Model Theory (Draft 20 Jul 00)

Wilfrid Hodges

1 The boundaries of the subject

In 1954 Alfred Tarski [210] announced that 'a new branch of metamathematics' had appeared under the name of *the theory of models*. The subject grew fast: the Omega Group bibliography of model theory in 1987 [148] ran to 617 pages. By the mid 1980s there were already too many dialects of model theory for anybody to be expert in more than a fraction. For example very few model theorists could claim to understand both the work of Zilber and Hrushovski at the edge of algebraic geometry, and the studies by Immerman and Vardi of classifications of finite structures. And neither of these lines of research had much contact with what English-speaking philosophers and European computational linguists had come to refer to as 'model-theoretic' methods or concepts.

Nevertheless all these brands of model theory had common origins and important family resemblances. Some other things called models definitely lie outside the family. For example this chapter has nothing to say about 'modelling', which means constructing a formal theory to describe or explain some phenomena. Likewise in cognitive science the 'mental models' of Gentner and Stephens [67] or Johnson-Laird [100] lie outside our topic.

All the flavours of model theory rest on one fundamental notion, and that is the notion of *a formula ϕ being true under an interpretation I*. The classic treatment is Tarski's paper [202] from 1933. In this paper Tarski supposes that we have a language L with a precisely defined syntax. Ignoring punctuation, the symbols of L are of two kinds: *constants* and *variables*. The constants have fixed meanings; they will usually include logical expressions such as 'and' and 'equals'. The variables have no meaning, but (to short-circuit Tarski's very careful formulation a little) we can assign an object to each variable, and ask whether a given formula ϕ of L becomes true when each variable is regarded as a name of its assigned object. The grammatical categories of the variables determine what kinds of object can be assigned to them; for example we can assign individuals to individual variables, classes of individuals to class variables, and so on. If A is an allowed assignment of objects to variables and A makes ϕ true, then A is said to *satisfy* ϕ, and to be a *model* of ϕ, and ϕ is said to be *true in* A. (In [202] Tarski says 'satisfy', and moves on to give a definition of 'true' in terms of 'satisfy'.)

One of Tarski's main aims in this paper was to show that for certain kinds of language L, the relation 'A satisfies ϕ' is definable using only set theory, the syntax of L and the notions expressed by the constants of L. Thus one speaks of *Tarski's definition of truth* (or of *satisfaction*).

In several papers around 1970, Tarski's student Richard Montague [138] set out to show that Tarski's treatment applies to some nontrivial fragments of English. This work of Montague is a paradigm of what philosophers and linguists call *model-theoretic semantics*. Tarski himself ([202] §6 end) foresaw this development, but he suspected that it could only be carried through by rationalising natural language to such an extent that it might not 'preserve its naturalness and [would] rather take on the characteristic features of the formalized languages'. For the rest of this chapter, we shall only be concerned with formulas of artificial languages.

Tarski's 1933 paper brought into focus a number of ideas that were in circulation earlier. The notion of an assignment satisfying a formula is implicit in George Peacock ([151], 1834) and explicit in George Boole ([25] p. 3, 1847), though without a precise notion of 'formula' in either case. The word 'satisfy' in this context may be due to Edward V. Huntington (for example in [97] 1902). Geometers had spoken of gypsum or paper 'models' of geometrical axioms since the 17th century; abstract 'models' appeared during the 1920s in writings of the Hilbert school (von Neumann [147] 1925, Fraenkel [59] p. 342, 1928).

In 1932 Kurt Gödel wrote to Rudolf Carnap that he was intending to publish 'eine Definition für 'wahr' ' ([73]). He never published it. We know that by 1931 Gödel already had a good understanding of definability of truth in systems of arithmetic (see Feferman [57]); there is no solid evidence on whether he had thought about general set-theoretic definitions of truth before Tarski's paper was published.

2 One structure at a time

In Tarski's paper [202] he gives four examples of truth definitions, one of which (in his §3) he describes as 'purely accidental' because it doesn't follow the general strategy of his main definition. In this example he considers what today we would call the structure A of all subsets of a given set a, with relation $\subseteq$; he discusses what can be said about A using the corresponding first-order language L. (This may be an anachronism; one could also describe his example as the structure consisting of the set a with no relations, and a corresponding monadic second-order language.) Tarski works out an explicit definition of the relation 'ϕ is true in A', where ϕ ranges over the sentences of L. To bring this example within his scheme, Tarski counts the symbol $\subseteq$ as a constant; the two quantifiers are constants too, understood as ranging over the set of subsets of a.

Tarski doesn't call A a structure—in fact he doesn't even have a symbol for it—but the following features make it an example of what later became known as a structure. It carries a *domain* or *universe* (the set of subsets of a); it also carries a binary relation on the domain, labelled by the symbol '$\subseteq$'. A structure can also carry labelled relations and functions of any finite arity on the domain, and labelled elements of the domain. Structures in this sense are an invention of the second half of the 19th century—for example Hilbert handled them freely in his Foundations of Geometry ([85] 1899). Richard Dedekind ([44] 1871 and [39] 1872) frequently used the name *System* for structures (and also for sets—apparently he thought of a structure as a set that comes with added features). Weber and Hilbert spoke of 'Systeme von Dingen', to distinguish from axiom systems. In model theory the name 'system' persisted until it was replaced by 'structure' in the 1950s, it seems under the influence of Abraham Robinson [162] and Bourbaki [28]. (Augustus De Morgan introduced the term 'universe' in 1846 [40], to mark the fact that one deals with a determinate set of things which may be different in different discourses. 'Gebiet'—which is 'domain' in German—is a loose term, but its use in structures may owe something to Hermann Grassmann [72] 1844; Dedekind used it as a synonym for 'System'.)

Tarski's choice of example was not an accident. Leopold Löwenheim [123] §4 (1915) had already studied the same example within the context of the Pierce-Schröder calculus of relatives, and he had proved a very suggestive result. In modern terms, Löwenheim had shown that there is a set of 'basic' formulas of the language L with the property that every formula ϕ of L can be reduced to a boolean combination ψ of basic formulas which is equivalent to ϕ in the sense that exactly the same assignments to variables satisfy it in A. Thoralf Skolem [191] §4, 1919 and Heinrich Behmann [18] 1922 had reworked Löwenheim's argument so as to replace the calculus of relatives by more modern logical languages. In 1927 C. H. Langford [114], [115] applied the same ideas to dense or discrete linear orderings.

Tarski realised that not only the arguments of Löwenheim and Skolem, but also the heuristics behind them, provided a general method for analysing structures. This method became known as the *method of quantifier elimination*. In his Warsaw seminar, starting in 1927, Tarski and his students applied it to a wide range of interesting structures. Important early examples were the field of real numbers (Tarski [201] 1931) and the set of natural numbers with symbols for 0, 1 and addition (Presburger [160] 1930). In both these cases the method yielded (a) a small and easily described set of basic formulas, (b) a description of all the relations definable in the structure by first-order formulas, (c) an axiomatisation of the set of all first-order sentences true in the structure and (d) an algorithm for testing the truth of any sentence in the structure. (Here (b) comes at once from (a). For (c), one would write down any axioms needed to reduce all formulas to boolean combinations of basic formulas, and all axioms needed to determine the truth of basic sentences. Then (d) follows since the procedure for reducing to basic formulas is effective.)

One should note two characteristic features of the method of quantifier elimination. First, the structure serves only to provide a stock of axioms. True, nontrivial theorems about the structure may be useful for finding the axioms. (For the field of real numbers, Tarski used Sturm's theorem.) But once the axioms are on the table, the rest of the work (and it can be heavy) is entirely syntactic and unlikely to appeal to model theorists. Tarski was aware of this. As late as 1978 he was defending the method of quantifier elimination against modern methods

> which often prove more efficient. ... It seems to us that the elimination of quantifiers, whenever it is applicable to a theory, provides us with direct and clear insight into both the syntactical structure and the semantical contents of that theory—indeed, a more direct and clearer insight than the modern more powerful methods to which we referred above.

(Doner, Mostowski and Tarski [43] p. 1f.)

Second, the method works on just one structure at a time. It involves no comparison of structures. Let me dwell on this for a moment. In 1936 [203] Tarski reworked and refined Langford's results on discrete linear orderings. Starting from an arbitrary discrete linear ordering A, he would follow the method until he came to a choice of axiom that was true in some discrete linear orderings and false in others. He found that he needed only the following axioms or their negations:

> There is a first element.
> There is a last element.
> There are at least n elements (n a positive integer).

From this he could read off exactly what are the possible completions of the theory of discrete linear orderings. In this way he got information not about a single structure but about a class of structures. (In the Appendix to [203], Tarski tells us that by 1930 he had defined the relation of *elementary equivalence*, in modern symbols $\equiv$: $A \equiv B$ if the same first-order sentences are true in A as in B.) Nevertheless the quantifier elimination itself involves only one structure at a time, and any comparisons come later.

There are two common misconceptions about the method of quantifier elimination. The first is that the basic formulas should be quantifier-free. This was never the intention, though it happened in some cases. (The name means that we reach the basic formulas by eliminating quantifiers, not that all quantifiers are removed.) Later model theorists said that a theory has *the property of quantifier elimination* if every formula is equivalent, in all models of the theory, to a quantifier-free formula with the same free variables. One must avoid confusing the property with the method.

The second misconception is that the main aim of the method was to solve decision problems. This is about half true. It was Behmann's paper [18] in 1922 that introduced the term 'Entscheidungsproblem' (cf. Mancosu [135]), long before Church and Turing in 1935–6 made the notion of an effectively decidable set precise. Also Skolem in [191] presented his quantifier elimination as a way of 'evaluating' (*auswerten*) formulas; and in [197] he emphasised decidability (though he had other aims too). But Langford's two papers [114], [115] show no awareness of decidability; he is concerned only with finding complete sets of axioms. Tarski's reworkings of the quantifier eliminations of Langford [203] and Skolem [204] are silent about decidability. Tarski [201] in 1931 and his student Presburger [160] in 1930 mention in passing that a decision procedure falls out of the quantifier elimination. Not until 1940 does Tarski come out with the following (not very model-theoretic) statement:

> It is possible to defend the standpoint that in all cases in which a theory is tested with respect to its completeness the essence of the problem is not in the mere proof of completeness but in giving a decision procedure (or in the demonstration that it is impossible to give such a procedure).

([206] Note 11, published in 1967 from the 1940 page proofs).

By contrast it always was one of the main aims of quantifier elimination, in both Skolem and Tarski, to answer the question 'What relations are definable in a given structure A by means of formulas of a given formal language?'. (In 1910 Hermann Weyl [222] had introduced the class of first-order definable relations over a relational structure, but without using a formal language.) This remained a central question of model theory in all periods.

For example, when the structure A is an algebraically closed field, we are asking what are the constructible sets of affine geometries over A. It took some time for model theorists to realise that this gave them an entry into algebraic geometry. Abraham Robinson realised it after a fashion, and with Peter Roquette he gave a model-theoretic proof of the finiteness theorem of Siegel and Mahler [170]. This was published in 1975, shortly after Robinson's premature death. But the real breakthrough came with the discovery of Morley rank in algebraically closed fields and the subsequent development of geometric model theory. Already Boris Zilber's 1977 paper [225] is largely about first-order definable relations.

Likewise it was a long time before model theorists realised what a strong tool they had in Tarski's description of the first-order definable sets in the field of real numbers. These sets are precisely the unions of finitely many sets, each of which is either a singleton or an open interval with endpoints either in the field or $\pm\infty$; in 1986 Anand Pillay and Charles Steinhorn [157] introduced the name *o-minimal* for ordered structures in which these are all the first-order definable sets. Lou van den Dries had pointed out in 1984 [45] that the o-minimality of the field of real numbers already gives strong information about definable relations of higher arity—in particular it allows one to recover a form of cell decomposition in the style known to geometers. Julia Knight, Pillay and Steinhorn [109] generalised this cell decomposition to all o-minimal structures. One of the outstanding results in this area of model theory was Alex Wilkie's proof in 1991 [223] that the field of real numbers with the exponentiation function added is o-minimal and has model-complete theory (i.e. the existential formulas form a basic set). Wilkie's method was quite different from Tarski's; the problem may be beyond the reach of methods as effective as Tarski's. But Wilkie could call on the advances in model theory made by Abraham Robinson and his generation, and also a substantial body of work on real-algebraic geometry by Khovanskii and others.

3 The metamathematics of first-order classes of structures

Strictly speaking, if one compares the sentences true in two structures A and B, one is no longer using the symbols of the structures as constants. This would have made it awkward to apply Tarski's [202] truth definition directly within model theory. But Tarski was aware that there was a problem here, and in [205] §37 (1941 edition AND THE POLISH??) he proposed a way of dealing with it. In a typical axiom system there are (again ignoring punctuation) not two but three kinds of symbol. First come the logical symbols for 'and', 'or' and so forth. Second there are variables. But thirdly there are what Tarski calls 'primitive terms'. These include the relation and function symbols of the axiom system, and in general they also include a term standing for the domain over which quantifiers are to range. The primitive terms have no meaning, so they are unsuitable to appear as constants in the 1933 truth definition. Therefore Tarski applies that definition to a sentence ϕ and a structure A indirectly: he first forms an expression ϕ' by replacing the primitive terms in ϕ by variables, and then he says that A is a *model of* ϕ if the assignment of the appropriate features of A to the corresponding variables in ϕ' satisfies ϕ'.

This is strangely roundabout. In a different way, so is Tarski's procedure of [208], where he replaces formulas by functions from structures to definable sets. But it must have become clear by 1950 that the truth definition should be rewritten to handle languages with three levels of symbol—logical, variable, primitive. Tarski's paper [210] already assumes such a truth definition, and Tarski and Robert Vaught wrote out the details in 1957 [213]. (Mostowski [143] §3 1952 is an anticipation but in a different idiom.) As urged by C. S. Peirce [154] in 1884, Tarski and Vaught took the quantifiers to range over the domain implicitly, so that no primitive symbol for the domain was needed. They referred to the remaining primitive symbols as the *non-logical constants*.

So now people had a direct model-theoretic definition of 'structure A is a model of sentence ϕ', and they could straightforwardly define the class $\mathrm{Mod}(T)$ of all structures which are models of a given set T of first-order sentences. In this notation, Tarski's definition ([204] p. 8, 1953) of the relation 'sentence ϕ is a *logical consequence* of theory T' becomes

$$\mathrm{Mod}(T) \subseteq \mathrm{Mod}(\{\phi\})$$

i.e. 'Every model of T is a model of ϕ'.

