathcal{O}$ is an orbit in $\mathcal{E}$ under recursive automorphisms.

(c) $\mathcal{O}$ is either the recursive sets or the creative sets.

Herrmann comments: The implications from (c) to (b) and from (b) to (a) are known by results of Harrington.

4.8. (Herrmann) Suppose that $\mathcal{X}$ is a nonempty subset of $\mathcal{E}$ with no finite and no co-finite element, that $\mathcal{X}$ is invariant under automorphisms of $\mathcal{E}$ and that $\mathcal{X}$ is Σ^0_3 in the indices. Conjecture: The following are equivalent.

(a) $\mathcal{X}$ is Σ^0_3-complete.

(b) $\mathcal{X}$ is one of the following: recursive, creative, the union of recursive and creative, nonsimple.

Herrmann comments: the previous conjecture follows from this one by analysis of cases.

4.9. (a) (Herrmann) The collection of hypersimple sets is orbit complete. That is, every simple set is automorphic to a hypersimple set.

(b) (Stole) The collection of dense simple sets is orbit complete. Here a set A is *dense simple* if the enumeration of the complement of A eventually dominates every total recursive function.

January 1996: Cholak has announced a positive solution to both statements.

4.10. (Herrmann) Questions on maximal sets:

(a) Are the maximal sets an orbit in $\langle \mathcal{E}, HS \rangle$, the lattice of recursively enumerable sets with an additional predicate for the property of hypersimplicity?

(b) Are every two maximal sets automorphic by an automorphism induced by a Δ_2^0 permutation of the natural numbers?

(c) Suppose that M_1 and M_2 are maximal recursively enumerable sets. Is it the case that $\langle [M_1]_m, \leq_1 \rangle$ and $\langle [M_2]_m, \leq_1 \rangle$ are isomorphic? Here $[M]_m$ is the many-one degree of M and $\leq_1$ is the ordering given by one-one reducibility.

(d) (Conjecture) For any maximal set M, the effective orbit of M (in $\mathcal{E}$) contains infinitely many orbits (of $\mathcal{E}$) under recursive permutation.

(e) Does every effective orbit of a maximal set have an element of every *high* (recursively enumerable) degree?

4.11. (Kummer) Let A be recursively enumerable. Let $\mathcal{L}(A)$ denote the lattice of recursively enumerable supersets of A and consider the quotient of $\mathcal{L}(A)$ by $\{B : B \in \mathcal{L}(A)\text{and } B - A \text{ is r.e.}\}$. Say that A is *D-h.h.-simple* if this quotient is a Boolean algebra and A is *D-maximal* if this quotient is the 2 element Boolean algebra. Are the Turing degrees of the D-h.h.-simple sets the same as the Turing degrees of the D-maximal sets?

5 Turing Degrees of n-R. E. Sets

Let $\mathcal{D}_n$ denote the collection of Turing degrees of n-recursively enumerable sets.

5.1. (Downey's Conjecture) $\mathcal{D}_n$ and $\mathcal{D}_m$ are elementarily equivalent whenever n and m are greater than or equal to 2.

5.2. (a) (Cooper) Is $\mathcal{R}$ definable in $\mathcal{D}_n$, for each $n \geq 2$? Is $\mathcal{D}_n$ definable in $\mathcal{D}$ for some $n \geq 2$?

(b) (Yi) Is there a definable intermediate element of $\mathcal{D}_n$?

Yi comments: A positive answer (relativized version) will refute Martin's Conjecture 2.2.

5.3. (a) (Downey and Shore) Is it true that for each finite lattice L, there is an n, which may depend on L, and a lattice embedding of L into $\mathcal{D}_n$? Is every finite lattice isomorphic to a sublattice of $\mathcal{D}_2$?

(b) (Yi) Let n be fixed. Characterize the finite filters in $\mathcal{D}_n$. Is every finite lattice isomorphic to a principal filter in $\mathcal{D}_n$?

Yi comments: The analysis of the finite filters in D_n is a necessary step in giving a decision procedure for the $\forall\exists$-theory of D_n (with $\leq$). A positive answer implies that he $\forall\exists$-theory of D_n decidable.

5.4. (a) (Li's Conjecture) For any n, $\mathcal{D}_n$ is definable in $\mathcal{D}(\leq 0')$ using finitely many parameters from $\mathcal{D}_{n+1}$.

(b) (Li) Is $\bigcup_{n\geq 1} \mathcal{D}_n$ definable in $\mathcal{D}(\leq 0')$ relative to finitely many parameters?

