Kleene's Normal Form Theorem and the First Incompleteness Theorem

Peter Smith

July 10, 2007

Here, by way of reminder, is a version of Kleene's theorem for one-place total functions:

> **Theorem 1** *There is a three-place p.r. function T and a one-place primitive recursive (p.r.) function U such that any one-place μ-recursive function can be given in the standard form.*
>
> $$f_e(n) =_{\text{def}} U(\mu z[T(e, n, z) = 0])$$
>
> *for some value of e.*

And now we'll show

> **Theorem 2** *Kleene's Normal Form Theorem plus Church's Thesis entails the First Incompleteness Theorem.*

Proof: Suppose that there is a p.r. axiomatized formal system of arithmetic S which is p.r. adequate (i.e. can represent all p.r. functions), is ω-consistent (and hence consistent), and is negation complete. Then for every sentence φ either $S \vdash \varphi$ or $S \vdash \neg\varphi$.

Since S is p.r. adequate, there will be a four-place formal predicate T which captures the p.r. function T that appears in Kleene's theorem. And now consider the following definition,

$$\overline{f}_e(n) = \begin{cases} U(\mu z[T(e, n, z) = 0]) & \text{if } \exists z(T(e, n, z) = 0) \\ 0 & \text{if } S \vdash \forall \mathsf{z} \neg \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \mathsf{z}, \mathsf{0}) \end{cases}$$

We'll show that, given our assumptions about S, this well-defines an effectively computable total function for any e.

Take this claim in stages. First, we need to show that our two conditions are exclusive and exhaustive:

1. The two conditions are mutually exclusive (so the double-barrelled definition is consistent). For assume that both (a) $T(e, n, k) = 0$ for some number k, and also (b) $S \vdash \forall \mathsf{z} \neg \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \mathsf{z}, \mathsf{0})$. Since the formal predicate T captures T, (a) implies $S \vdash \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \overline{\mathsf{k}}, \mathsf{0})$. Which contradicts (b), given that S is consistent.

2. The two conditions are exhaustive. Suppose the first of them doesn't hold. Then for every k, it isn't the case that $T(e, n, k) = 0$. So for every k, $S \vdash \neg \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \overline{\mathsf{k}}, \mathsf{0})$. By hypothesis S is ω-consistent, so we can't also have $S \vdash \exists \mathsf{z} \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \mathsf{z}, \mathsf{0})$. Hence by the assumption of negation-completeness we must have $S \vdash \neg \exists \mathsf{z} \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \mathsf{z}, \mathsf{0})$, which is equivalent to the second condition.

Which proves that, given our initial assumptions, our conditions well-define a total function $\overline{f}_e$.

Now we prove that $\overline{f}_e$ is effectively computable. Given values for e and n just start marching through the numbers $k = 0, 1, 2, \ldots$ until we find the first k such that either $T(e, n, k) = 0$ (and then we put $\overline{f}_e(n) = U(\mu z[T(e, n, z) = 0])$), or else k is the super g.n. of a proof in S of $\forall \mathsf{z} \neg \mathsf{T}(\overline{\mathsf{e}}, \overline{\mathsf{n}}, \mathsf{z}, 0)$ (and then we put $\overline{f}_e(n) = 0$). Each of those conditions can be effectively checked to see whether it obtains – in the second case because S is p.r. axiomatized, so we can effectively check whether k codes for a sequence of expressions which is indeed an S-proof. And it follows from what we've just shown that eventually one of the conditions must hold.

Two more observations (still with our original assumptions in play):

3. Suppose f_e is μ-recursive, then $f_e(n) = U(\mu z[T(e, n, z) = 0])$ and the condition $\exists z(T(e, n, z) = 0)$ obtains for every n. And so in that case $f_e = \overline{f}_e$. Hence a list of the $\overline{f}_e$ will include all the μ-recursive functions.

4. Since, given e, we know how to compute the computable function $\overline{f}_e$, the diagonal function $d(n) =_{\mathrm{def}} \overline{f}_n(n) + 1$ is also effectively computable. But then d is a computable total function distinct from all the $\overline{f}_e$, hence distinct from any μ-recursive function.

So we've just shown that – given our original assumptions – there is a computable total function d which isn't μ-recursive, contradicting Church's Thesis.

Hence, if we do accept Church's Thesis, then it follows from Kleene's Theorem that, if S is a p.r. axiomatized, p.r. adequate, ω-consistent theory, it can't also be negation complete – which is (the core of) the First Incompleness Theorem – and proved without appeal to the construction of a provability predicate, or appeal to the diagonalization lemma.

Since Church's Thesis is here being used in labour-saving mode (to link two formal results together) we could of course sharpen the argument so as not to go via Church's Thesis: but this version is more transparent. And, I'm rather tempted to add, if you don't find it a delight, then maybe you aren't quite cut out for this logic business after all!