Changes of viewpoint usually generate new terminology. In 1950 there was still no agreed name for a set of sentences of a formal language. For example one finds them referred to as Axiomsysteme, sets of axioms, sets of postulates, sets of laws, sets of propositions, and less often Theorien (e.g. in Hilbert [86]). In 1935/6 Tarski's 'deductive theories' were sets of expressions with a logic attached ([203]). During the mid 1950s the word *theory* came to be accepted as the standard word for a set of sentences of a logical language. (But not by Robinson, who continued to write of 'sets of axioms' or 'sets of sentences' throughout his career.)

A number of earlier metamathematical results fell into place at once within this general picture.

(a) In 1906 Gottlob Frege [64] attacked the idea of using a set of formal axioms to define a class of structures. One thing that he found particularly scandalous was that mathematicians should in good faith use the same symbol to mean different things (in different structures) within one and the same discourse:

> In der Tat, wenn es sich darum handelte, sich und andere zu täuschen, so gäbe es kein besseres Mittle dazu, als vieldeutige Zeichen. (Indeed, if it were a matter of deceiving oneself and others, there would be no better means than ambiguous signs. [64] p. 307.)

But in the new truth definition the relation 'symbol S names relation R in structure A' has an unambiguous and purely mathematical content, so that even the most punctilious model theorist can use it with a pure heart. (Frege had other objections too, and they need other answers.)

(b) In the years around 1900, Giuseppe Peano ([151] 1891), Hilbert ([85] 1899), Huntington ([97] 1902) and their colleagues considered a number of questions of the form: Is axiom ϕ in the axiom set T deducible from the other axioms in T? (Hilbert's cited work includes the famous example of Euclid's parallel postulate.) To show the answer No, one proves that some model of $T \setminus \{\phi\}$ is not a model of ϕ, and one usually does this by describing such a model explicitly. The truth definition allows us to give a purely mathematical explanation of what is happening in arguments of this kind.

I add a word of caution here. In 1936 [204] Tarski proposed a definition of 'logical consequence' between sentences *without non-logical constants*. The notion that he called 'logical consequence' in 1953 [212] was the natural analogue for languages with three levels of symbol, but since it applies to a different class of languages, it can hardly pick up the same relation. Mathematical logicians who use Tarski's 1953 terminology are sometimes said by philosophers to be endorsing 'the model-theoretic theory of logical consequence'. This seems to be a case of cross purposes; mathematical logicians who use the phrase are simply referring to a useful set-theoretical notion, not taking sides in a contentious conceptual analysis.

The description 'model-theoretic theory' does certainly apply to Tarski's 1936 [204] proposal, in the same broad sense that Tarski's 1933 truth definition is model-theoretic. Tarski also used the word 'model' in the 1936 paper, but (uncharacteristically) to stand for a *reinterpretation* of terms that already carry a meaning. His proposal was that a sentence ϕ should count as logically true if and only if every model (in this odd sense) of ϕ is true. He noted that to make this definition work, one would need to restrict the class of terms that can be reinterpreted, so that in some appropriate sense the reinterpreted sentence has the same 'form' as the original. (And for logical consequence he proposed a similar definition.)

Part of Tarski's 1936 proposal was several hundred years old; already in the twelfth century Peter Abelard [1] p. 255 had noted that a 'perfect' syllogism remains valid when its terms are altered. Unknown to Tarski, Bernard Bolzano [24] §§147f in 1837 had suggested using this idea as a *definition* of analyticity: in Bolzano's terminology a true proposition P is *logically analytic* if and only if we get a true proposition whenever we vary the non-logical ideas that appear in P. Tarski's 1936 proposal adds one further ingredient that is missing in Bolzano: in the reinterpretation, terms should be allowed to stand for any objects of suitable type in the universe, regardless of whether they already have names in the language. This was a byproduct of the truth definition, but for Tarski it was one of the most important features of his proposal.

(c) In his paper of 1915 on the calculus of relatives [123], Löwenheim showed that every sentence of first-order logic, if it has a model, has a model with at most countably many elements. His proof has several interesting features, including his introduction of function symbols to reduce the satisfiability of a sentence

$$\forall x \exists y \ \phi(x, y)$$

to the satisfiability of the sentence

$$\forall x \ \phi(x, F(x)).$$

Thus it seems that Löwenheim invented Skolem functions, if we forgive him his bizarre explanation of the passage from the first sentence to the second. (He uses infinite, possibly uncountable, strings of existential quantifiers!)

Skolem tidied up Löwenheim's argument and strengthened the result in two alternative ways. In 1920 [192] he showed, using the axiom of choice and a coherent account of Skolem functions, that if T is a countable first-order theory (in fact he allows countable conjunctions and disjunctions too, and infinite quantifier strings) with a model A, then T has a model B with at most countably elements; the proof shows that B can be taken as an elementary substructure of A, but at this date Skolem lacked even the notion of substructure. Two years later [193] he showed without the axiom of choice that every countable first-order theory with a model has an at most countable model. He used this result to construct a countable (and hence nonstandard) model of Zermelo-Fraenkel set theory.

These results of Skolem seem to have attracted little attention before the 1950s. But from the new perspective of the 1950s they were central results about classes of the form Mod(T). In 1957 Tarski and Vaught [213] proved the stronger result that if L is a first-order language with at most λ formulas, and A is a structure for L of cardinality $> \lambda$, then A has an elementary substructure of cardinality λ. This result was known as the *Downward Löwenheim-Skolem-Tarski Theorem* (though it soon became usual to drop Tarski's name from the list in the interests of brevity).

(d) In his 1930 PhD thesis [68], Kurt Gödel reworked Skolem's argument from [193] and used it to show that if T is any syntactically consistent theory in a countable first-order language, then relations and functions can be defined on the natural numbers, corresponding to the relation and function symbols of T, so that the defined relations and functions make the natural numbers $\mathbb{N}$ into a model of T (perhaps with an equivalence relation for equality). Thus one could build a structure by 'interpreting' the language of the structure in another given structure.

Mostowski [142] 1948, Tarski [212] 1953, Ershov [56] 1974 and others developed this idea, usually under the name of *interpretation*. Since their aim was to prove undecidability results, they thought in terms of the syntactic interpretation of the theory T rather than in terms of building a new structure. But when stability theorists found they needed a notion of one structure being interpretable in another, Ershov's notion was the one they used. Shelah [185] (Chapter III §6) described how one might think of the elements of structures interpretable in a structure M as *imaginary elements* of M.

If the theory T in Gödel's argument was itself arithmetically definable, then one could place bounds on the arithmetical complexity of the interpreting relations and functions. So Gödel's paper was a first step towards effective model theory. Various people studied effective models at various times and places—the text of Goncharov and Ershov [71] has nearly 400 references—but it never became a mainstream topic.

(e) It was natural to look for upward versions of Skolem's results. Tarski claimed in 1934 (in a note added by the editors to the end of Skolem [195]) that in 1927/8 he had proved that every consistent first-order theory with no finite model has a model with uncountably many elements. (Vaught [214] p. 160 reports the few facts that are known about this early proof by Tarski.)

Anatolii Mal'tsev [130] 1936 stated that every first-order theory with an infinite model has models of 'all cardinalities' (presumably he meant all high enough cardinalities). His proof rested on the Compactness Theorem, on which see (f) below. Given the Compactness Theorem, it is easy to prove that if L is a first-order language with at most λ formulas, and A is a structure for L of cardinality $< \lambda$, then A has an elementary extension of cardinality λ. This result became known as the *Upward Löwenheim-Skolem-Tarski Theorem*—though again Tarski's name was often dropped. The irony was that it was Skolem [196], not Tarski, who refused to accept that the theorem was true (though he allowed that it might be deducible within some formal set theories).

(f) The Compactness Theorem for first-order logic states that if T is a first-order theory such that every finite subset of T has a model, then T has a model. Results equivalent to this appear in papers of Gödel (i) for countable languages, [68] in 1930 and (ii) for propositional languages of arbitrarily cardinality, [69] in 1932. In 1941 Mal'tsev [132] stated the full theorem, referring to his paper [130] of 1936 for the proof. In fact [130] doesn't state the theorem, though it does contain a correct proof in two parts. First Mal'tsev states and proves the Compactness Theorem for a propositional language of arbitrary cardinality. Then he shows how this theorem can be lifted to a theory S in an arbitrary first-order language, by first passing to Skolem normal form and then introducing individual constants as 'witnesses' (to use the modern jargon) in order to replace S by a propositional theory—let us call it T—with the properties: (i) If every finite subset of S is satisfiable then every finite subset of T is satisfiable, (ii) if T is satisfiable then S is satisfiable. (i) is routine to check from Mal'tsev's construction, but he never states it explicitly in the paper; this is hardly a ground for denying him the theorem.

The familiar proof of the Compactness Theorem by way of a completeness proof for a proof calculus for first-order languages of arbitrary cardinality is a slight revision (due to Gisbert Hasenjaeger [78]) of the proof given independently by Leon Henkin [81] in 1949, from his PhD thesis. The detour through a proof calculus is very easily avoided by replacing 'T is syntactically consistent' throughout the argument by 'All finite subsets of T have models'.

Any proof of the Compactness Theorem has to use some method for building structures of arbitrarily high cardinality. Interpretation in $\mathbb{N}$, as in Gödel's argument, is no longer an option. Both Mal'tsev and Henkin take the elements of their models to be individual constants (or equivalence classes of these constants, to give '=' its standard meaning). Since there are uncountably many of them, these constants can only be linguistic objects in an abstract sense. In fact the central point seems to be that both Mal'tsev and Henkin introduced a well-ordered sequence of objects and then used it as a template to build the model around. Constructions of this kind were popular in the 1970s, particularly in connection with Jensen's prediction principles (such as diamond) that allow one to 'predict' how the model will sit around the template; see for example Shelah on the Whitehead problem [184] 1974.

Frayne, Morel and Scott [63] 1962 discovered another proof of the Compactness Theorem along completely different lines. They used ultraproducts (on which see the next section), after Tarski had noticed that reduced products can be used to prove compactness for sets of Horn sentences.

The name 'Compactness Theorem' is from Tarski's 1950 Congress address, [208]. Until the late 1940s the Compactness Theorem went almost unnoticed in the West. This shouldn't be a surprise; at first sight the Compactness Theorem has nothing useful to say about the kinds of question we considered in section 2 above. But as soon as attention shifted to axiomatically defined classes, it became clear that the Compactness Theorem was a powerful tool for building structures within such a class.

The credit for this realisation goes to Mal'tsev and independently Abraham Robinson. We have already noted that Mal'tsev [130] used compactness to derive a form of upward Löwenheim-Skolem theorem. More startling, he used compactness in two group-theoretic papers [131] 1940 (implicitly) and [132] 1941 (explicitly) to prove local theorems in group theory. The result of [131] is the now well-known theorem that if all the finitely generated subgroups of a group G have faithful linear representations of degree n over fields, then so does G. These were the first examples of results interesting to mathematicians in other fields, but proved by model theory. Unfortunately the first notice of [132] in the West seems to have been in 1959 ([134]). But meanwhile Robinson had independently proved the Compactness Theorem in his PhD thesis [163] and had used it to deduce several mathematical statements such as the following ([162] p. 7, which I state in more modern terms): 'Every purely transcendental extension K of the field R of rational numbers has a field extension which is an elementary extension of R and adds no new element algebraic over the transcendence basis of K.' Section 5 will examine how Robinson developed this line of work.

4 Expressive power

As soon as one has in place a serviceable definition of the relation 'Structure A is a model of first-order sentence ϕ', certain things become almost routine. One can define the class Mod(T) of all models of a theory; dually one can define the set Th($\mathbf{K}$) of all first-order sentences true throughout the class $\mathbf{K}$ of structures. Together Mod and Th form a Galois connection. A class of the form Mod(T) is said to be an *EC class* or an *elementary class* if T is finite, and an EC_Δ *class* or an *elementary class in the wider sense* in the general case. Thus Tarski [210] in 1954, except that he had 'arithmetical' for 'elementary'.

An obvious question to ask is: (1) Given a first-order theory T, what can one say in general about its class of models? For example, are there interesting general constructions that will give models of T with interesting properties? The dual question would be: Given a class of structures, what can one say in general about the set of sentences true in all of them? But this is a boring question (or at least it was until computer scientists found themselves looking for efficient ways to determine, for a given structure A and sentence ϕ, whether ϕ is true in A). So instead one asked: (2) Given two structures A and B, how can one determine whether the same sentences are true in both of them?

These are technical questions calling for technical answers. During the 1950s and early 1960s a stream of answers emerged. The text of Chang and Keisler [33], published in 1973, is a compendium of the main achievements.

There were two other trends in the model theory of this period, neither of them much connected with the questions (1) and (2), though they had a large influence on the ways in which the technical breakthroughs were exploited. The first trend was that people began to take a close interest in languages which are stronger than first-order but more tractable than second-order, for example infinitary languages and languages with cardinality quantifiers (see section 9 below). So when someone had introduced a technique for first-order languages, he or she could move on to testing the same technique on stronger and stronger languages. Often a variant of the technique would still work, but set theoretic assumptions and arguments would begin to appear. An observation of William Hanf [76] helped to organise this area: he noted that for any reasonable language L there is a least cardinal κ (which became known as the *Hanf number* of L) such that if a sentence of L has a model of cardinality at least κ then it has arbitrarily large models. A great deal of work and ingenuity went into finding the Hanf numbers of a range of languages.

One effect of this trend was that the centre of gravity of research on questions (1) and (2) moved steadily away from first-order languages and towards infinitary languages during the period from 1950 to 1970, bringing a heady dose of set theory into the subject. Allow me two anecdotes. In about 1970 a Polish logician reported that a senior colleague of his had advised him not to publish a textbook on first-order model theory, because the subject was dead. And in 1966 David Park, who had just completed a PhD in first-order model theory with Hartley Rogers at MIT, visited the research group in Oxford and urged us to get out of first-order model theory because it no longer had any interesting questions. (Shortly afterwards he set up in computer science.)

The second trend was that this was the period in which model theorists began to take maps between structures seriously (see section 5). The effects of this trend were both subtler and more profound. Most of the rest of this chapter will be devoted to them.

Here follow three of the answers to (1).

(1a) *Ultraproducts.* In 1955 Jerzy Łoś [121] described a construction which combined a family of structures by means of an ultrafilter D on the power set algebra of the index set. He stated that a first-order sentence ϕ is true in the constructed structure if and only if the set of indices at which ϕ is true lies in D. This became known as *Łoś's Theorem*, and the construction itself gained the name of *ultraproduct*; an ultraproduct of isomorphic structures was called an *ultrapower*. (It came to light that ultraproducts or their close relatives had been used earlier by Skolem [194] 1934, Hewitt [84] 1948 and Arrow [3] 1950. Skolem's application was model-theoretic, to build a structure elementarily equivalent to the natural numbers with $+$ and $\cdot$ but not isomorphic to them.)