5.5. (Communicated by Lempp) Find an elementary difference between the d.r.e. and the 3-r.e. degrees in the language of partial ordering with a unary predicate for the r.e. degrees.

Lempp Comments: Success here would provide evidence against Downey's conjecture 5.1.

6 Strong Reducibilities

6.1. (Downey) Which lattices can be realized as intervals in the recursively enumerable *wtt*-degrees?

6.2. (a) (Downey) Let U denote a finite upper semilattice (not necessarily with a least element) which is an upwards closed segment in some countable distributive upper semilattice. Is there a recursively enumerable *tt*-degree whose recursively enumerable m-degrees form a structure isomorphic to U?

Downey comments: Cholak and Downey have shown the answer is yes for lattices.

(b) (Downey) Consider the same questions in the infinite case.

6.3. (Downey) Let Q_1 and Q_2 denote the degrees of the recursive sets under any pair of resource bounded T-reducibilities such as $\leq_T^p$ or $\leq_T$ with an elementary time bound. Are Q_1 and Q_2 isomorphic?

6.4. (Downey) Are the P-time degrees of all sets isomorphic to the *tt*-degrees above $0'$?

6.5. (Downey) Let $A, B \subseteq \Sigma^* \times \omega$. Say that $A \leq_T^u B$ if there are a constant α and a procedure Φ such that for each x and k, $\langle x, k\rangle \in A$ iff $\Phi(B; \langle x, k\rangle) = 1$ subject to the following constraints.

- The computation of $\Phi(B; \langle x, k\rangle)$ only uses oracle questions from the set

$$B^{(\leq g(k))} = \{\langle z, k'\rangle : \langle z, k'\rangle \in B \text{ and } k' \leq g(k)\}.$$

- The computation of $\Phi(B; \langle x, k\rangle)$ only uses time $g(k)|x|^\alpha$.

Consider the following questions.

(a) Are the $\leq_T^u$-degrees of recursive sets dense?

(b) What is the structure of the $\leq_T^u$-degrees of recursive sets? For instance does an exact pair theorem hold?

(c) What is the situation for the analogous m-degrees?

6.6. (Downey) Are the P-time degrees of some elementary recursive class such as the exponential sets undecidable?

6.7. (Downey) Is there a set A such that the lattice of sets which are in NP relative to A has an undecidable theory?

6.8. (Nies) Is $\mathcal{D}_{tt}(\leq_{tt} 0')$ isomorphic to $\mathcal{R}_{tt}$?

6.9. (Nies) Is every incomplete recursively enumerable *tt*-degree branching?

Nies comments: This can be shown for T-incomplete recursively enumerable *tt*-degrees.

August 1997: Fejer and Shore have announced a positive solution.

6.10. (Nies) Is the 4-element Boolean algebra isomorphic to an initial segment of the recursively enumerable *btt*-degrees ?

Nies comments: By a result of Haught and Harrington, this would give a Σ_2-elementary difference in the language of partial orders to $\mathcal{R}_{tt}$.

7 Enumeration Degrees

Let $\mathcal{E}$ denote the partial ordering of the enumeration degrees.

7.1. (a) (Slaman) Is every countable upper semilattice with 0 and 1 isomorphic to an interval in $\mathcal{E}$?

(b) (Slaman) Is $\exists\forall$-theory of $\mathcal{E}$ decidable?

7.2. (a) (Cooper) Is the jump definable within $\mathcal{E}$?

(b) (Cooper) Are the total degrees definable within $\mathcal{E}$?

7.3. (Cooper) Does every infinite ascending sequence in $\mathcal{E}$ have a minimal upper bound?

7.4. (Sorbi) Does there exist a quasiminimal e-degree with an uncountable set of minimal (Turing) degrees above it? (Has applications for the Medvedev lattice.) *May 1994: Slaman and Sorbi have announced an affirmative solution.*

8 Definability and Proof Theory in Second Order Arithmetic

Let C be the first order scheme

If φ defines a total injective function then its range is unbounded.

Let RT^2 be the second order statement that every partition of the pairs of integers into 2 pieces has an infinite homogeneous set.

8.1. (a) (Seetapun) Does $RCA_0 + RT^2 \vdash WKL_0$?

(b) (Seetapun) If T is an infinite recursive binary tree does there exist a recursive partition of pairs for which every homogeneous set computes a path through T?

