Incomputability,
Fifty Years after Alan Turing

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A beginning ... and an end?

"I am making a collection of experiments in the order I mean to do them in. I always seem to want to make things from the thing that is commonest in nature and with the least waste in energy."

— Alan Turing's letter home, March 1925, quoted in Andrew Hodges, *Alan Turing: The Enigma*, Vintage edn., 1992, p.19

"With so few messages from the unseen mind to work on, [Alan Turing's] inner code remains unbroken. According to his imitation principle, it is quite meaningless to speculate upon his unspoken thoughts. Wovon man nicht sprechen kann, darüber muss man schweigen. But Alan Turing could not possess the philosopher's detachment from life. It was, as the computer might put it, the unspeakable that left him speechless."

— A. Hodges, *Alan Turing: The Enigma*, final paragraph, p.527

Incomputable: "unable to be calculated or estimated, ORIGIN early 17th cent." (New Oxford Dictionary of English, 1998)

Note: In everyday usage, incomputability is a barrier to *human activity* — something we all experience in everyday life

- *Coping strategies?*

I. Imitation:

- Strategies learned from copying what works for other people (appeal to cultural resources)

II. Reductionism and Thought Experiments:

- Complex phenomena broken down into basic ones, aiming at solutions we recognise as being *scientific* ('computing with awareness')

- For Alan Turing, strategy I was not a favoured option!

- The experience of incomputability an intimate one, his scientific and personal life inextricably intertwined — his mind a ready-made laboratory for exploring the computable/ incomputable interface

> *"You could take a safe bet that if you ventured on some self-evident proposition, as for example that the earth was round, Alan would produce a great deal of incontrovertible evidence to prove that it was almost certainly flat, ovular, or much the same shape as a Siamese cat which had been boiled for fifteen minutes at a temperature of one thousand degrees Centigrade."*
>
> — attributed to Alan's brother, John Turing, 1928-ish, in Andrew Hodges, *Alan Turing: The Enigma*, p.33

... *and finding the barrier all-too-real*

- On Thurs. 13 February 1930, Turing's school friend Christopher Morcom died — an unexpected blow to the young Alan Turing (see Hodges), a seismic shattering of his personal Laplacian universe ...

- *Quantum uncertainty* — a new resonance between the personal and the scientific (following Eddington, *The Nature of the Physical World*, eg.)

- Tentative steps towards view of a *Universe as information*, subsuming matter

- The *necessity* of incomputability

•• A fluidity of thought, anticipating current concerns (and confusions!) about computability

Note: At this point *the mind is not seen as mechanical*

"It used to be supposed in Science that if everything was known about the Universe at any particular moment then we can predict what it will be through all the future. … More modern science however has come to the conclusion that when we are dealing with atoms and electrons we are quite unable to know the exact state of them … The conception then of being able to know the exact state of the universe then really must break down on the small scale. This means then that the theory which held that as eclipses etc. are predestined so were all our actions breaks down too. We have a will which is able to determine the action of … atoms … in … the brain, …

… matter is meaningless in the absence of spirit … as regards the actual connection between spirit and body I consider that the body by reason of being a living body can 'attract' and hold on to a 'spirit' … The body provides something for the spirit to look after and use."

— written by Turing for Mrs. Morcom, around 1932, and quoted in Hodges, pp. 63–64

That remarkable 1936 paper

- Typically, Turing considers with a *more basic* question than that asked by other authors — not "What is a computable function?" — but

> 'The real question at issue is "What are the possible processes which can be carried out in computing a [real] number?" '
>
> —- A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. Lond. Math. Soc. (2) **42** (1936–7), p.249

- The Turing machine may not be the only — or even the first — general notion of computability

- But it was the most convincingly formulated and presented — it won over Gödel (see John Dawson, *Logical Dilemmas*, pp. 101–102)

- It also takes account of the scientist's need to describe the world in terms of *real numbers*

Kurt Gödel, and Church's thesis — the final word :

> *"Over the years G habitually credited A. M. Turing's paper of 1936 as the definitive work in capturing the intuitive concept [of computability], and did not mention Church or E. Post in this connection. He must have felt that Turing was the only one who gave persuasive arguments to show the adequacy of the precise concept ... In particular, he had probably been aware of the arguments offered by Church for his 'thesis' and decided that they were inadequate. It is clear that G and Turing (1912–1954) had great admiration for each other, ... "*

— from Hao Wang, *Reflections on Kurt Gödel*, MIT Press, 1987, p.96

Idea: Gödel number (code) programs, and build a *universal* Turing machine U capable of *decoding* and *implementing* any coded program.