We noted in the previous section that one can prove the Compactness Theorem directly by ultraproducts. Ultrapowers also give a direct construction of arbitrarily large elementary extensions of an infinite structure. In 1961 Dana Scott [177] used a measurable cardinal to construct an inner model of set theory that was an ultrapower of the universe. This ensured that ultraproducts remained an essential tool of set theory long after most model theorists had lost interest in them.

The further development of ultraproducts within model theory largely revolved around what one might call their injective-like properties: various commutative diagrams could be completed by adding a mapping from some given structure to an ultraproduct of other given structures. The simplest example was *Frayne's Lemma* ([63] 1962): If A is elementarily equivalent to B then A is elementarily embeddable in some ultrapower C of B. Frayne's proof of this result constructed the embedding and the ultrapower C simultaneously.

Jerome Keisler reorganised arguments of this kind in a style that was to prove important—Morley and Vaught [141] speak of Keisler's ' "one element at a time" property'. Keisler himself (footnote on Theorem 2.2 of his doctoral dissertation, [104] 1961) compared his procedure with the element-at-a-time methods used by Cantor [30] and Hausdorff [79] to build up isomorphisms between densely ordered sets. Frayne's Lemma is a good example to illustrate the point. Let the sequence $(a_i : i < \kappa)$ list all the elements of A. The aim then is to construct, by induction on $\alpha \leq \kappa$, a sequence $(c_i : i < \alpha)$ of elements of the ultrapower C so that the map $a_i \mapsto c_i$ is a *partial elementary map*, i.e. any formula satisfied by the a_i in A is also satisfied by the corresponding c_i in C. We can construct such a sequence provided that we know that C has the following property: Given any set X of fewer than κ elements of C and any complete 1-type over these elements (i.e. set of formulas $\phi(x)$ with elements of X as parameters, which is maximal consistent with the theory of C), there is an element of C realising this type. Keisler said that if C had (essentially) this property, then it was *κ-replete*, and he then showed how to construct κ-replete ultrapowers. Keisler's definition was slightly inconvenient—he counted the formulas rather than the parameters—and a corrected version took the name *κ-saturation*. A structure of cardinality κ was called *saturated* if it was κ-saturated. (The name is from Vaught [215] in the case $\kappa = \omega$.)

Thus it turned out that ultraproducts were useful largely because of their high saturation. Highly saturated elementary extensions are easy to construct directly by realising types and taking unions of elementary chains. Since saturation is a simpler concept than ultraproducts, ultraproducts largely dropped out of use within the model theory community once the early enthusiasm had worn off. But there remain some important theorems for which ultraproducts give the only known reasonable proofs; one is Keisler's theorem [106] that uncountably categorical theories fail to have the finite cover property. Many algebraists still value ultraproducts for their transparency.

(1b) *Omitting types.* Ultraproducts give highly saturated models. One also wanted a way of building models that are very unsaturated. In 1959 Vaught [215] gave the classic omitting types theorem for countable models of complete first-order theories. This theorem allows one to omit countably many types at once; Vaught attributes this feature to Ehrenfeucht. The paper also contains Vaught's Conjecture as a question: 'Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?' (The Conjecture is that there is no such theory. It is still open in spite of some impressive work by Rubin, Steel, Buechler, Newelski and others on special cases.)

There were several close variants of omitting types. The Henkin-Orey theorem [149] was one that appeared before Vaught's paper, while Robinson's finite forcing [14] and Grilliot's theorem [74] on constructing families of models with few types in common were two that came later. Martin Ziegler [224] made finite forcing more palatable by recasting it in terms of Banach-Mazur games; the same recasting works for all versions of omitting types.

Finite forcing builds existentially closed models; these were introduced into model theory by Michael Rabin [161] 1962 and Per Lindström [119] 1964. During the 1970s Belegradek, Ziegler, Shelah and others put a good deal of energy went into constructing existentially closed groups, after Macintyre [125] 1972 had shown that they have remarkable definability properties.

(1c) *Indiscernibles.* In 1956 Ehrenfeucht and Mostowski [50] showed, using Ramsey's Theorem, that if T is a complete first-order theory with infinite models and $(X, <)$ is a linearly ordered set, then T has a model A whose domain includes X, and for each finite n, any two strictly increasing n-tuples from X satisfy the same formulas in A. (In short, $(X, <)$ is an *indiscernible sequence* in A—though Ehrenfeucht and Mostowski said 'homogeneous set'.) If A is the closure of X under Skolem functions (as we can always arrange), A is said to be an *Ehrenfeucht-Mostowski model* of T.

Ehrenfeucht-Mostowski models have tightly controlled properties. For example they realise few types (see their use in section 6 below). By choosing $(X, <)$ and $(X', <')$ sufficiently different, we can often ensure that the Ehrenfeucht-Mostowski models constructed over these two ordered sets are not isomorphic; this is the basic idea underlying most of Shelah's constructions of large families of nonisomorphic models (again see section 6). One can also construct Ehrenfeucht-Mostowski models of infinitary theories, using various theorems of the Erdős-Rado partition calculus in place of Ramsey's

Theorem. As a byproduct we get a versatile way of building *two-cardinal models*, i.e. models of first-order theories in which some definable parts have one infinite cardinality and others have another infinite cardinality, as Morley showed in 1965 [140]. (Vaught had obtained two-cardinal results earlier by other methods.)

In 1964 Frederick Rowbottom showed that if the set-theoretic universe contains a measurable cardinal (or even an Erdős cardinal), then the constructible universe forms an Ehrenfeucht-Mostowski model on a class of ordinals which includes all uncountable cardinals. This result had enormous repercussions in set theory, long before its late publication in [174].

(2) In his 1950 Congress address Tarski [208] showed how to give a precise mathematical definition of elementary equivalence, using essentially the set-theoretic definition of satisfaction. This demonstrated that elementary equivalence is a sound notion, but it gave no new information about the notion. It was natural to ask if the same relation could be defined by a completely different approach.

Any two elementarily equivalent saturated structures of the same cardinality are isomorphic. In 1961 Keisler [104] exploited this to show, with the help of the generalised continuum hypothesis, that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers. Ten years later Shelah [180] proved the same theorem without assuming the generalised continuum hypothesis. Simon Kochen in 1961 [110] gave another characterisation of elementary equivalence, using direct limits of ultrapowers.

In spite of their elegance, these characterisations of $\equiv$ had few practical consequences. A much more serviceable answer came from another source. Roland Fraïssé described a hierarchy of interrelated families of partial isomorphisms between structures [62]. In terms of this hierarchy he gave necessary and sufficient conditions for two relational structures to agree in all prenex first-order sentences with at most n alternations of quantifier, for each finite n. So $A \equiv B$ if A and B agree in this sense for all finite n. Fraïssé's paper was unfortunately hard to read, and his ideas became known through a paper of Ehrenfeucht ([49], 1961) who had come on them independently. Soon afterwards they were rediscovered again by the Kazak mathematician Taĭmanov [200].

In Ehrenfeucht's version, two players play a game to compare two structures A and B. The players alternate; in each step, the first player chooses an element of one structure and the second player then chooses an element of the other structure. The second player loses as soon as the elements chosen from one structure satisfy a quantifier-free formula not satisfied by the corresponding elements from the other structure. This is the *Ehrenfeucht-Fraïssé back-and-forth game* on the two structures. For a first-order language with finitely many relation and individual constant symbols and no function symbols, one could show that A and B agree in all sentences of quantifier rank at most k if and only if the second player has a strategy that keeps her alive for at least k steps. Hence A is elementarily equivalent to B if and only if for each finite k, the second player can guarantee not to lose in the first k steps.

With this equipment it's very easy to show that if G, G' are elementarily equivalent groups and H, H' are elementarily equivalent groups, then the product group $G \times H$ is elementarily equivalent to $G' \times H'$. This is perhaps the best way to view the technique of Feferman and Vaught [58] 1962 for computing the set of sentences true in an arbitrary product of structures. A similar technique worked with ordered sums of structures.

The beauty of this idea of Fraïssé and Ehrenfeucht was that nothing tied it to first-order logic. Ehrenfeucht himself [49] used it to prove the equivalence of various ordinal numbers as ordered sets with predicates for $+$ and $\cdot$, in a language with a second-order quantifier ranging over finite sets. Carol Karp [103] adapted it to infinitary logics, and it reappeared in Chang's construction of Scott sentences [32]. Later (see section 9) it became one of the central tools of computer science logic.

5 Maps between structures

During the period 1930–1950, mathematicians generally had begun to take a closer interest in the maps between structures. This was the period that saw the invention of category theory. The trend naturally made its way into model theory.

For example Garrett Birkhoff [21] published his famous characterisation of the classes of models of sets of identities in 1935. Birkhoff's paper uses a number of straightforward model-theoretic facts about mappings, for example that universally quantified equations are preserved under taking homomorphic images; in 1951 Marczewski [136] extended this result to all positive first-order sentences and asked for a converse. Tarski [210] reported that his own work on formulas preserved in substructures (the Łoś-Tarski Theorem) was done in 1949–50. This work of Łoś and Tarski, together with Marczewski's question, launched a flood of preservation theorems for all kinds of mapping. Keisler had a rich crop to survey already in 1965 [105]. But theorems of this kind continued to be proved up to the end of the 20th century (e.g. [10] in 1999).

Definability theorems are a close relative of preservation theorems, and they tend to have similar proofs. Instead of maps between structures, they talk about *reducts*, where one or more symbols are stripped away from the language (as with some forgetful functors). The first and most famous definability theorem of first-order model theory was *Beth's Theorem*, [20] 1953, which says that if in models of a theory T a certain symbol R is not definable in terms of the other symbols, then one can find two models of T which are identical in everything except the interpretation of the symbol R; in other words, Padoa's method always works.

An unkind comment—though it has some truth—is that in practice one only ever uses the trivial direction of a preservation or definability theorem. These theorems contribute more to the inner structure of model theory than they do to applications. But during the 1950s the maps between structures came to play a deeper role in model theory, not just as possible topics but as essential tools of the subject. One can trace this development to two model theorists, Abraham Robinson and Roland Fraïssé. I begin with Robinson.

Robinson's 1950 Congress address [162] uses embeddings, homomorphisms, elementary embeddings and unions of chains. Elementary embeddings were a new idea. Robinson had no name for them and not much of a general theory. In fact it was Robinson's style, then and later, to use maps without even having a symbol for them. To find an embedding of A in B, he would prove that B is a model of the *diagram* of A; in Robinson's sense a diagram was a set of sentences describing a structure, not a display of arrows. It was only in 1957 that Tarski and Vaught [213] defined elementary embeddings and made them generally available.

Robinson had a particular genius for finding model-theoretic statements—usually about mappings—that are equivalent to significant facts in this or that branch of algebra or field theory. To measure his style one should compare his treatment of algebraically closed fields with that of Tarski [207]. Tarski had applied the method of quantifier elimination to the theory T of algebraically closed fields and established that the quantifier-free formulas form a basic set, and that T has countably many completions, namely one for each characteristic.

Robinson made the following observations: (1) If A and B are algebraically closed fields of the same characteristic, then by downward Löwenheim-Skolem and compactness one can find fields A', B' elementarily equivalent to A, B respectively, and both of transcendence degree ω. By Steinitz' Theorem A' and B' are isomorphic, and hence A and B are elementarily equivalent. This gives Tarski's conclusion on completions of T ([162]). (2) We say that a theory S is *model-complete* if every embedding between models of S is elementary. A sufficient condition for model-completeness is that every embedding between models of S preserves universal formulas ([166]). It then follows at once from the Hilbert Nullstellensatz that the theory T of algebraically closed fields is model-complete ([165]). (3) Suppose that whenever A and B are models of a theory S and X is a nonempty substructure of both A and B, the formulas satisfied by elements of X in A are the same as those satisfied by the same elements in B; then S has the property of quantifier elimination (i.e. the quantifier-free formulas are a basic set). In the case of our theory T, standard facts about amalgamation of fields allow us to embed both A and B in an algebraically closed field C, forming a commutative diagram over X. By model-completeness the embeddings of A and B in C are elementary, so the criterion for quantifier elimination holds ([168] §4.3). Thus Robinson reached all Tarski's conclusions about algebraically closed fields by general model-theoretic principles and standard algebraic facts about fields. This is typical of Robinson's reorganisation of the subject.

Robinson certainly didn't confine himself to finding new proofs of known theorems (though he was never inhibited about doing precisely this). Among his many contributions were some new examples of model-complete theories. One of these was the theory of differentially closed fields of characteristic 0 [165] p. 134 in 1956, as if to predict how useful this theory would be for Hrushovski's proof of the geometric Mordell-Lang Conjecture some thirty-five years later (section 9 below).

Better known than all of these was *nonstandard analysis*, which appeared almost without warning in 1961 [167]. Robinson used compactness to form an elementary extension $^\star\mathbb{R}$ of the field $\mathbb{R}$ of real numbers (with any further structure attached) containing infinitesimal elements. He noted that if a theorem of real analysis can be written as a first-order sentence ϕ, then to prove ϕ it suffices to use the infinitesimals to show that ϕ is true in $^\star\mathbb{R}$ (a typical example of what Robinson called a *transfer argument*).

We turn to Fraïssé. In 1953 he published a short paper [60], with a fuller account a year later [61]. He limited himself to structures with just one relation symbol and no other nonlogical constants (though his arguments are valid for structures with finitely many relation symbols and individual constants). Taking the ordered set of rational numbers as a paradigm, he made two important observations. (a) We can characterise those classes of finite structures which are of the form: all finite structures embeddable in a given countable structure A. (Following Fraïssé I shall call these γ-*classes*—it is not a standard name.) (b) A γ-class has the amalgamation property if and only if A can be chosen to be homogeneous, and in this case A is determined up to isomorphism by the γ-class. (A class $\mathbf{K}$ has the *amalgamation property* if for all embeddings $e_1 : A \to B_1$ and $e_2 : A \to B_2$ within $\mathbf{K}$ there are embeddings $f_1 : B_1 \to C$ and $f_2 : B_2 \to C$, also within $\mathbf{K}$, such that $f_0 e_0 = f_1 e_1$. A is *homogeneous* if every isomorphism between finite substructures of A extends to an automorphism of A.)

Thus Fraïssé introduced the amalgamation property to model theory (though the name came later, from Jónsson).

Fraïssé's (a) introduced into model theory a kind of Galois theory of structures: it invited one to think of a structure as built up by a pattern of amalgamated extensions of smaller structures. This idea became important in stability theory.

Fraïssé's (b) provided a way of building countable structures by assembling a suitable γ-class of finite structures. His version of the idea was modest, but it was widely used as a source of ω-categorical structures. It was also enough to provide a framework for constructions by Ehud Hrushovski which solved key problems posed by Zilber ([92], [92]).