8.2. (a) (Slaman) Is there an n such that $P^- + I\Sigma_n + C \vdash PA$? *December 1993: Kaye has announced a negative solution.*

(b) (Slaman) Does $RCA_0 + RT^2 \vdash C$? Does $RCA_0 + RT^2 \vdash PA$? *Cholak, Jockusch, and Slaman have announced a negative solution.*

(c) (Slaman) Are $RCA_0 + RT^2$ and RCA_0 equiconsistent?

8.3. (Jockusch) Consider the statement "Every infinite partially ordered set has an infinite subset which is either a chain or an antichain." Is this statement equivalent to RT^2, working in the base theory RCA_0?

8.4. (a) (Simpson) Does $RCA_0 + WKL$ prove that there is a real of minimal Turing degree?

(b) (Simpson) Is there an infinite recursive binary tree T such that every infinite path through T computes a set of minimal Turing degree? *July 1995: Groszek and Slaman have announced a positive solution to 8.4b.*

8.5. (Downey) A Π^0_1-class P is *thin* if for all other Π^0_1-classes Q contained in P there is a clopen set U such that Q is equal to $P \cap U$.

   (a) Characterize the degrees of members of thin Π^0_1-classes.

   (b) If P_1 and P_2 are two thin perfect classes is there an automorphism of the lattice of Π^0_1-classes taking P_1 to P_2? What if both P_1 and P_2 are classes with a unique point of Cantor rank one?

8.6. (Downey) Ketonen described a set of invariants that classify countable Boolean algebras. What is the proof theoretical strength of this result?

8.7. Is every Scott set the standard part of a nonstandard model of PA?

8.8. (Jockusch) Suppose that P is an infinite partially ordered set (coded on the natural numbers). Is it the case that either there is an infinite chain within P which is recursive in P', or there is an infinite antichain within P which is recursive in P'?

   This question is closely related to Jockusch's Question 8.3.

   1997: Herrmann has announced a negative solution.

9 Higher Recursion Theory

9.1. (Sacks) Does every countable set of hyperdegrees have a minimal upper bound?

9.2. (Sacks) Is there a role for the infinite injury method in E-recursion theory?

9.3. Suppose that α is Σ_1-admissible. Is there a subset of α of minimal α-degree?

10 Definability in Algebra and Analysis

10.1. (a) (Downey and Kurtz) Given a Π^0_1-class P is there a torsion free Abelian group G whose cone of orderings is in effective correspondence with P? *August 1997: Reed Solomon has announced a negative solution. According to Solomon, every such group has either two orderings or continuum many.*

   (b) (Downey and Kurtz) Let G be a recursive group. Supposing that G is orderable, is G isomorphic to a recursively orderable group?

10.2. (Pour-El) Let $\mathbf{Q}$ and $\mathbf{R}$ denote the rational and real numbers. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. A *presentation* of f is a set $\{\langle q, a_{q,n}\rangle : q \in \mathbf{Q} n \in \omega\} \subseteq \mathbf{Q}^2$ such that for all $q \in \mathbf{Q}$ and $n \in \omega$, $|f(q) - a_{q,n}| < 2^{-n}|$. Then, $D(f)$ is the set of Turing degrees of presentations of f. Clearly $D(f)$ is upwards closed in $\mathcal{D}$; does it have a least element?

11 Recursive and Decidable Models

Thanks to Bakhadyr Khoussainov for supplying annotations for the problems in this section.

11.1. (Downey) A structure $\mathcal{A}$ is *n-recursive* if the collection of n-quantifier statements true in $\mathcal{A}$ is decidable.

   (a) For each n, is there a finitely presented group that is n-recursive but not $n+1$-recursive?

(b) Is there a finitely presented group $\mathcal{A}$ such that for each n, $\mathcal{A}$ is n-recursive but $\mathcal{A}$ is not decidable?

11.2. (Goncharov) Let T be ω_1-categorical with recursive models.

(a) Is its prime model recursive? *February 1995: Khoussainov, Nies and Shore have announced a negative solution.*

(b) Let N and M be recursive models of T such that N is prime and M is autostable. Is N also autostable?

[[Definitions needed for the above problem. Fix a computable language

$$\sigma = (P_0^{n_0}, P_1^{n_1}, \ldots, F_0^{m_0}, F_1^{m_1}, \ldots),$$

where $P_i^{n_i}$ is a predicate symbol of arity n_i, $F_j^{m_j}$ is a functional symbol of arity $m_j \geq 0$, and $P_0^{m_0}$ is the equality sign.