At a stroke, Turing anticipated:

- *Interpretive programs* — U unscrambles codes into implementable information (quintuples)
- The *Stored Program Computer* — distinction between program and data evaporates
- The *versatility* of today's computers — with *hardware* solutions replaced by equivalent software

How universal? Turing's machines were framed in terms of the computing capabilities of a human clerk — those of *machines in general* remained to be fully pinned down.

Did Turing Invent the Computer?

- Main early influence on design through von Neumann — see his EDVAC report (1945), and the credit given in his Hixon lecture (1948)

- Also, key role in design of 'Colossi' (circa 1944)

But relevant for us: Turing machines . . . still provide the standard setting for the definition of the complexity of computation in terms of bounds on time and space;

• together with the neural nets of McCulloch and Pitts they provided the foundations of the theory of automata;

• together with the generated sets of Post [1943] they provided the foundation for the theory of formal grammars.

— Robin Gandy, Preface to the 1936–37 papers, *Collected Works of A.M. Turing: Mathematical Logic*, North-Holland, 2001, p.17.

- Turing's paper was not immediately useful, of course —

- But it did relate to a grand scientific enterprise traceable back to Newton and beyond — no less than that of capturing the algorithmic content of the natural world

> *When we say that we understand a group of natural phenomena, we mean that we have found a constructive theory which embraces them*
>
> — Albert Einstein, *Out of My Later Years*, Philosophical Library, New York, 1950, p.54

- And for Hilbert, this included mathematics and its epistemology

> **Hilbert's Programme** (1904–1928): Capture mathematics in complete, consistent theories.

Hilbert's 'Entscheidungsproblem' — Find an algorithm for deciding if a given sentence is logically valid or not

grew out of the view that …

For the mathematician there is no Ignorabimus, and, in my opinion, not at all for natural science either. … The true reason why [no one] has succeeded in finding an unsolvable problem is, in my opinion, that there <u>is no</u> unsolvable problem. In contrast to the foolish Ignorabimus, our credo avers:

We must know,
We shall know.

— David Hilbert, Königsberg, 8 September 1930, in opening address to the Society of German Scientists and Physicians

Für den Mathematiker gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. . . . Der wahre Grund, warum [keiner] nicht gelang, ein unlösbares Problem zu finden, besteht meiner Meinung nach darin, daß es ein unlösbares Problem überhaupt nicht gibt. Statt des törichten Ignorabimus heiße im Gegenteil unsere Losung:

Wir müssen wissen,
Wir werden wissen!

— from *David Hilbert Gesammelte Abhandlungen*, v.3, Verlag von Julius Springer, Berlin 1935, p.379

Natural Examples of Incomputable Objects

- Church showed no such algorithm existed. And so did Turing — essentially showing:

> (1) The set of inputs $n \in \mathbf{N}$ on which U halts is computably enumerable but not computable, and hence:
>
> (2) The logically valid sentences form an incomputable c.e. set.

- And a whole repertoire of incomputable c.e. sets appeared, and any reasonably rich theory turned out to be undecidable

- Of interest to 'real' mathematicians — unsolvability of the word problem for groups (Post/ Markov 1947, Turing 1950, Novikov/ Boone 1955)

There was a second 1937 paper, of course —

Whether calculating mentally or with pencil and paper, Turing was methodical only by fits and starts, and often made mistakes. [When I came to know him later the phrase 'What's a factor of two between friends?' had become a catchword.] But he understood very well what it meant to be totally methodical. Indeed an acceptance — sometimes ready, sometimes reluctant — of the dichotomy between the clearly perceived ideal and the confused actuality was fundamental in Turing's thought.

— Robin Gandy, Preface to the 1936–37 papers, *Collected Works of A.M. Turing: Mathematical Logic*, North-Holland, 2001, p.9.