In 1956/7 Bjarni Jónsson, who had reviewed Fraïssé's [60], wrote two papers [101], [102] removing the limitation to finite and countable structures in Fraïssé's construction of homogeneous structures. The cost he had to pay was that the generalised continuum hypothesis was needed at some cardinals. Michael Morley realised almost at once (and later published with Vaught who had come to similar conclusions independently, [141] 1962) that, thanks to the Compactness Theorem, Jónsson's assumptions on the γ-class are verified if one considers the class of all 'small' subsets of models of a complete theory T and replaces embeddings by partial elementary maps. (In fact Morley and Vaught used a trick from Skolem [192], adding relation symbols so that partial elementary maps become embeddings.) It also emerged that the resulting homogeneous structures were exactly the saturated models of T.

In the 1970s there was some debate about how best to handle the Morley-Vaught γ-class. Gerald Sacks [176] proposed one should think of it as a category with partial elementary maps as morphisms. Shelah [185] (Chapter I §1) went straight to a very large saturated model C (but we never ask exactly how large); in his picture the γ-class is simply the class of all small subsets of the domain of C, and the partial elementary maps are the restrictions of automorphisms of C. Shelah's view prevailed. The structure C was known as the *big model* or (following John Baldwin) the *monster model*. When geometric model theory came on the scene, people noted that by going with Shelah the model-theoretic community had opted for the analogue of André Weil's 'universal domain' [221] Chapter IX §1, rather than the more recent category-theoretic language of Grothendieck.

The Morley-Vaught theory tells us that under suitable set-theoretic assumptions, every structure has a saturated elementary extension. These set-theoretic assumptions were always a stumbling block, and so weak forms of saturation were devised that served the same purposes without special assumptions. For example every structure has an elementary extension that is special [34]. Every countable structure has a recursively saturated elementary extension [15].

In 1964 Jan Mycielski [146] noticed that Kaplansky's notion of an algebraically compact abelian group (today more often called a pure-injective abelian group) has a purely model-theoretic characterisation that is a close analogue of saturation. With colleagues in Wrocław, Mycielski developed this observation into a theory of *atomic compact structures*, which was useful on the borderline between model theory and universal algebra.

Since atomic compact structures have a large amount of symmetry, they tend to have neat algebraic structural descriptions too; in fact this was the reason for Kaplansky's interest in them. To some extent the same holds for saturated structures, and even for κ-saturated structures when κ is large enough. For example in 1970 Paul Eklof and Edward Fisher [53] noted that every ω_1-saturated abelian group is algebraically compact, and so one can read off the results of Wanda Szmielew's quantifier elimination for abelian groups [199] rather easily from Kaplansky's structure theory. Likewise Ershov [54] used ω_1-saturated boolean algebras to recover Tarski's quantifier elimination results for boolean algebras. Clean methods of this kind quickly became standard practice.

Fraïssé's contribution was ignored in many accounts of the history. After some careful acknowledgments by Morley and Vaught [141], [61] disappears clean from the record, remembered only by a few people working on ω-categoricity. Maybe Fraïssé had less of a flair for publicity than Robinson. Another factor may have been that many people at the time regarded the whole line of work from Fraïssé to Morley and Vaught as trivial. Thus Saunders Mac Lane [128] reports that when Morley first brought him the material that led to [141],

> . . . I said, in effect: "Mike, applications of the compactness the-

orem are a dime a dozen. Go do something better."

Mac Lane adds that Morley's Theorem (see section 6 below) was the fruit of this advice.

Another factor that may have obscured the role of Fraïssé and of Jónsson was that model theorists formed the habit of hiding amalgamations. In 1956 Robinson [164] proved a fundamental amalgamation theorem of first-order logic. But he never stated it; you have to extract it from Robinson's proof of a less interesting theorem (the Joint Consistency Theorem). When stability theory returned to Fraïssé's view of a structure as built up from amalgamations of extensions of smaller structures, the convention was always to reduce to amalgamations of the form 'Amalgamate $Y \supset X$ and $X \cup \{b\}$ over X', in line with Keisler's 'one element at a time' approach. Amalgamations of this kind were called *extending the type of* b *over* X *to* Y. In 1983 Shelah [186] restored the amalgamation viewpoint with a vengeance: to construct structures of cardinality ω_n from countable pieces, he formed n-dimensional amalgams. But this work of Shelah had little influence in first-order model theory.

In any event, some of the best tools of 1960s model theory were a blend of the Robinson line and the Fraïssé line. For example Robinson's criterion for a theory T to have quantifier elimination was rewritten as: 'Given models B and C of T with a common nonempty substructure A, and an enough-saturated elementary extension D of B, there exists an elementary embedding of C into D making the diagram commute.' To taste, one could require D to be an ultrapower of A.

James Ax and Simon Kochen put the new machinery to work in 1965/6 [5], [6], [7] by finding a complete set of axioms for the field of p-adic numbers (uniformly for any prime p) and then showing that this theory has elimination of quantifiers. Their method was completely different from the method of quantifier elimination, and it seems likely that any proof by that method would have been hopelessly unwieldy. Instead they considered saturated valued fields of cardinality ω_1. One can say a good deal in algebraic terms about the structure of such fields; Ax and Kochen were able to show that under certain conditions, any two such fields are isomorphic. They then wrote down these conditions as a first-order theory T. Assuming the generalised continuum hypothesis, any two countable models A, B of T have saturated elementary extensions of cardinality ω_1, which are isomorphic, so that A and B must be elementarily equivalent. This proves the completeness of T (and hence its decidability since the axioms are effectively enumerable); a similar argument using saturated structures shows that T is model-complete, and one more push shows that Robinson's criterion for quantifier elimination is satisfied. There are various tricks that one can use to eliminate the generalised continuum hypothesis.

This work of Ax and Kochen, together with very similar but independent work of Yuri Ershov, [55], marked the beginning of a long line of research in the model theory of valued fields. But it hit the headlines because it gave a proof of an 'almost everywhere' version of a conjecture of Emil Artin on C_2 fields. Since counterexamples to the full conjecture appeared at about the same time, 'almost everywhere' was about as much as one could hope for, short of an explicit list of the exceptions.

Around 1970 category theory was developing fast. Now that model theory took maps between structures seriously, it was reasonable to try to develop a categorical model theory where the maps were the main levers. Michael Makkai and colleagues did some groundwork (e.g. [129]), but the subject never really took off. Perhaps model theorists enjoy handling elements and dislike morphisms between theories. Nevertheless two papers did show that useful ideas might come from category theory. One was by Daniel Lascar [116] 1982, who had visited Makkai and discussed with him the category of elementary embeddings between models of a complete theory; Lascar's enquiries threw up several useful ideas, including a notion of *Lascar strong type* that came to play a role in the study of simple theories. And Edmund Robinson [171] 1986 put categorical logic to good use in a model-theoretic study of the p-adic spectra of commutative rings.

6 Categoricity and classification theory

E. V. Huntington [97] called attention to the fact that some theories have only a single model up to isomorphism; he called such theories 'sufficient'. The second-order Peano axioms for number theory (Dedekind [39]) are a familiar example (though not Huntington's). Oswald Veblen ([219] 1904) proposed the name *categorical* for theories with this property. As Veblen noted, a categorical theory is necessarily complete in the sense that it entails, for every sentence ϕ of the appropriate language, either ϕ or $\neg\phi$. (Bolzano knew a version of this in 1837: [24] §110.)

No first-order theory with infinite models can be categorical, by the Upward Löwenheim-Skolem Theorem. But often in algebra we make do with less: for example if A and B are vector spaces over the same field k (and we can express this with first-order sentences true in A and B), then A and B are isomorphic whenever they have the same dimension. In particular if A and B have the same cardinality and it is greater than that of k, then they have the same dimension and so are isomorphic. So it is natural to say (as did Vaught in 1954 [214]) that a theory T is λ-*categorical* if it has, up to isomorphism, exactly one model of cardinality λ. Then—as Vaught noted, and we saw in section 5 that Robinson [168] had already used a version of the argument—a first-order theory which has no finite models and is λ-categorical for some λ must be complete.

In 1959 Lars Svenonius [198] showed that among countable structures, the models of ω-categorical theories are precisely those structures whose automorphism group has finitely many orbits of n-element sets, for each finite n. Permutation groups with this property are said to be *oligomorphic*. Svenonius' characterisation crossed the boundary between two different branches of mathematics, with consequences that we come back to in section 8. Other model theorists (notably Czesław Ryll-Nardzewski [175]) gave purely model-theoretic equivalents of ω-categoricity.

In 1955 Łoś [121] asked: If T is a complete theory in a countable first-order language, and T is λ-categorical for some uncountable λ, then is T λ-categorical for every uncountable λ? With hindsight we can see that this was an extraordinarily fortunate question to have asked in 1955, for two main reasons. The first was that at just this date the tools for starting to answer the question were becoming available. If T is λ-categorical and A, B are models of T of cardinality λ which are respectively highly saturated and Ehrenfeucht-Mostowski, then A and B are isomorphic and we deduce that models of T of cardinality λ have very few types to realise. This is strong information. Thus Łoś's question 'stimulated quite a bit of the work concerning models of arbitrary complete theories' (Vaught [216]).

Second, Łoś's question was unusual in that it called for a description of *all* the uncountable models of a theory. The answer would involve finding a *structure theorem* to explain how any model of the theory is put together. This pointed in a very different direction from Tarski's [210] 'mutual relations between sentences of formalized theories and mathematical systems in which these sentences hold'. One mark of the change of focus was that expressions like *uncountably categorical* (i.e. λ-categorical for all uncountable λ) and *totally categorical* (i.e. λ-categorical for all infinite λ), which originally applied to theories, came to be used chiefly for *models* of those theories. For example Walter Baur wrote in 1975 [16] of '$\aleph_0$-categorical modules'.

In 1965 Michael Morley answered Łoś's question in the affirmative [139]; this is *Morley's Theorem.* Amid all the literature of model theory, Morley's paper stands out for its clarity, its elegance and its richness in original ideas.

Morley's central innovation was *Morley rank*, which assigns an ordinal rank to each definable relation in any model of a theory T λ-categorical for some uncountable λ. (In Morley's presentation the rank was assigned to complete types, but later workers generally used the induced rank on formulas or definable relations.) In an algebraically closed field the Morley rank of an algebraic set is equal to its Krull dimension; Morley certainly had some such correlation in mind, as he signalled by giving the name *totally transcendental* to theories that assign a Morley rank to all definable relations in their models. Morley conjectured that the Morley rank of any uncountably categorical structure (i.e. the Morley rank of the formula $x = x$) is always finite; this was proved soon afterwards by Baldwin [8], and independently by Zilber.

Baldwin and Lachlan [9] in 1970 reworked and strengthened Morley's results. Building on the unpublished dissertation of William Marsh [137], they showed that each model of an uncountably categorical theory carries a definable *strongly minimal set* with an abstract dependence relation that defines a dimension for the model. Once the strongly minimal set is given, the rest of the model is assembled around it in a way that is unique up to isomorphism. They also showed that the number of countable models of such a theory, up to isomorphism, is either 1 or ω.

A few young researchers set to work to extend Morley's result to uncountable first-order languages. One of them was Frederick Rowbottom, who in 1964 [173] introduced the name 'λ-stable' for theories with few types over sets of λ elements; hence the name *stability theory* for this general area.

In 1969 [178] Saharon Shelah began to publish in stability theory. With his formidable theorem-proving skill he reshaped the subject almost from the start (and many other model theorists fled from the field rather than compete with him). By 1971 he had proved the uncountable analogue of Morley's Theorem [180]. But more important, he had formulated a plan of action.

Ehrenfeucht [49] had already noticed that a theory which defines an infinite linear ordering on n-tuples of elements must have a large number of non-isomorphic models of the same cardinality. Shelah saw this result as marking a division between 'good' theories that have few models of the same cardinality, and 'bad' theories that have many. Shelah's strategy was to hunt for possible bad features that a theory might have (like defining an infinite linear ordering), until the list was so comprehensive that a theory without any of these features is pinned down to the point where we can list all of its models in a structure theorem. As Shelah once explained it in conversation, the outcome should be to show that whenever **K** is the class of all models of a complete first-order theory, 'if **K** is good, it is very very good, but if **K** is bad it is horrid'. Shelah coined the word *nonstructure* for the horrid case, and he suggested several definitions of nonstructure [187]. In one definition, a nonstructure theorem finds a family of 2^λ models of cardinality λ, none of which is elementarily embeddable in any other. In another definition, a nonstructure theorem finds two nonisomorphic models of cardinality λ that are indistinguishable by strong infinitary languages.

Pursuing this planned dichotomy, Shelah wrote some dozens of papers and one large and famously difficult book ([181] 1978; the second edition in 1990 reports the successful completion of the programme for countable first-order theories in 1982). Shelah also wrote a number of papers on analogous dichotomies for infinitary theories or abstract classes of structures (e.g. [185]). His own name for this area of research was *classification theory.* The name applies at two levels: first-order theories classify structures, and Shelah's theory classifies first-order theories.

Shelah himself often said that his main interests lay on the nonstructure side ([188] p. 154):

> I was attracted to mathematics by its generality, its ability to give information where apparently total chaos prevails, rather than by its ability to give much concrete and exact information where we a priori know a great deal.

But even though he downplayed it himself, in his work on the structure side of the dichotomy he vastly expanded the range of the new tools introduced by Morley. Thus Shelah's *superstable*, *stable* and *simple* were successive weakenings of Morley's 'totally transcendental', and his *forking* was a powerful abstract notion of dependence for stable theories. (Shelah didn't say much about simple theories. But when some examples became important in work of Hrushovski in the early 1990s, Byung-Han Kim [107] showed that forking behaves well in them too.) Shelah's *regular types* are a generalisation of strongly minimal sets, in the sense that they carry an abstract dependence relation that gives them a dimension. In a superstable structure, the relations between the regular types determine for example whether we can expand one part of the structure while keeping another part fixed.

Shelah also showed that for models of a stable theory, any complete type is in a certain sense 'definable' by first-order formulas ([181] 1971, independently proved by Lachlan [111] 1972). He showed that the definition can always be taken over a *canonical base* which is a family of imaginary elements of the model (in the sense mentioned in (d) of section 3 above). A special case of his construction is André Weil's ([221] p. 68) field of definition of a variety, except that the field of definition consists of ordinary elements, not imaginary ones. Bruno Poizat explained this in 1985 ([159] §16e) by showing that algebraically closed fields have *elimination of imaginaries*, in the sense that their genuine elements can stand in for their imaginary ones.