A constructivization or constructive enumeration of a structure $\mathcal{A}$ of the language σ is a mapping ν from ω onto the domain of the structure such that the set

$$\{(i, l_1, \ldots, l_{n_i}) : \mathcal{A} \models P_i^{n_i}(\nu l_1, \ldots, \nu l_{n_i})\}$$
$$\cup \{j, p_1, \ldots, p_{m_{j+1}} : \mathcal{A} \models F_j^{m_j}(\nu p_1, \ldots, \nu p_{m_j}) = \nu p_{m_{j+1}}\}$$

is recursive. The structure $\mathcal{A}$ is **constructivizable** if it has a constructivization. A pair $(\mathcal{A}, \nu)$ is called **constructive structure**. Two constructivizations ν and μ of the same structure $\mathcal{A}$ are called **autoequivalent** if there exist a recursive function f and an automorphism α of $\mathcal{A}$ such that $\alpha\nu(i) = \mu f(i)$ fo all $i \in \omega$. Two constructivization ν and μ are called **strongly equivalent** if there exists a recursive function g such that $\nu(i) = \mu g(i)$ for all $i \in \omega$. The structure $\mathcal{A}$ is called **autostable** if any two constructivizations of $\mathcal{A}$ are autoequivalent. The structure $\mathcal{A}$ is **constructively stable** if any two constructivizations of $\mathcal{A}$ are strongly equivalent.

The following facts can be checked. The structure $\mathcal{A}$ has a constructivization if and only if $\mathcal{A}$ has a recursive presentation. The structure $\mathcal{A}$ is recursively categorical if and only if $\mathcal{A}$ is autostable. If structure is constructively stable, then it is autostable. However, the converse is not correct. For example, any structure with more than countably many automorphisms is not constructively stable.

The notions of constructivization, constructive stability, and autostability are due to Mal'cev.]]

11.3. (Goncharov) Is the free product of groups of recursive automorphisms [[of recursive models]] a group of recursive automorphism for some recursive model?

11.4. (Goncharov) Characterize the models which have an effectively infinite class of constructive enumerations.

[[Definitions needed for the above problem. A structure $\mathcal{A}$ is **effectively infinite** if there exists a procedure which applied to any uniformly computable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of pairwise non-recursively isomorphic recursive presentations of $\mathcal{A}$ produces a presentation $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is not recursively isomorphic to $\mathcal{A}_i$ for all $i \in \omega$.

(Khoussainov comments: Goncharov proved that nonrecursively categorical linear orderings, boolean algebras, Abelian groups are examples of effectively infinite structures.)]]

11.5. (Goncharov) Characterize theories for which any constructivizable model is decidable.

11.6. (Goncharov) Characterize the families of recursively enumerable sets with exactly one up to equivalence computable enumeration.

[[Definitions needed for the above problem. Let S be a family of r.e. sets. **A computable enumeration** of S is any mapping ν from ω onto S for which the set $\{(n,m)|m \in \nu(n)\}$ is r.e. Say that a computable enumeration ν_1 is **reducible** to a computable enumeration ν_2 ($\nu_1 \leq \nu_2$) if there exists a recursive function f such that $\nu_1 = \nu_2 f$. If $\nu_1 \leq \nu_2$ and $\nu_2 \leq \nu_1$, then the enumerations ν_1 and ν_2 are called **equivalent**. Thus, $\leq$ defines a preorder on the class $C(S)$ of computable enumerations of S. The partially ordered set, denoted by $R(S)$, naturally defined by this preorder is called **Rogers Semilattice**.]]

11.7. (Goncharov) Let $R(S)$ be the Rogers semilattice of computable enumerations for a family of recursively enumerable sets S.

(a) If there are two different minimal elements in $R(S)$, are there infinitely many minimal elements?

(b) Are there infinitely many minimal elements in $R(S)$, if the family S has two nonequivalent positive computable enumerations.

[[Definition needed for the above problem. A computable enumeration ν of S is **positive** if the set $\{(n,m)|\nu n = \nu m\}$ is r.e.]]

11.8. (Morley) Is every countable model of an Ehrenfeucht theory with a decidable saturated model also decidable? *August 1995: Goncharov has announced a negative solution.*

11.9. (Dobrica) Does the Rogers semilattice of computable enumerations of some class of recursive models up to recursive isomorphisms have 0, 1 or an infinite number of elements?