And it was Church who first proved Church's theorem —

> *If he had been a more conventional worker, he would not have attacked the Hilbert problem without having read up all of the available literature, including Church's work. He then might not have been pre-empted — but then, he might never have created the new idea of the logical machine, with its simulation of 'states of mind', which not only closed the Hilbert problem but opened up quite new questions.*
>
> — A. Hodges, *Alan Turing: The Enigma*, p.114

But incomputability into the long grass . . .

- 1936 not only sees a new clarity about what 'incomputability' really is — but the emergence of a conceptual framework which actually took us <u>away</u> from the real world and the uncertainties facing working scientists

- We see the birth of *Recursion Theory*, and new notions inimical to vague intuitions

- We see an emphasis on purely *mathematical issues*, extending to logic in general — guided by an introjected real world rather than an actual one

- The growing belief that mathematics — and science in general — *could carry on much as before* without ever bumping into incomputable objects

- Discovery (J. Myhill) that all the unsolvable problems discovered in the 1930s were *the same*

- Richness of the *computable* universe revealed — even see *reverse mathematic* emerge as an attempt to rescue Hilbert's programme

The Search for Really Natural Examples

Natural? — "existing in or caused by nature"

— *The New Oxford Dictionary of English*, 1998 edition

- Negative solution to Hilbert's Tenth Problem (Davis, Matiyasevich, Putnam, Robinson, 1972) — Everyday mathematics leads us unavoidably to incomputable sets

- Pour-El and Richards differential equation with computable boundary conditions leading to incomputable solutions

- The predictive incompleteness of quantum theory — bypassed by quantum computing (despite recent claims, e.g. Tien Kieu, 2003) …

Von Neumann's axioms distinguished the **U** *(unitary evolution) and* **R** *(reduction) rules of quantum mechanics. Now, quantum computing so far (in the work of Feynman, Deutsch, Shor, etc.) is based on the* **U** *process and so computable. It has not made serious use of the* **R** *process: the unpredictable element that comes in with reduction, measurement, or collapse of the wave function.*

— Andrew Hodges: *What would Alan Turing have done after 1954?*, in *Alan Turing: Life and legacy of a great thinker*, Christoff Teuscher (ed.), Springer, 2004

- Kreisel, 1970 — a collision problem related to the 3-body problem which might give "an analog computation of a non-recursive function (by repeating collision experiments sufficiently often)"

- More recently:

Painlevé Problem (1897): *Do noncollision singularities exist for the N-body problem for any $N \geq 4$?*

"Yes" — Jeff Xia, 1988, Saari and Xia *Off to Infinity in Finite Time*, Not. Amer. Math. Soc. 42, 1995

• **1931–36** — No Turing machine can prove all the true sentences of arithmetic (Turing's work had enabled a precise notion of *formal system* to use in Gödel's theorem)

• But — a human observer can transcend what any *given* such machine can prove

•• **Question:** Is there a mathematical analysis throwing light on the *apparent* ability of the human mind to transcend the mechanical — by an extended constructivism?

• **1937** — On the way back to America:

> *. . . now [Turing] gave the impression that he had long been happy with the Russellian view, that at some level the world must evolve in a mechanistic way. . . . Symbolically, the Research fountain pen that Mrs Morcom had given him in 1932 was lost on the voyage.*
>
> — A. Hodges, *Alan Turing: The Enigma*, p.137

That opaque 1939 paper —

- **Idea** — Use the constructive ordinals $\mathcal{O}$ of Church and Kleene (1937) to inductively extend theories via Gödel-like unprovable sentences

- And — partial success — get a hierarchy containing proofs for all true Π^0_1 sentences of arithmetic

- Although turns out that *different* hierarchies can be complete, or invariant, but not both

- And — main disappointment — does not seem to work for Π^0_2 sentences (that waits for Feferman and the use of *stronger reflection principles*)

Notes: • Turing is specially interested in Π^0_2 sentences — he shows that most mathematically interesting problems, such as *the Riemann hypothesis*, are met at that level

• And in invariance, since that allows one to unambiguously *classify* problems according to their 'depth' (i.e., level of ordinal notation)

Oracle machines and relativisation invented —

• In investigating the 2-quantifier sentences, Turing seeks a constructive derivation of a non-Π^0_2 problem — and in so doing invents *relativisation* (using *oracle machines*) and the *Turing jump*!