A byproduct of the work of Morley and Shelah was a series of papers determining what structures in various natural classes were categorical, totally transcendental and so forth. The first nontrivial paper of this kind was by Joseph Rosenstein, [172] 1969. But certainly the most influential was a paper of Angus Macintyre in 1972 [124], where he showed that an infinite field is totally transcendental if and only if it is algebraically closed. In the aftermath of Macintyre's paper, Zilber proved (1977 [225]) that any totally transcendental skew field is an algebraically closed field, and Cherlin and Shelah showed (1979 [37]) that the same is true for superstable skew fields. In the course of this and related work, both Zilber and Cherlin independently noticed that a group definable in an uncountably categorical structure has many of the typical features of an algebraic group. Cherlin [35] in 1979 conjectured that every simple group of finite Morley rank is up to isomorphism an algebraic group over an algebraically closed field. This became known as *Cherlin's Conjecture*; Zilber [225] had conjectured the same thing for the special case of simple groups interpretable in uncountably categorical structures.

Stable groups turned out to have an unexpectedly large amount of structure, much of which carried over to modules (which are always stable). Various authors (among them Macintyre, Garavaglia, Baldwin, Saxl, Belegradek) noticed chain conditions that hold in some or all stable groups. In 1985 Poizat [159] created a rich theory of stable groups by generalising ideas from [225] and [37]. Poizat's framework allows one to rely on intuitions from algebraic geometry in handling stable groups; for example their behaviour is strongly influenced by their generic elements. At the same time Alexandr Borovik [27] started to bring recent group theory to bear on Cherlin's Conjecture. Stable groups attracted other workers and remained a lively topic to the end of the century, though Cherlin's Conjecture is still open.

7 Geometric model theory

Geometric model theory classifies structures in terms of their combinatorial geometries and the groups and fields that are interpretable in the structures. The roots of this theory go back to work of Lachlan, Cherlin and above all Zilber in stability theory in the 1970s, and for this reason the theory is also known as *geometric stability theory* (the title of Pillay's text [156]). But by the early 1990s it emerged that the same ideas sometimes worked well in structures that were by no means stable.

An abstract dependence relation gives rise to a combinatorial geometry—in what follows I say just 'geometry'. In this geometry certain sets of points are closed, i.e. they contain all points dependent on them. Zilber [227] classified geometries into three classes: (a) *trivial* or *degenerate*, where all sets of points are closed; (b) nontrivial locally modular, which are not trivial but if a finite number of points are fixed (i.e. made dependent on the empty set), then the resulting lattice is modular—for brevity this case is often called *modular*; (c) the remainder, known briefly as *non-modular*. Classical examples are: for (a), the dependence relation where an element is dependent only on sets containing it; for (b), linear dependence in a vector space; for (c), algebraic dependence in an algebraically closed field.

This classification made its way into model theory rather indirectly. Zilber was working on a proof that no complete totally categorical theory is finitely axiomatisable. (His first announcement of his proof of this result in 1980 [226] was flawed by a writing-up error which is repaired in [231].) In work on ω-categorical stable theories Lachlan [112] had introduced a combinatorial structure which he called a *pseudoplane*. A key step in Zilber's argument was to show that no totally categorical structure contains a definable pseudoplane. From this he deduced that the geometry of the strongly minimal set must be either trivial or modular, and his main result followed in turn from this. Cherlin, on reading [226] and seeing the error, went to the classification of finite simple groups and proved directly [36] that the strongly minimal set must be either trivial or modular. This result has a purely group-theoretic formulation. In fact several people discovered it independently, and it became known as the *Cherlin-Mills-Zilber theorem* in honour of three of them. Zilber's proof, once repaired, reaches the result without the classification of finite simple groups.

In the light of Zilber's work on uncountable categoricity and its extension by Cherlin, Harrington and Lachlan [36], model theorists looked to see what other structures might have modular geometries. One particularly influential result was proved independently by Hrushovski and Pillay, and published jointly [95]: a group G is modular (i.e. has only modular or trivial geometries) if and only if all definable subsets of G^n are boolean combinations of cosets of subgroups.

We saw that Zilber first applied his trichotomy of geometries by showing that in the structures he was considering, the non-modular case never occurred. Zilber now proposed to apply the same trichotomy to another question, namely his question (mentioned in the previous section) whether every simple group interpretable in an uncountably categorical structure must be an algebraic group over an algebraically closed field. Algebraically closed fields themselves have non-modular geometry; at the 1984 International Congress Zilber [229] conjectured the converse, viz. that any uncountably categorical structure with non-modular geometry must be—up to interpretability both ways—an algebraically closed field. This was known as *Zilber's Conjecture*.

A word about Zilber's motivation may be in order. Macintyre said in 1988 [127] that 'Purely logical classification[s] give only the most superficial general information' (and attributed the point to Kreisel). Zilber was convinced that the opposite must be true: if classical mathematics rightly recognises certain structures as 'good', then it should be possible to say in purely model-theoretic terms what makes these structures good. In fact Zilber in conversation quoted Macintyre's paper [124] as an example of how a purely model-theoretic condition (total transcendence) can be a criterion for an algebraic property (algebraic closure). Zilber was also convinced that being a model of an uncountably categorical countable first-order theory is an extremely strong property with rich mathematical consequences, among them strong homogeneity and the existence of a definable dimension.

In 1988 Hrushovski refuted Zilber's Conjecture [92]. But for both Hrushovski and Zilber this meant only that the right condition hadn't yet been found. Since it seemed to be particularly hard to recover the Zariski topology from purely model-theoretic data, a possible next step was to axiomatise the Zariski topology. This is not straightforward: it has to be done in all finite dimensions simultaneously, since the closed sets in dimension n don't determine those in dimension $n + 1$. But Hrushovski described a set of axioms, and Zilber and Hrushovski found that by putting together what they knew, they could prove [96] that Zilber's Conjecture holds for models of the axioms. In particular, if the geometries are non-modular then the model is isomorphic to the topology of a smooth curve over an algebraically closed field, and the field is interpretable in the topology.

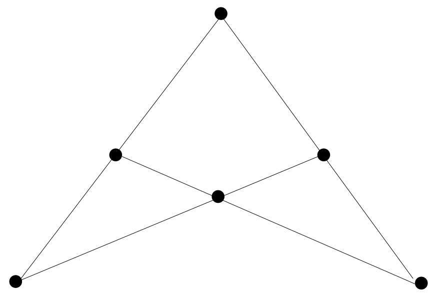
In 1996 Hrushovski [90] published a proof of the geometric Mordell-Lang Conjecture in all characteristics. Key ingredients of his argument were the results on the Zariski topology and on weakly normal groups, and earlier results on the stability of separably closed and differentiably closed fields. Hrushovski went on to apply a similar treatment to the Manin-Mumford Conjecture. This case was a little different: the structures in question were unstable. But Hrushovski showed that they inherited enough stability from a surrounding algebraically closed field; and in any case they were 'simple' in Shelah's classification. About the same time, Yakob Peterzil and Sergei Starchenko [155] found that Zilber's geometric trichotomy gave a strong classification of o-minimal fields. These fields are a very long way from being stable.

By the mid 1990s workers in geometric model theory had built up a large body of expertise. One theme that is worth exploring in this is the role of groups interpretable in a structure.

When Baldwin and Lachlan [9] in 1971 had shown that every uncountably categorical structure consists of a strongly minimal set D and other elements attached around it, they found they needed to say something about the way these other elements are attached. Because of categoricity, something in the theory has to prevent the set of attached elements being larger than D. The simplest guess would be that each attached element has to satisfy an algebraic formula (i.e. one satisfied by only finitely many elements) with parameters in D. Baldwin and Lachlan finished their paper with a complicated example to show that this need not hold. Later Baldwin realised that an easy example was already to hand: a direct sum G of countably many cyclic groups of order p^2 for a prime p. The socle (the set of elements of order at most p) is strongly minimal, in fact a vector space over the p-element field. An element a of order p^2 is described by saying what pa is; but if b is any element of the socle then some automorphism of G fixes the socle pointwise and takes a to $a + b$. In fact the orbit of a over the socle is parametrised by elements of the socle. This parametrisation keeps the orbit from having cardinality greater than that of the socle.

Zilber [231] realised that this was a common pattern in uncountably categorical structures. Each such structure is a finite tower; at the bottom is a strongly minimal set, and as we go up the tower, the orbit of an element over the preceding level in the tower is always parametrised by some group interpretable in that preceding level. He called these groups *binding groups.* There are some cohomological constraints, which allowed Ahlbrandt and Ziegler [2] to begin cataloguing the possibilities.

Later Zilber ([228] Lemma 3.3) called attention to the combinatorial configuration



which occurs in modular strongly minimal sets. (The blobs are points of the geometry. All points are pairwise independent. A line between three points means they form a dependent set.) He showed how to construct a group from the configuration; but since this was in the middle of an argument by reductio ad absurdum and quite strong assumptions were in force, it was less than the definitive result. Hrushovski [89] looked closer and showed, using Zilber's configuration, that every modular regular type has an infinite group interpretable in it (in a generalised sense).

Soon after this, Hrushovski [94] and Laskowski [118] used this result of Hrushovski to show that the power of groups to bind structures together could be used to settle unsolved problems of stability theory. This made it clear that henceforth the interpretability of groups in a structure would have to be considered one of the tools of model theory. A number of researchers went to work to discover and characterise the groups interpretable in various structures of model-theoretic interest.

8 Model theory within mathematics

Model-theoretic writers before 1950 rarely said what they thought about the relation between their work and the rest of mathematics. The natural assumption was that they were exploring properties of notions in the foundations of mathematics: axiom, independence, consistency, model, truth. Tarski ([211] p. 152) saw his 1933 truth definition as a contribution to one of 'the classical problems of philosophy'.

Listening to the contributions of Robinson and Tarski to the 1950 International Congress, we begin to hear a new tune. Both speakers set out to convince their audience that

> contemporary symbolic logic can produce useful tools—though by no means omnipotent ones—for the development of actual mathematics, more particularly for the development of algebra and, it would appear, algebraic geometry (Robinson [162] p. 694)

and that model theory has applications

> which may be of general interest to mathematicians and especially to algebraists; in some of these applications the notions of [model theory] itself are not involved at all' (Tarski [208] p. 717).

Over the next few decades, claims of this kind became commonplace, especially in applications for research grants.

There were some high points in the search for applications 'in which the notions of model theory itself are not involved'. Nonstandard analysis was one. Another was the work of Ax, Kochen and Ershov on Artin's Conjecture on C_2 fields (see section 5 above). In 1984 Jan Denef [41] used Macintyre's quantifier elimination for p-adic fields in order to prove a special case of a conjecture of Serre on Poincaré series; then with van den Dries, Denef proved the full conjecture [42] in 1988. Murthy and Swan in 1976 [145] gave an application of a criterion of Eklof [51] 1975 for a fact of geometry to be independent of the choice of universal domain. Shelah [182] applied stability theory to show that in characteristic 0 each differential field has a unique prime differential closure. And although it took some time to filter out of Russia, Mal'tsev's early work on local theorems in group theory was a significant example. In all these cases a device from model theory—usually involving compactness—was used to prove a result definitely belonging to some other area of mathematics.

There was a revealing episode in the late 1960s. James Ax [4] gave a brief and neat model-theoretic proof of the 'somewhat unexpected fact that an injective morphism of an algebraic variety into itself is surjective' (Armand Borel's description). Borel [27] and Shimura promptly found proofs not using model theory; then M. Raynaud pointed out (in a footnote to Borel's paper) that Grothendieck had essentially proved the result already in 1967 by sheaf methods. The outcome seemed to be that the model-theoretic proof was elegant but unnecessary, and Ax's result owed more to his skill as a mathematician than it did to his model-theoretic approach. It seems not to have inspired other geometers or number theorists to learn model theory.

There is a similar story to tell about Robinson's use of nonstandard analysis to solve the problem whether square roots of compact operators on Hilbert spaces have invariant subspaces ([19], 1966). In a telephone interview with Dauben ([38] p. 327), Chang rightly said 'Major credit must go to Robinson'; but he said 'to Robinson', not 'to nonstandard analysis'.

One-off successes of this kind were never a likely way to win converts to model theory from outside. Sometimes model theorists claimed to have better model-theoretic proofs of known results from other branches of mathematics; this was even less calculated to win friends and influence people.

In fact the idea of 'applying model theory to actual mathematics' was already growing rusty by 1970. By that date most model theorists regarded model theory itself as a part of actual mathematics. And in any case the natural ebb and flow of mathematical research throws up much subtler relationships than 'applying area X in area Y'. For example two areas may overlap; questions or methods of common interest form a weak overlap, and a stronger overlap is where the same researchers place themselves in both fields.

Certainly model theory had for a time a strong overlap of this kind with set theory. Tarski noted already in 1950 that ([208] p. 705) 'set-theoretical constructions and methods play an essential part in the development of the general theory of arithmetical classes'. During the 1960s there was a good deal of interplay between model theory and set theory. Set theory gave conditions for the existence of various kinds of model, and one of the first applications of Ronald Jensen's fine structure theory was to a two-cardinal question in model theory [99]. In return model theory gave set theory indiscernibles and ultrapowers. Gaifman, Rowbottom and Silver were three of the people who could be found on both sides of the divide.

Though few would have predicted it, this proved to be only a temporary alliance. In the early 1970s something of a revulsion against combinatorial set theory began to express itself among model theorists. Within first-order model theory the developments of sections 3 to 5 above seemed to have run their natural course, and the future lay in applications or in infinitary analogues. The model theory of infinitary logic seemed in danger of lapsing into axiomatic or descriptive set theory. (One could quote the work of Shelah, Eklof and Mekler [52] on almost free abelian groups as an example of the former, and Vaught [217] and Barwise [11] to illustrate the latter.) The one exception to prove the rule was Shelah, who brought in theorems of Fődor and Solovay as essential tools of nonstructure theory, and invented proper forcing for applications in model theory. Shelah's vigorous use of combinatorial set theory, beautiful mathematics though it was, unfortunately did little to encourage younger researchers to move into the area opened up by Morley, until Lascar and Poizat [117] published their elegant and set-theory-free introduction to stability theory in 1979.

The tide might yet turn between model theory and set theory. During the 1990s several authors pointed to links between descriptive set theory and current first-order model theory, not least among them Vaught's Conjecture [17].