[[Definition needed for the above problem. Let S be a class of recursively presentable structures. **A computable enumeration** of S is a mapping ν from ω onto S such that there exists a procedure which applied to every $n \in \omega$ produces a recursive presentation (constructivization) of $\nu(n)$. Say that a computable enumeration ν_1 of S is **reducible** to a computable enumeration ν_2 ($\nu_1 \leq \nu_2$) if there exists a procedure which applied to any $\nu_1(n)$ produces a $\nu_2(m)$ and a recursive isomorphism from $\nu_1(n)$ to $\nu_2(m)$. If $\nu_1 \leq \nu_2$ and $\nu_2 \leq \nu_1$, then the enumerations ν_1 and ν_2 are called **equivalent**. Thus, one can define a partially ordered set $R(S)$, called **Rogers Semilattice** of computable enumerations.]]

11.10. (Goncharov) Is there a nontrivial Rogers semilattice of computable enumerations of some class of recursive models up to equivalence which is a lattice?

11.11. (Goncharov) Let (M,ν) be a finitely generated positive model of finite signature σ. Is there a finite extension σ_1 and an extension $((M,\sigma_1),\nu)$ of this signature such that $((M,\sigma_1),\nu)$ is autoequivalent to a free system in some quasivariety with finite system axioms (conditional equations)?

[[Definitions needed for the above problem. Fix a finite functional language σ. Let $\mathcal{A}$ be a structure of this language. A mapping ν from ω onto the domain of a structure $\mathcal{A}$ is **positive, or equivalently r.e.** if all basic operations of $\mathcal{A}$ are recursive with respect to ν and the equality relation is r.e. If ν is positive, then the pair $(\mathcal{A},\nu)$ is called **positive structure**.

An identity is an expression $t = q$, where t, q are terms. **An equational implication** is an expression of the form $I_1 \& \ldots \& I_n \to I_{n+1}$, where each I_k is an identity. The class of models which satisfy a given set S of identities (equational implications) is called **variety** (**quasivariety**) defined by S.

It is known that every variety or quasivariety defined by an S has a **free system** over any given set X, that is there is an algebra $F(X)$ unique up to isomorphisms such that $F(X)$ is generated by X and any map from X to any algebra from the variety can be extended uniquely up to a homomorphism.

An algebra $\mathcal{A}$ is **specified by equations (equational implications)** if there exist a **finite extension** σ' of σ, an appropriate expansion $\mathcal{A}'$ of $\mathcal{A}$, and a finite set S of identities (equational specifications) of the extended language σ' such that $\mathcal{A}'$ is isomorphic to the free algebra $F(\{c_1, \ldots, c_n\})$, where $c_1, \ldots, c_n$ are all constants appearing in S. The above question asks if any finitely generated positive algebra can be specified by equational implications.

(Khoussainov comments: Bergstra and Tucker proved that any recursive algebra is equationally specified. Kassimov proved that there exists a finitely generated positive algebra which can not be equationally specified. If one omitted the hypothesis that the model is finitely generated, then the problem would have a negative solution.)]]

11.12. (Goncharov) Let M be an autostable model and fix $a_1, \ldots, a_n \in M$. Is the extension $(M, a_1, \ldots, a_n)$ of M by constants for $a_1, \ldots, a_n$ also autostable? *February 1995: Cholak, Goncharov, Khoussainov and Shore have announced a negative solution.*

11.13. (Khoussainov and Shore). Let $k > 1$. Does there exist a recursive model with exactly two (n) recursive presentations which are not isomorphic via a function recursive in 0^k ?

(Khoussainov comments: All the known examples of recursive models with finite number of recursive isomorphism types are Δ^0_3-categorical, that is any two recursive presentations of each of these models are isomorphic via a function recursive in $0''$. On the other hand, S. Goncharov proved that if a nonrecursively categorical model is Δ^0_2-categorical, then the number of its recursive isomorphism types is infinite.)

11.14. (Shinoda and Slaman) Is there a countable model M and a set of reals X of measure 1 contained in the set of 1-random reals such that for each real y, there is an element of X which is recursive in y if and only if there is a presentation of M which is recursive in y?

11.15. (Rosenstein) Suppose that P is a recursive linear order. Are the following conditions equivalent?

- Every recursive copy of P has a recursive self-embedding.
- Every recursive copy of P has a recursive dense subset.
- There is an interger n and an infinite subinterval of P in which all discrete intervals have length less than n.

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