• **1944–48** — Post's wonderful 1944 paper, and 1948 short abstract, describe some first far-reaching consequences (Post's theorem, 'Turing reducible', degrees of unsolvability), and clarifies what is happening in the 1939 paper

•• These three publications establish the still crucial theme of the *interrelationship between computability and information content*

•• And establish the *Turing universe* of algorithmically related reals as the standard model for computationally complex environments

Computing the incomputable?

- Turing claims to clarify the relationship between 'ingenuity' (subsumed within the ordinal logics) and 'intuition' (needed to identify good ordinal notations — $\emptyset^{(\omega)}$ level intuition!)

- **But 'hypercomputation'** — even though anticipated and of subsequent relevance — is tangential to Turing's thoughts here (cf. Davis)

> "Mathematical reasoning may be regarded ... as the exercise of a combination of ... *intuition* and *ingenuity*. ... In pre-Gödel times it was thought by some that all the intuitive judgements of mathematics could be replaced by a finite number of ... rules. The necessity for intuition would then be entirely eliminated.
>
> In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity, and this in spite of the fact that our aim has been in much the same direction."
>
> – A.M. Turing, *Systems of logic based on ordinals*, pp.134–5

A few years later, Hadamard recounts:

"At first Poincaré attacked [a problem] vainly for a fortnight, attempting to prove there could not be any such function . . . [quoting Poincaré:]

> *Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it . . . I did not verify the idea . . . I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience sake, I verified the result at my leisure."*
>
> — Jacques Hadamard, *The Psychology of Invention in the Mathematical Field*, Princeton Univ. Press, 1945

- Who else but Turing would have attempted a mathematical explanation at that time . . .

Post was less subtle in 1941, anticipating Penrose:

> " ... we may write
>
> *The Logical Process is Essentially Creative*
>
> This conclusion, ... makes of the mathematician much more than a kind of clever being who can do quickly what a *machine* could do ultimately. We see that a *machine* would never give a complete logic; for once the machine is made *we* could prove a theorem it does not prove."
>
> — E. L. Post, *Absolutely unsolvable problems and relatively undecidable propositions – Account of an anticipation*, in collected works of Post (ed. Davis), 1994, p.429

- By the 1990s this sense of context is lost:

  " ... *Soare insists ... that computability includes relative computability, i.e. relative recursiveness as a means of classifying non-recursive sets. This terminology strikes me as wrong-headed, as if one were to insist that biology includes the study and classification of inanimate objects.*"

  — S. G. Simpson, communication to F.O.M. list, 18 Aug. 1998

Hierarchies and proof theoretic consequences

- Feferman et al — replace consistency principles with much more powerful reflection principles (see *Reflecting on incompleteness*, J.S.L., 1991)

> ... *Turing anticipated ... the classification by ordinals of the provably (total) recursive functions of various formal systems, obtained later by prooof-theoretical work.*
>
> — S. Feferman, *Turing in the Land of O(z)*, 1988, p.127

- And anticipates Paris-Harrington, 1977

- In section 10 ('The continuum hypothesis. A digression.') Turing indicates how to replace ω_1 by the constructive ordinals, and the subsets of ω by the computable reals — and anticipates subsequent hierarchies of computable functions

- Later — Feferman's work on 'autonomous' ordinal logic puts Penrose's brave but flawed speculations on computability and the mind in context

- Turing's universe of computably related reals provides a basic model of scientific descriptions of a computationally complex real universe (Cooper 1998, Copeland 1997, Cooper-Odifreddi 2003 …)

- With a corresponding algorithmic content, implicit infinities completed and based on the reals

- In particular, get an explanation of the *emergence* phenomenon via appropriate notions of mathematical definability and invariance

- Mathematically –

> … *eventually, the idea of transforming computability from an absolute notion into a relative notion would serve to open up the entire subject of generalized recursion theory.*
>
> — S. Feferman, *Turing in the Land of O(z)*, 1988, p.127

Pathology — or Real World Complexity?

- **Memorable images** —

- **Sacks** (in Bressanone, 1979) — 'Ordinary recursion theory' illustrated by slide of *cultural revolution turmoil* (obsessive, formless activity)

- **Gandy** (Varna, 1989) — Communicates the structure of the Turing degrees via desperate *scribbles on a blackboard*

- The *bi interpretability conjecture* and attempts to prove Turing rigidity ...