The model theory of sets had a kind of younger brother in the model theory of arithmetic, which provided a door between model theory and proof theory. From Skolem to the end of the 20th century, many model theorists examined models of (first-order) Peano arithmetic and other fragments of first-order arithmetic. Often these were test cases for general theorems of model theory. But John Shepherdson, in an influential paper of a more distinctive kind [190] 1965, proposed using models of fragments of arithmetic as a way of studying the proof-theoretic strength of these fragments. This led to a line of papers in which the model theory and the proof theory of arithmetic were closely intertwined. Most famously, Jeff Paris and Leo Harrington [150] in 1977 used a model-theoretic construction to show that the consistency of Peano arithmetic follows from Peano arithmetic together with a result intermediate between the finite and infinite Ramsey theorems (thus providing the first 'natural' arithmetical theorem undecidable from the Peano axioms).

The frontier with universal algebra was also friendly. Model theory and universal algebra had a common interest in at least the classes axiomatised by universal Horn theories (for example Mal'tsev [133] 1973 or more recently Hart et al. [77] 1994), and they both talked about embeddings, homomorphisms and direct products in a very general setting. But there was not much traffic across this frontier. The equation "model theory = universal algebra + logic" from Chang and Keisler [33] never meant much in practice.

There was also a brief and small-scale liaison with graph theory. Tony Gardiner [66], when in 1976 he classified the finite homogeneous simple graphs, had no idea that he was classifying those finite simple graphs whose complete first-order theory has the quantifier elimination property. When model theorists noticed it, they saw scope for a range of similar classifications. One of the first was the classification of countable homogeneous graphs by Lachlan and Woodrow [113] in 1980 (a paper that set new standards in beauty for model-theoretic illustrations). But no longterm links were established; Gardiner's paper had already almost exhausted the matters of common interest.

Although set theory, universal algebra and graph theory are unambiguously parts of 'actual mathematics', none of them lies close to the central areas of algebra and algebraic geometry that Robinson and Tarski hoped to engage with. Before I turn to these areas, I note what happened to nonstandard analysis almost as soon as Robinson introduced it. The subject had (it seems) very little to give back to model theory, and almost at once it formed its own largely separate community.

Turning to algebra, one definite success—though on a small scale—was the study of countable models of ω-categorical theories. As we noted in section 6, this is simply the study of oligomorphic permutation groups. But the equivalence was not cashed in until Dugald Macpherson, a combinatorialist writing his DPhil under Peter Cameron, approached Lachlan in 1981. Zilber's work on ω-categorical groups (see section 6 above) became known soon after, and this brought the group theorist Peter Neumann and his DPhil student David Evans onto the scene. This was enough to form the seed of a research community in the area of overlap. The resulting harvest included

[29], [83], two works that are equally in model theory and in group theory. At the end of the century the pace of work had died down, but there was still unfinished business, for example on covers and on smoothly approximable structures.

Relations with field theory were harder to pin down, and I have to rely on personal impressions. After the work of Ax, Kochen and Ershov reported in section 5, not many field theorists persevered in the area. Michael Fried and Moshe Jarden were an exception, as witness their book [65] 1986. But on the model theory side, a number of people built up a great deal of expertise in one or other area of field theory and function theory. (Names that come to mind are Chatzidakis, Delon, van den Dries, Gardener, Haskell, Macintyre, Marker, Scanlon, Speissegger, Wilkie, among others.) This group were capable of taking on classical open problems and solving them using model-theoretic tools; a typical example was the solution by van den Dries, Macintyre and Marker of an old problem of Hardy, [47]. Members of this group were highly respected by 'classical' mathematicians who knew little about model theory. But the communities never really merged. One noticed that at conferences where members of this group met, there were rarely more than one or two classical geometers or function theorists—and not for lack of invitations.

In 1996 [91] Hrushovski published the first proof of the geometric Mordell-Lang Conjecture in all characteristics. The language of his proof was model-theoretic, [ASK McQUILLAN FOR A COMMENT?] At the end of the century it was still unclear whether Hrushovski's work had opened up an area that model theorists and algebraic geometers would be able to cultivate together. There were some signs that members of the two communities wanted a closer contact. A number of joint papers by model theorists and geometers appeared. Hrushovski himself acted as an interpreter between the communities, for example in [94].

9 Other languages, other structures

In 1885 C. S. Peirce [153], fresh from inventing quantifiers, mentioned that the universal and the existential quantifier are not the only examples. He gave the example of the quantifier 'For two-thirds of all x'. Unfortunately nobody picked up Peirce's idea, until in 1957 Mostowski [144] called attention to the quantifiers 'For at least $\aleph_\alpha$ x'. Mostowski's paper was timely, because it was useful to have in the 1960s a variety of extensions of first-order logic for testing out new constructions.

In 1969 [120] Per Lindström published another timely paper, in which he gave model-theoretic necessary and sufficient conditions for a logic to have the same expressive power as first-order logic. His result suggested that it might be possible to fit the various logics studied during the previous decade into some higher organisation of logics, within a *generalised* (or *abstract*) *model theory*. Alas, the facts weren't there to support such a theory. The 1970s saw some valiant efforts in this direction, and by the mid 1980s a large amount was known about many different logics extending first-order logic (see Barwise and Feferman [13]). But the most quotable outcome of any generality was that very few logics apart from first-order logic satisfy the Craig interpolation lemma.

The mathematical logicians within computer science shrugged their shoulders and asked what is the interest of a logic in which it's impossible to express everyday notions like connectedness, even on finite structures. Thus for example Gurevich [75] 1984:

> The question arises how good is first-order logic in handling finite structures. It was not designed to deal exclusively with finite structures. ... One would like to enrich first-order logic so that the enriched logic fits better the case of finite structures.

One solution was first-order logic with a fixed-point operator added, as in Chandra and Harel [31] 1982 and Blass, Gurevich and Kozen [23] 1985. The model theory of this logic and its relatives were studied mostly by computer scientists, but this seems to be purely an accident of history; these languages would have been good to have available in the 1960s. For other various other reasons, logicians studied logics with only a finite number of variables. (Barwise [12] 1977 was the first of several people who independently described these logics and their associated pebble games.) Other logicians studied logics of a modal kind, with or without fixed-point operators. Ehrenfeucht-Fraïssé back-and-forth games adapt smoothly to these contexts, and there are computer science applications. During the 1990s the group of Jouko Väänänen in Helsinki took a particular interest in game-theoretic and combinatorial aspects of non-classical logics both large and small (e.g. [98] 1990, [80] 1996).

One of the early variants of first-order logic was still showing signs of life in the 1990s. In 1961 Henkin [82] described an extension of first-order logic where the dependencies between the quantifiers are not controlled by the order of the quantifiers, for example as in

> For all x there is y such that for all z there is w depending only on z such that $R(x, y, z, w)$.

Sometimes, as in this example, the dependencies between the quantifiers could be written as a branching diagram; in which case one spoke of *branching quantifiers*. Henkin proposed a semantics in terms of Skolem functions, and Hintikka [87] later revised it into the form of games of imperfect information. There was some debate whether these logics had useful applications in computer science (e.g. Blass and Gurevich [22] 1986) or in linguistics. But in themselves they had a problematic model theory, because the semantics determines only when a sentence is true in a structure, and not whether an assignment of elements satisfies a formula in a structure. Thus one could ask about classes defined by axioms in such a logic, but not about relations definable by a formula in a structure. Work to clarify this continued to the end of the century.

References

[1] Peter Abelard, *Dialectica*, ed. L. M. de Rijk, Van Gorcum 1970.

[2] Gisela Ahlbrandt and Martin Ziegler, What's so special about $(\mathbb{Z}/4\mathbb{Z})^{(\omega)}$?, *Archive for Mathematical Logic* 31 (1991) 115–132.

[3] Kenneth J. Arrow, A difficulty in the concept of social welfare, *Journal of Political Economy* 58 (1950) 328–346.

[4] James Ax, The elementary theory of finite fields, *Annals of Mathematics* 88 (1968) 239–271.

[5] James Ax and Simon Kochen, Diophantine problems over local fields: I, *American Journal of Mathematics* 87 (1965) 605–630.

[6] James Ax and Simon Kochen, Diophantine problems over local fields: II, A complete set of axioms for p-adic number theory, *American Journal of Mathematics* 87 (1965) 631–648.

[7] James Ax and Simon Kochen, Diophantine problems over local fields: III, Decidable fields, *Annals of Mathematics* 83 (1966) 437–456.

[8] John T. Baldwin, α_T is finite for $\aleph_1$-categorical T, *Transactions of American Mathematical Society* 181 (1973) 37–51.

[9] John T. Baldwin and A. H. Lachlan, On strongly minimal sets, *Journal of Symbolic Logic* 36 (1971) 79–96.

[10] Alexandru Baltag, Interpolation and preservation for pebble logics, *Journal of Symbolic Logic* 64 (1999) 846–858.

[11] Jon Barwise, *Admissible sets and structures*, Springer, Berlin 1975.

[12] Jon Barwise, On Moschovakis closure ordinals, *Journal of Symbolic Logic* 42 (1977) 292–296.

[13] Jon Barwise and Solomon Feferman eds., *Model-Theoretic Logics*, Springer, New York 1985.

[14] Jon Barwise and Abraham Robinson, Completing theories by forcing, *Annals of Mathematical Logic* 2 (1970) 119–142.

[15] Jon Barwise and John Schlipf, An introduction to recursively saturated and resplendent models, *Journal of Symbolic Logic* 41 (1976) 531–536.

[16] Walter Baur, $\aleph_0$-categorical modules, *Journal of Symbolic Logic* 40 (1975) 213–220.

[17] Howard Becker and Alexander S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, Cambridge University Press, Cambridge 1996.

[18] Heinrich Behmann, Beiträge zur Algebra der Logik,insbesondere zum Entscheidungsproblem, *Mathematische Annalen* 86 (1922) 163–229.

[19] Allen R. Bernstein and Abraham Robinson, Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos, *Pacific Journal of Mathematics* 16 (1966) 421–431.

[20] Evert W. Beth, On Padoa's method in the theory of definition, *Indagationes Mathematicae* 15 (1953) 330–339.

[21] Garrett Birkhoff, On the structure of abstract algebras, *Proceedings of Cambridge Philosophical Society* 31 (1935) 433–454.

[22] Andreas Blass and Yuri Gurevich, Henkin quantifiers and complete theories, *Annals of Pure and Applied Logic* 32 (1986) 1–16.

[23] Andreas Blass, Yuri Gurevich and Dexter Kozen, A zero-one law for logic with a fixed-point operator, *Information and Control* 67 (1985) 70–90.

[24] Bernard Bolzano, *Wissenschaftslehre*, Sulzbach 1837.

[25] George Boole, *The Mathematical Analysis of Logic*, MacMillan, Barclay and MacMillan, Cambridge 1847.

[26] George Boole, *An Investigation of the Laws of Thought*, Dover, New York 1958 (original 1854).

[27] Alexandr Borovik, The theory of finite groups and uncountably categorical groups (Russian), *Preprint of Academy of Sciences of the USSR, Siberian Division* 1984.

[28] Nicolas Bourbaki, *Éléments de mathématique*, Livre II : Algèbre. Ch. 1: Structures algébrique, Hermann, Paris 1958.

[29] Peter Cameron, *Oligomorphic permutation groups*, Cambridge University Press, Cambridge 1990.

[30] Georg Cantor, Beiträge zur Begründung der transfiniten Mengenlehre, I, *Mathematische Annalen* 46 (1895) 481–512.

[31] Ashok Chandra and David Harel, Structure and complexity of relational queries, *Journal of Computation and Systems Science(?)* 25 (1982) 99–128.

[32] Chen Chung Chang, Some remarks on the model theory of infinitary languages, in *The syntax and semantics of infinitary languages*, ed. Jon Barwise, Lecture Notes in Mathematics 72, Springer, Berlin 1968, pp. 36–63.

[33] Chen Chung Chang and H. Jerome Keisler, *Model Theory*, North-Holland, Amsterdam 1973.

[34] Chen Chung Chang and H. Jerome Keisler, *Continuous Model Theory*, Princeton University Press, Princeton NJ 1966.

[35] Gregory L. Cherlin, Groups of small Morley rank, *Annals of Mathematical Logic* 17 (1979) 1–28.

[36] Gregory L. Cherlin, Leo Harrington and Alistair H. Lachlan, $\aleph_0$-categorical, $\aleph_0$-stable structures, *Annals of Pure and Applied Logic* 28 (1985) 103–135.

[37] Gregory L. Cherlin and Saharon Shelah, Superstable fields and groups, *Annals of Mathematical Logic* 18 (1980) 227–270.

[38] Joseph Warren Dauben, *Abraham Robinson*, Princeton University Press, Princeton NJ 1995.

[39] Richard Dedekind, *Stetigkeit und Irrationale Zahlen*, Vieweg, Brunswick 1872.

[40] Augustus De Morgan, On the syllogism: I. On the structure of the syllogism, *Transactions of Cambridge Philosophical Society* 8 (1846) 379–408.

[41] Jan Denef, The rationality of the Poincaré series associated to the p-adic points on a variety, *Inventiones Mathematicae* 77 (1984) 1–23.

[42] Jan Denef and Lou van den Dries, p-adic and real subanalytic sets, *Annals of Mathematics* 128 (1988) 79–138.

[43] John E. Doner, Andrzej Mostowski and Alfred Tarski, The elementary theory of well-ordering—a metamathematical study, in *Logic Colloquium '77*, ed. Angus Macintyre, Leszek Pacholski and Jeff Paris, North-Holland, Amsterdam 1978, pp. 1–54.

[44] P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, herausgegeben und mit Zusätzen versehen von R. Dedekind, Vieweg, Brunswick 1871.

[45] Lou van den Dries, Remarks on Tarski's problem concerning ($\mathbb{R}$, $+$, $\cdot$, exp), in *Logic Colloquium '82*, ed. G. Lolli et al., North-Holland, Amsterdam 1984, pp. 97–121.

[46] Lou van den Dries, *Tame Topology and O-minimal Structures*, Cambridge University Press, Cambridge 1998.

[47] Lou van den Dries, Angus Macintyre and David Marker, Logarithmic-exponential power series, *Journal of London Mathematical Society* 56 (1997) 417–434.

[48] Andrzej Ehrenfeucht, On theories categorical in power, *Fundamenta Mathematicae* 44 (1957) 241–248.

[49] Andrzej Ehrenfeucht, An application of games to the completeness problem for formalized theories, *Fundamenta Mathematicae* 49 (1961) 128–141.

[50] Andrzej Ehrenfeucht and Andrzej Mostowski, Models of axiomatic theories admitting automorphisms, *Fundamenta Mathematicae* 43 (1956) 50–68.

[51] Paul C. Eklof, Categories of local functors, in *Model theory and algebra: A memorial tribute to Abraham Robinson*, ed. D.H. Saracino and V.B. Weispfenning, Lecture Notes in Math. 498, Springer, Berlin 1975, pp. 91–116.

[52] Paul C. Eklof, Infinitary model theory of abelian groups, *Israel Journal of Mathematics* 25 (1976) 97–107.

[53] Paul C. Eklof and Edward R. Fisher, The elementary theory of abelian groups, *Annals of Mathematical Logic* 4 (1972) 115–171.

[54] Yuri L. Ershov, Decidability of the elementary theory of distributive lattices with relative complements and the theory of filters (Russian), *Algebra i Logika* 3 (3) (1964) 17–38.

[55] Yuri L. Ershov, On the elementary theories of local fields (Russian), *Algebra i Logika* 4 (2) (1965) 5–30.

[56] Yuri L. Ershov, Theories of nonabelian varieties of groups, in *Proceedings of Tarski Symposium*, American Mathematical Society, Providence RI 1974, pp. 255–264.

[57] Solomon Feferman, 'Kurt Gödel: Conviction and Caution', in *In the Light of Logic*, Oxford University Press, New York 1998.

[58] Solomon Feferman and Robert L. Vaught, The first order properties of products of algebraic systems, *Fundamenta Mathematicae* 47 (1959) 57–103.

[59] Adolf Fraenkel (= Abraham Fraenkel), *Einleitung in die Mengenlehre*, Springer, Berlin 1928.

[60] Roland Fraïssé, Sur certaines relations qui généralisent l'ordre des nombres rationnels, *Comptes Rendus de l'Académie des Sciences de Paris* 237 (1953) 540–542.

[61] Roland Fraïssé, Sur l'extension aux relations de quelques propriétés des ordres, *Annales Scientifiques de l'École Normale Supérieure* 71 (1954) 363–388.

[62] Roland Fraïssé, Application des γ-opérateurs au calcul logique du premier échelon, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 2 (1956) 76–92.

[63] T. E. Frayne, A. C. Morel and D. S. Scott, Reduced direct products, *Fundamenta Mathematicae* 51 (1962) 195–228.

[64] Gottlob Frege, Über die Grundlagen der Geometrie, *Jahresbericht der Deutschen Mathematiker-Vereinigung* 15 (1906) 293–309; 377–403; 423–430; tr. in *On the foundations of geometry, and Formal theories of arithmetic*, tr. E.W. Kluge, Yale University Press, New Haven 1971.

[65] Michael Fried and Moshe Jarden, *Field arithmetic*, Springer, Berlin 1986.

[66] Anthony Gardiner, Homogeneous graphs, *Journal of Combinatorial Theory B* 20 (1976) 94–102.

[67] Dedre Gentner and Albert L. Stevens, *Mental Models*, Lawrence Erlbaum, Hillsdale NJ 1983.

[68] Kurt Gödel, Die Vollständigkeit der Axiome des logischen Funktionenkalküls, *Monatshefte für Mathematische Physik* 37 (1930) 349–360.

[69] Kurt Gödel, Eine Eigenschaft der Realisierungen des Aussagenkalküls, *Ergebnisse eines Mathematischen Kolloquiums* 3 (1932) 20–21.

[70] Kurt Gödel, *Collected Works* Vol. I, ed. Solomon Feferman et al., Oxford University Press, New York 1986.

[71] Sergei S. Goncharov and Yuri L. Ershov, *Konstruktivnye Modeli*, Siberian School of Algebra and Logic, with Plenum Publishing Corporation, New York 1999.

[72] Hermann Günther Grassmann, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik dargestellt und durch Anwendungen auf die übrigen Zweige der Mathematik, wie auch auf Statik, Mechanik, die Lehre fon Magnetismus und die Krystallonomie erläutert*, Stettin 1844.

[73] Ivor Grattan-Guinness, In memoriam Kurt Gödel: his 1931 correspondence with Zermelo on his incompletability theorem, *Historia Mathematica* 6 (1979) 294–304.

[74] Thomas J. Grilliot, Omitting types: application to recursion theory, *Journal of Symbolic Logic* 37 (1972) 81–89.

[75] Yuri S. Gurevich, Toward logic tailored for computational complexity, in *Computation and proof theory* (Aachen, 1983), Lecture Notes in Mathematics 1104, Springer, Berlin 1984, pp. 175–216.

[76] William P. Hanf, Models of languages with infinitely long expressions, in *Abstracts of contributed papers from the First Logic, Methodology and Philosophy of Science Congress*, Vol. 1, Stanford University 1960, p. 24.

[77] Bradd Hart, Sergei Starchenko and Matt Valeriote, Vaught's conjecture for varieties, *Transactions of American Mathematical Society* 342 (1994) 173–196.

[78] Gisbert Hasenjaeger, Eine Bemerkung zu Henkin's Beweis für die Vollständigkeit des Prädikatenkalküls der ersten Stufe, *Journal of Symbolic Logic* 18 (1953) 42–48.

[79] Felix Hausdorff, Grundzüge einer Theorie der geordneten Mengen, *Mathematische Annalen* 65 (1908) 435–505.

[80] Lauri Hella, Kerkko Luosto and Jouko Väänänen, The hierarchy theorem for generalized quantifiers, *Journal of Symbolic Logic* 61 (1996), 802–819.

[81] Leon Henkin, The completeness of the first-order functional calculus, *Journal of Symbolic Logic* 14 (1949) 159–166.

[82] Leon Henkin, Some remarks on infinitely long formulas, in *Infinitistic methods: Proc. symposium on foundations of math.*, Pergamon Press and Państwowe Wydawnictwo Naukowe 1961, pp. 167–183.

[83] Bernhard Herwig and Daniel Lascar, Extending partial automorphisms and the profinite topology on free groups, *Transactions of American Mathematical Society* 352 (2000) 1985–2021.

[84] E. Hewitt, Rings of real-valued continuous functions, *Transactions of American Mathematical Society* 64 (1948) 45–99.

[85] David Hilbert, *Foundations of Geometry* (translated and revised; original from 1899), Open Court, La Salle 1971.

[86] David Hilbert, Letter to Gottlob Frege, c. 1900, in Gottlob Frege, *Kleine Schriften* ed. Ignacio Angelelli, Georg Olms, Hildesheim 1967, pp. 410–413.

[87] Jaakko Hintikka, *The Principles of Mathematics Revisited*, Cambridge University Press, Cambridge 1996.

[88] Wilfrid Hodges, *Model Theory*, Cambridge University Press, Cambridge 1993.

[89] Ehud Hrushovski, Locally modular regular types, in *Classification Theory*, ed. J. T. Baldwin, Lecture Notes in Mathematics 1292, Springer, Berlin 1987, pp. 132–164.

[90] Ehud Hrushovski, Unidimensional theories are superstable, *Annals of Pure and Applied Logic* 50 (1990) 117-138.

[91] Ehud Hrushovski, Strongly minimal expansions of algebraically closed fields, *Israel Journal of Mathematics* 79 (1992) 129–151.

[92] Ehud Hrushovski, A new strongly minimal set, *Annals of Pure and Applied Logic* 62 (1993) 147–166.

[93] Ehud Hrushovski, The Mordell-Lang conjecture for function fields, *Journal of American Mathematical Society* 9 (1996) 667–690.

[94] Ehud Hrushovski, Geometric model theory, *Documenta Mathematica* Extra volume ICM 1998, I, pp. 281–302.

[95] Ehud Hrushovski and Anand Pillay, Weakly normal groups, in *Logic Colloquium '85*, ed. Paris Logic Group, Elzevier, New York 1987, pp. 233–244.

[96] Ehud Hrushovski and Boris Zilber, Zariski geometries, *Journal of the American Mathematical Society* 9 (1996) 1–56.

[97] Edward V. Huntington, A complete set of postulates for the theory of absolute continuous magnitude, *Transactions of American Mathematical Society* 3 (1902) 264–279.

[98] Tapani Hyttinen and Jouko Väänänen, On Scott and Karp trees of uncountable models, *Journal of Symbolic Logic* 55 (1990) 897–908.

[99] Ronald Björn Jensen, The fine structure of the constructible universe, *Annals of Mathematical Logic* 4 (1972) 229–308; erratum *Annals of Mathematical Logic* 4 (1972) 443.

[100] P. N. Johnson-Laird, *Mental Models*, Cambridge University Press, Cambridge 1983.

[101] Bjarni Jónsson, Universal relational systems, *Mathematica Scandinavica* 4 (1956) 193–208.

[102] Bjarni Jónsson, Homogeneous universal relational systems, *Mathematica Scandinavica* 8 (1960) 137–142.

[103] Carol R. Karp, Finite-quantifier equivalence, in *The theory of models*, Proceedings of 1963 International Symposium at Berkeley, ed. J. W. Addison et al., North-Holland, Amsterdam 1965, pp. 407-412.

[104] H. Jerome Keisler, Ultraproducts and elementary classes, *Indagationes Mathematicae* 23 (1961) 477–495.

[105] H. Jerome Keisler, Some applications of infinitely long formulas, *Journal of Symbolic Logic* 30 (1965) 339–349.

[106] H. Jerome Keisler, Ultraproducts which are not saturated, *Journal of Symbolic Logic* 32 (1967) 23–46.

[107] Byunghan Kim, Forking in simple unstable theories, *Journal of London Mathematical Society* 57 (1998) 257–267.

[108] Stephen Cole Kleene, *Introduction to Metamathematics*, North-Holland, Amsterdam 1952.

[109] Julia Knight, Anand Pillay and Charles Steinhorn, Definable sets in ordered structures II, *Transactions of American Mathematical Society* 295 (1986) 593–605.

[110] Simon Kochen, Ultraproducts in the theory of models, *Annals of Mathematics* 74 (1961) 221–261.

[111] Alistair H. Lachlan, A property of stable theories, *Fundamenta Mathematicae* 77 (1972) 9–20.

[112] Alistair H. Lachlan, Two conjectures regarding the stability of ω-categorical theories, *Fundamenta Mathematicae* 81 (1974) 133–145.

[113] Alistair H. Lachlan and Robert E. Woodrow, Countable homogeneous undirected graphs, *Transactions of American Mathematical Society* 262 (1980) 51–94.

[114] C. H. Langford, Some theorems on deducibility, *Annals of Mathematics* 28 (1927) 16–40.

[115] C. H. Langford, Theorems on deducibility (Second paper), *Annals of Mathematics* 28 (1927) 459–471.

[116] Daniel Lascar, On the category of models of a complete theory, *Journal of Symbolic Logic* 47 (1982) 249–266.

[117] Daniel Lascar and Bruno Poizat, An introduction to forking, *Journal of Symbolic Logic* 44 (1979) 330–350.

[118] Michael Chris Laskowski, Uncountable theories that are categorical in a higher power, *Journal of Symbolic Logic* 53 (1988) 512–530.

[119] Per Lindström, On model-completeness, *Theoria* 30 (1964) 183–196.

[120] Per Lindström, On extensions of elementary logic, *Theoria* 35 (1969) 1–11.

[121] Jerzy Łoś, On the categoricity in power of elementary deductive systems and some related problems, *Colloquium Mathematicum* 3 (1954) 58–62.

[122] Jerzy Łoś, Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres, in *Mathematical Interpretation of Formal Systems*, ed. Th. Skolem et al., North-Holland, Amsterdam 1955, pp. 98–13.

[123] Leopold Löwenheim, Über Möglichkeiten im Relativkalkül, *Mathematische Annalen* 68 (1915) 447–470.

[124] Angus Macintyre, On ω_1-categorical theories of fields, *Fundamenta Mathematicae* 71 (1971) 1–25.

[125] Angus Macintyre, On algebraically closed groups, *Annals of Mathematics* 96 (1972) 53–97.

[126] Angus Macintyre, On definable subsets of p-adic fields, *Journal of Symbolic Logic* 41 (1976) 605–610.

[127] Angus Macintyre, Trends in logic, in *Logic Colloquium '88*, ed. R. Ferro et al., North-Holland, Amsterdam 1989, pp. 365f.

[128] Saunders Mac Lane, The education of Ph.D.s in mathematics, in *A Century of Mathematics in America, Part III* ed. Peter Duren, American Mathematical Society, Providence RI 1989, pp. 517–523.

[129] Michael Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, Contemporary Mathematics 104, American Mathematical Society 1989.

[130] Anatolii I. Mal'tsev, Untersuchungen aus dem Gebiete der mathematischen Logik, *Matematicheskie Sbornik* 1 (43) (1936) 323–336; tr. in [134], pp. 1–14.

[131] Anatolii I. Mal'tsev, On the faithful representation of infinite groups by matrices (Russian), *Matematicheskie Sbornik* 8 (50) (1940) 405–422; tr. as A.M.S. Translations (2) 45 (1965) 1–18.

[132] Anatolii I. Mal'tsev, A general method for obtaining local theorems in group theory (Russian), Ucenie Zapiski Ivanov. Ped. Inst. (Fiz-mat. Fakul'tet) 1 (1941) no. 1, 3–9; tr. in [134], pp. 15–21.

[133] Anatolii I. Mal'tsev, *Algebraicheskie sistemy*, Nauka, Moscow 1970; tr. as Mal'cev, *Algebraic Systems*, Springer, Berlin 1973.

[134] Anatolii I. Mal'tsev, *The metamathematics of algebraic systems, Collected papers: 1936–1967*, tr. Benjamin Franklin Wells III, North-Holland, Amsterdam 1971.

[135] Paolo Mancosu, Between Russell and Hilbert: Behmann on the foundations of mathematics, *Bulletin of Symbolic Logic* 5 (1999) 303–300.

[136] E. Marczewski, Sur les congruences et les propriétés positives d'algèbres abstraites, *Colloquium Mathematicum* 2 (1951) 220–228.

[137] William E. Marsh, On ω_1-categorical and not ω-categorical theories, Dissertation, Dartmouth College 1966.

[138] Richard Montague, *Formal Philosophy*, ed. with introduction by Richmond H. Thomason, Yale University Press, New Haven and London 1974.

[139] Michael Morley, Categoricity in power, *Transactions of American Mathematical Society* 114 (1965) 514–538.

[140] Michael Morley, Omitting classes of elements, in *The theory of models*, Proceedings of 1963 International Symposium at Berkeley, ed. J. W. Addison, Leon Henkin and Alfred Tarski, North-Holland, Amsterdam 1965, pp. 265–273.

[141] Michael Morley and Robert Vaught, Homogeneous universal models, *Mathematica Scandinavica* 11 (1962) 37–57.

[142] Andrzej Mostowski, *Logika Matematyczna*, Monografie Matematyczne 18, Warsaw 1948.

[143] Andrzej Mostowski, On models of axiomatic systems, *Fundamenta Mathematicae* 39 (1952) 133–158.

[144] Andrzej Mostowski, On a generalization of quantifiers, *Fundamenta Mathematicae* 44 (1957) 12–36.

[145] M. Pavaman Murthy and Richard G. Swan, Vector bundles over affine surfaces, *Inventiones Mathematicae* 36 (1976) 125–165.