- A Turing universe framed by failed mathematical ambitions — and isolated from its natural home, the complexity of the material world

•• A reluctance to stray beyond purely technical questions — unlike Turing himself ...

> *Turing, as is well known, had a mechanistic conception of mind, and that conviction led him to have faith in the possibility of machines exhibiting intelligent behavior.*
>
> — S. Feferman, *Turing in the Land of O(z)*, 1988, pp.131–2

- But in Turing's real world the balance between logic and science sometimes shifts

- At one extreme we have Turing, founder of AI and seminal influence on its methodology (the Turing Test) ...

- At the other his interest in quantum theory — his writings for Mrs Morcom, his late postcards to Robin Gandy

- And in between he considered possibilities — coming out of his 1944-48 experiences of the ACE ('Automatic Computing Engine') project— such as machines which make mistakes ...

> *... if a machine is expected to be infallible, it cannot also be intelligent. There are several theorems which say almost exactly that.*
>
> — A.M. Turing, talk to the London Mathematical Society, February 20, 1947, quoted in Hodges, p.361

- ... and *learn* (but no mention of oracles!) —

> *No man adds very much to the body of knowledge. Why should we expect more of a machine? Putting the same point differently, the machine must be allowed to have contact with human beings in order that it may adapt itself to their standards.* — A.M. Turing, same talk, Hodges, p.361

•• Is the logic of a Turing machine sufficient to capture the workings of a human brain?

•• What is the nature of the mechanical in the physical world? And what relationship does this have to the mind?

Turing's seminal 1950 AI paper —

> *" I propose to consider the question, '<u>Can machines think</u> ?' "* — AMT, *Computing machinery and intelligence*, in *Mind*, 1950, pp.433–460

- Not "Is the human mind a Turing machine"
- Basis for AI —

> **Turing Test:** *" 'Are there imaginable digital computers which would do well in the imitation game ?' "* — *Mind*, 1950, p.442

- Limited case for machine intelligence —

"I believe that in about fifty years' time it will be possible to programme computers . . . to make them play the imitation game so well that an average interrogator will not have more than 70 per cent. chance of making the right identification after five minutes of questioning. The original question . . . I believe to be too meaningless to deserve discussion." — *Mind*, 1950, p.442

Background 1939 experiences:

— <u>Gödel's theorem</u> gives illusion people transcend computers (feeds into Turing's *'The Mathematical Objection'*)

— <u>Ordinal logics</u> falsely suggest machines transcend the computable (intuition reducible)

- Also considers objection based on *'Continuity of the Nervous System'*
- Again mentions *'Learning Machines'* . . .

> " . . . *when reading turing's 1939 paper i DID have the impression that he thought that by an oracle he meant a human being, and thus that non-computable functions could be humanly computable. the oracle device could be thought of as a formalization of a human-machine interaction, in which the calls to the oracle(s) would be a kind of human help received by the machine. if this interpretation were correct, then it would mean that turing did not accept the church-turing thesis that recursive = human computability. quite the contrary, actually.*"
>
> — e-mail from George Odifreddi, 1 May, 2004

- An emerging coming together of logicians, computer scientists, theoretical physicists, people from the life sciences, humanities and beyond

- Incomputability in Nature, n-body problem, quantum phenomena, computing with reals and scientific computing

- Computing beyond the Turing barrier, analog computers and hypercomputation

- Emergence and its mathematical models — even here anticipated by Turing (Odifreddi tells "*[Gerald] Edelman quotes Turing as a precursor of his work on morphogenesis*")

The 1936 paradigm shift renewed — That of a coming together of science and mathematics to replace the Laplacian model of science with one whose complexities match those of the real world.

Turing's final thoughts on the mind as machine?

> *"The results which have been described in this article are mainly of a negative character, setting certain bounds to what we can hope to achieve purely by reasoning. These, and some other results of mathematical logic may be regarded as going some way towards a demonstration, within mathematics itself, of the inadequacy of 'reason' unsupported by common sense."*
>
> — final paragraph of Alan Turing, *Solvable and Unsolvable Problems*, Penguin Science News 31, 1954, p.23

During this spring [Turing] spent some time inventing a new quantum mechanics ... he produced a slogan 'Description must be non-linear, prediction must be linear'.

— Letter from Robin Gandy to Max Newman, June 1954