[146] Jan Mycielski, Some compactifications of general algebras, *Colloquium Mathematicum* 13 (1964) 1–9.

[147] John von Neumann, Eine Axiomatisierung der Mengenlehre, *Journal für Reine und Angewandte Mathematik* 154 (1925) 219–240; correctin *ibid.* 155 (1925) 128; tr. in *From Frege to Gödel*, ed. Jean van Heijenoort, Harvard University Press, Cambridge 1967, pp. 393–413.

[148] *Omega Bibliography of Mathematical Logic* Vol. III: Model Theory, ed. Gert H. Müller et al., Springer, Heidelberg 1987.

[149] Stephen Orey, On ω-consistency and related properties, *Journal of Symbolic Logic* 21 (1956) 246–252.

[150] Jeff Paris and Leo Harrington, A mathematical incompleteness in Peano arithmetic, in *Handbook of Mathematical Logic*, ed. Jon Barwise, North-Holland, Amsterdam 1977, pp. 1133–1142.

[151] George Peacock, *Report on the Recent Progress and Present State of certain Branches of Analysis*, Report of the Third Meeting of the British Association for the Advancement of Science, John Murray, London 1834.

[152] Giuseppe Peano, Sul concetto di numero, *Rivista di Matematica* 1 (1891) 87–102.

[153] Charles S. Peirce, On the algebra of logic, *American Journal of Mathematics* 7 (1885) 180–202.

[154] Charles S. Peirce, On the algebra of logic (Second paper), unpublished manuscript (1884), in *Writings of Charles S. Peirce, Volume 5, 1884–1886*, ed. C. W. J. Kloesel et al., Indiana University Press, Bloomington Indiana 1993, pp. 111-115.

[155] Yakob Peterzil and Sergei Starchenko, Geometry, calculus, and Zilber's conjecture, *Bulletin of Symbolic Logic* 2 (1996) 72–83.

[156] Anand Pillay, *Geometric Stability Theory*, Clarendon Press, Oxford 1996.

[157] Anand Pillay and Charles Steinhorn, Definable sets in ordered structures I, *Transactions of American Mathematical Society* 295 (1986) 565–592.

[158] Bruno Poizat, Une théorie de Galois imaginaire, *Journal of Symbolic Logic* 48 (1983) 1151–1170.

[159] Bruno Poizat, *Cours de Théorie des Modèles*, Nur Al-Mantiq Wal-Ma'rifah, Paris 1985.

[160] M. Presburger, Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt, in *Sprawozdanie z I Kongresu Mat. Krajów Słowiańskich*, Warszawa 1930, pp. 92–101.

[161] Michael O. Rabin, Non-standard models and independence of the induction axiom, in *Essays on the foundations of mathematics*, ed. Y. Bar-Hillel et al., North-Holland, Amsterdam 1962, pp. 287–299.

[162] Abraham Robinson, On the application of symbolic logic to algebra, in *Proceedings of the International Congress of Mathematicians*, Cambridge Mass. 1950, vol. 1, pp. 686–694; reprinted in Abraham Robinson [169] pp. 3–11.

[163] Abraham Robinson, *On the Metamathematics of Algebra*, North-Holland, Amsterdam 1951.

[164] Abraham Robinson, A result on consistency and its application to the theory of definition, *Indagationes Mathematicae* 18 (1956) 47–58; reprinted in [169] pp. 87–98.

[165] Abraham Robinson, *Complete Theories*, North-Holland, Amsterdam 1956.

[166] Abraham Robinson, Completeness and persistence in the theory of models, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 2 (1956) 15–26; reprinted in [169] pp. 108–119.

[167] Abraham Robinson, Non-standard analysis, *Indagationes Mathematicae* 23 (1961) 432–440.

[168] Abraham Robinson, *Model Theory*, North-Holland, Amsterdam 1963.

[169] Abraham Robinson, *Selected Papers of Abraham Robinson*, ed. H. J. Keisler et al., Vol. 1, Model theory and algebra, North-Holland, Amsterdam 1979.

[170] Abraham Robinson and Peter Roquette, On the finiteness theorem of Siegel and Mahler concerning diophantine equations, *Journal of Number Theory* 7 (1975) 121-176.

[171] Edmund P. Robinson, The p-adic spectrum, *Journal of Pure and Applied Algebra* 40 (1986) 281–296.

[172] Joseph G. Rosenstein, $\aleph_0$-categoricity of linear orderings, *Fundamenta Mathematicae* 64 (1969) 1–5.

[173] Frederick Rowbottom, The Łoś conjecture for uncountable theories, *Notices of American Mathematical Society* 11 (1964) 248.

[174] Frederick Rowbottom, Some strong axioms of infinity incompatible with the axiom of constructibility, *Annals of Mathematical Logic* 3 (1971) 1–44.

[175] Czesław Ryll-Nardzewski, On the categoricity in power $\aleph_0$, *Bulletin de l'Académie Polonaise des Sciences Série Scientifique et Mathématique* (CHECK) 7 (1959) 545–548.

[176] Gerald E. Sacks, *Saturated Model Theory*, W. A. Benjamin, Reading Mass 1972.

[177] Dana S. Scott, Measurable cardinals and constructive sets, *Bulletin de l'Académie Polonaise des Sciences Série Scientifique, Mathématique, Astronomique, Physique* CHECK 9 (1961) 521–524.

[178] Saharon Shelah, Stable theories, *Israel Journal of Mathematics* 7 (1969) 187–202.

[179] Saharon Shelah, The number of non-isomorphic models of an unstable first-order theory, *Israel Journal of Mathematics* 9 (1971) 473–487.

[180] Saharon Shelah, Every two elementarily equivalent models have isomorphic ultrapowers, *Israel Journal of Mathematics* 10 (1971) 224–233.

[181] Saharon Shelah, Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory, *Annals of Mathematical Logic* 3 (1971) 271–362.

[182] Saharon Shelah, Uniqueness and characterization of prime models over sets for totally transcendental first-order theories, *Journal of Symbolic Logic* 37 (1972) 107–113.

[183] Saharon Shelah, Categoricity of uncountable theories, in *Proceedings of Tarski Symposium,* ed. L. Henkin et al., American Mathematical Society, Providence RI 1974, pp. 187–203.

[184] Saharon Shelah, Infinite abelian groups, Whitehead problem and some constructions, *Israel Journal of Mathematics* 18 (1974) 243–256.

[185] Saharon Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, North-Holland, Amsterdam 1978.

[186] Saharon Shelah, Classification theory for nonelementary classes I, The number of uncountable models of $\psi \in L_{\omega_1\omega}$, *Israel Journal of Mathematics* 46 (1983) 212–240 and 241–273.

[187] Saharon Shelah, Classification of first order theories which have a structure theorem, *Bulletin of American Mathematical Society* 12 (1985) 227–232.

[188] Saharon Shelah, Taxonomy of universal and other classes, in *Proceedings of International Congress of Mathematicians, 1986*, ed. Andrew M. Gleason, American Mathematical Society pp. 154–162.

[189] Saharon Shelah, Classification theory for non-elementary classes II, Abstract Elementary Classes, in *Classification Theory*, ed. J. T. Baldwin, Lecture Notes in Mathematics 1292, Springer, Berlin 1987, pp. 419–497.

[190] John C. Shepherdson, Non-standard models for fragments of number theory, in *The theory of models*, Proceedings of 1963 International Symposium at Berkeley, ed. J. W. Addison et al., North-Holland, Amsterdam 1965, pp. 342–358.

[191] Thoralf Skolem, Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen, *Videnskapsselskapets Skrifter*, I. Matem.-naturv. klasse no. 3 (1919).

[192] Thoralf Skolem, Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen, *Videnskapsselskapets Skrifter*, I. Matem.-naturv. klasse I no. 4 (1920) 1–36.

[193] Thoralf Skolem, Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre, in *Proceedings of 5th Scandinavian Mathematical Congress*, Helsinki 1922, pp. 217–232.

[194] Thoralf Skolem, Über einige Satzfunktionen in der Arithmetik, *Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo* I, no. 7 (1931) 1–28; reprinted in [197] pp. 281–306.

[195] Thoralf Skolem, Über die Nichtcharakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen, *Fundamenta Mathematicae* 23 (1934) 150–161.

[196] Thoralf Skolem, A critical remark on foundational research, *Kongelige Norske Videnskabsselskabs Forhandlinger, Trondheim* 28, no. 20 (1955) 100–105; reprinted in [197] pp. 581–586.

[197] Thoralf Skolem, *Selected Works in Logic*, ed. Jens Erik Fenstad, Universitetsforlaget, Oslo 1970.

[198] Lars Svenonius, $\aleph_0$-categoricity in first-order predicate calculus, *Theoria* 25 (1959) 82–94.

[199] Wanda Szmielew, Elementary properties of Abelian groups, *Fundamenta Mathematicae* 41 (1955) 203–271.

[200] A. D. Taĭmanov, Characteristics of axiomatisable classes of models (Russian), *Algebra i Logika* 1 (4) (1962) 5–31.

[201] Alfred Tarski, Sur les ensembles définissables de nombres réels I, *Fundamenta Mathematicae* 17 (1931) 210–239.

[202] Alfred Tarski, revised version translated as The Concept of Truth in Formalized Languages, [211] pp. 152–278.

[203] Alfred Tarski, Grundzüge des Systemenkalkúls, *Fundamenta Mathematicae* 25 (1935) 503–526 and 26 (1936) 283–301; tr. as Foundations of the calculus of systems, in [211] pp. 342–383.

[204] Alfred Tarski, O pojęciu wynikania logicznego, *Przegląd Filozoficzny* 39 (1936) 58–68; tr. as On the concept of logical consequence, in [211] pp. 409–420.

[205] Alfred Tarski, *Introduction to Logic and to the Methodology of Deductive Sciences*, Oxford University Press, New York 1994 (Polish original 1936).

[206] Alfred Tarski, *The Completeness of Elementary Algebra and Geometry*, Institut Blaise Pascal, Paris 1967 (a reprint of a work scheduled to appear in 1940 but prevented by the war).

[207] Alfred Tarski, Arithmetical classes and types of algebraically closed and real-closed fields, *Bulletin of American Mathematical Society* 55 (1949) 64.

[208] Alfred Tarski, Some notions and methods on the borderline of algebra and metamathematics, in *Proceedings of International Congress of Mathematicians*, Cambridge Mass. 1950, vol. 1, pp. 705–720.

[209] Alfred Tarski, *A decision method for elementary algebra and geometry* (prepared for publication with the assistance of J.C.C. McKinsey), 2nd edition, University of California Press, Berkeley 1951.

[210] Alfred Tarski, Contributions to the theory of models I, *Indagationes Mathematicae* 16 (1954) 572–581.

[211] Alfred Tarski, *Logic, Semantics, Metamathematics*, trans. J. H. Woodger, second edition ed. and introduced by John Corcoran, Hackett, Indianapolis 1983.

[212] Alfred Tarski, A. Mostowski and R. M. Robinson, *Undecidable Theories*, North-Holland, Amsterdam 1953.

[213] Alfred Tarski and Robert L. Vaught, Arithmetical extensions of relational systems, *Compositio Mathematica* 13 (1957) 81–102.

[214] Robert L. Vaught, Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability, *Indagationes Mathematicae* 16 (1954) 467–472.

[215] Robert L. Vaught, Denumerable models of complete theories, in *Infinitistic methods*, Proc. Symp. Foundations of Math., Warsaw 1959, Państwowe Wydawnictwo Naukowe, Warsaw 1961, pp. 303–321.

[216] Robert L. Vaught, Models of complete theories, *Bulletin of American Mathematical Society* 69 (1963) 299–313.

[217] Robert L. Vaught, Descriptive set theory in $L_{\omega_1\omega}$, in *Cambridge Summer School in Mathematical Logic*, ed. A. R. D. Mathias and H. Rogers, Lecture Notes in Mathematics 337, Springer, Berlin 1973, pp. 574–598.

[218] Robert L. Vaught, Alfred Tarski's work in model theory, *Journal of Symbolic Logic* 51 (1986) 869–882.

[219] Oswald Veblen, A system of axioms for geometry, *Transactions of American Mathematical Society* 5 (1904) 343–384.

[220] Hao Wang, A survey of Skolem's work in logic, in [197], pp. 17–52.

[221] André Weil, *Foundations of Algebraic Geometry*, American Mathematical Society Colloquium Publications vol. 29, American Mathematical Society, New York 1946.

[222] Hermann Weyl, Über die Definitionen der mathematischen Grundbegriffe, *Mathematisch-naturwissenschaftliche Blätter* 7 (1910) 93–95 and 109–113.

[223] Alex J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, *Journal of American Mathematical Society* 9 (1996) 1051–1094.

[224] Martin Ziegler, Algebraisch abgeschlossene Gruppen, in *Word Problems II, The Oxford Book*, ed. S.I. Adian et al., North-Holland, Amsterdam 1980, pp. 449–576. 321–329.

[225] B. I. Zil'ber, Groups and rings whose theory is categorical (Russian), *Fundamenta Mathematicae* 9 (1977) 173–188; tr. in *American Mathematical Society Translations* 149 (1991) 1–16.

[226] Boris I. Zil'ber, Strongly minimal countably categorical theories (Russian), *Sibirskii Matematicheskii Zhurnal* 21 (1980) 98–112.

[227] Boris I. Zil'ber, Totally transcendental structures and combinatorial geometries (Russian), *Doklady Akademii Nauk SSSR* 259 (1981) 5, pp. 1039–1041.

[228] Boris I. Zil'ber, Strongly minimal countably categorical theories (Russian), *Sibirskii Matematicheskii Zhurnal* Part II, 25 No. 3 (1984) 71–88; Part III, 25 No. 4 (1984) 63–77.

[229] Boris I. Zil'ber, The structure of models of uncountably categorical theories, in *Proceedings of the International Congress of Mathematicians, Warszawa 1983*, vol. I, PWN, Warsaw and North-Holland, Amsterdam 1984, pp. 359–368.

[230] Boris I. Zil'ber, Finite homogeneous geometries, in *Proceedings of Sixth Easter Conference in Model Theory*, ed. Bernd Dahn et al., Humboldt-Univ. 1988, pp. 186–208.

[231] Boris I. Zil'ber, *Uncountably categorical theories*, trans. D. Louvish, Translations of Mathematical Monographs 117, American Mathematical Society, Providence RI 1993.

Several model theory texts give more detailed historical information about particular theorems; for example Chang and Keisler [33], Hodges [88], Pillay [156]. There are surveys on Skolem by Hao Wang [220] and on Tarski by Vaught [218]. Sadly we still lack an authoritative survey of Abraham Robinson's contributions to model theory.