Notes Concerning the Gentzen's Calculus

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1 Introduction

You can consider this document as my *personal* notes related to Gentzen's calculus presented in [1, §1.4]. We have revisited the original text several times every time getting a little bit further.

We have tried to use this calculus to create proof-trees of various sequents I have considered interesting. These proofs were created ad-hoc, in the creation time without respect to the completeness theorem. So I struggled and applied the rules creatively.

After reading the Theorem 1.4.1—completeness of the Gentzen's calculus I have realized the uniqueness of this calculus. The proof of the completeness of this calculus actually gives us an algorithm how to create a proof-tree for any tautological sequent. Thus, dumb computer can be programmed to generate proofs without a single bit of creativity. That is fantastic because looking at different calculi (Z calculus, Hilbert's calculus) does not suggest existence of such a unique calculus as this with such a good properties. Though, the initial reason why we started to deal with Gentzen's calculus was the fact that its axiom and derivation rules were (from human's irrational point of view) more natural. That is they could be really used in practice—as I did in our initial experiments. These rules are usable and more natural than Hilbert's axioms. How would be able to deal with an axiom such as $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$? I must admit, I haven't tuned my mind to do that[1]. Somehow, Gentzen's calculus is more compatible with my thinking. If I looked at the particular rules for sufficiently long time $\smile$ then I was able to adopt them as the rules of my rational thinking in the area of propositional logic. This naturalness goes hand in hand with proof of soundness of this calculus. I have proved soundness of the particular derivation rules and it gave me more courage to use those rules in the exercises.

Here you can find completion of the proof of the Theorem 1.4.1—completeness of the Gentzen's calculus. The book contains partial proof of this theorem. It skips over the "boring details". I have tried to complete it on my own. The result is indeed somewhat "verbose".

I wasn't able to resist the temptation not to implement the proof-tree-generator for any given tautological sequent. This little procedure is indeed beautiful. Some of its results are observable at the end of this document.

2 Soundness

Nodes in the proof-tree are not particular formulæ but sequents of formulæ. Sequent is denoted as $\langle \Gamma \Rightarrow \Delta \rangle$. Both, Γ as well as Δ, are sets of formulæ. Γ is called the *antecedent* and the Δ is called

[1] But it seems that there are others which are very well capable of doing exactly this. See for example [2]. Look at [3]. Wow. Left for future lives—if only there were any $\frown$.

the *succedent* of a given sequent. These sets might contain (even zero) number of formulæ. Some example sequents follow:

$$
\begin{aligned}
\langle & \;\Rightarrow\; \rangle & (1)\\
\langle \varphi \,\&\, \neg\varphi & \;\Rightarrow\; \rangle & (2)\\
\langle & \;\Rightarrow\; \varphi \vee \neg\varphi\rangle & (3)\\
\langle \varphi & \;\Rightarrow\; \varphi\rangle & (4)\\
\langle \neg\varphi \vee \psi & \;\Rightarrow\; \varphi \rightarrow \psi\rangle & (5)\\
\langle \varphi \rightarrow \psi, \psi \rightarrow \varphi & \;\Rightarrow\; \varphi \equiv \psi\rangle & (6)
\end{aligned}
$$

Some of the above sequents are tautological and some of them are not. The semantics of the sequent $\langle\Gamma \Rightarrow \Delta\rangle$ is the follows. We consider a sequent to be *tautological* if and only if the following is true: *If all the formulæ in the antecedent are true, then at least one formula in the succedent is true.* Therefore, if we know that some sequent is tautological, then if all the formulæ in its antecedent are true then we are sure that some formula from its succedent must be true[2].

Gentzen's Calculus has a single axiom:

$$
\text{A:} \qquad / \;\; \langle\Gamma, \varphi \Rightarrow \Delta, \varphi\rangle
$$

and a whole bunch of derivation rules[3]:

$$
\begin{array}{rrcl}
\text{W:} & \langle\Gamma \Rightarrow \Delta\rangle & / & \langle\Gamma \Rightarrow \Delta, \varphi\rangle\\
\text{W:} & \langle\Gamma \Rightarrow \Delta\rangle & / & \langle\Gamma, \varphi \Rightarrow \Delta\rangle\\
\vee\text{−r:} & \langle\Gamma \Rightarrow \Delta, \varphi\rangle & / & \langle\Gamma \Rightarrow \Delta, \varphi \vee \psi\rangle\\
\vee\text{−r:} & \langle\Gamma \Rightarrow \Delta, \varphi\rangle & / & \langle\Gamma \Rightarrow \Delta, \psi \vee \varphi\rangle\\
\&\text{−l:} & \langle\Gamma, \varphi \Rightarrow \Delta\rangle & / & \langle\Gamma, \varphi \,\&\, \psi \Rightarrow \Delta\rangle\\
\&\text{−l:} & \langle\Gamma, \varphi \Rightarrow \Delta\rangle & / & \langle\Gamma, \psi \,\&\, \varphi \Rightarrow \Delta\rangle\\
\&\text{−r:} & \langle\Gamma \Rightarrow \Delta, \varphi\rangle, \langle\Gamma \Rightarrow \Delta, \psi\rangle & / & \langle\Gamma \Rightarrow \Delta, \varphi \,\&\, \psi\rangle\\
\neg\text{−l:} & \langle\Gamma \Rightarrow \Delta, \varphi\rangle & / & \langle\Gamma, \neg\varphi \Rightarrow \Delta\rangle\\
\neg\text{−r:} & \langle\Gamma, \varphi \Rightarrow \Delta\rangle & / & \langle\Gamma \Rightarrow \Delta, \neg\varphi\rangle\\
\rightarrow\text{−r:} & \langle\Gamma, \varphi \Rightarrow \Delta, \psi\rangle & / & \langle\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi\rangle\\
\rightarrow\text{−l:} & \langle\Gamma \Rightarrow \Delta, \varphi\rangle, \langle\Pi, \psi \Rightarrow \Lambda\rangle & / & \langle\Gamma, \Pi, \varphi \rightarrow \psi \Rightarrow \Delta, \Lambda\rangle\\
\equiv\text{−l:} & \langle\Gamma, \varphi \rightarrow \psi, \psi \rightarrow \varphi \Rightarrow \Delta\rangle & / & \langle\Gamma, \varphi \equiv \psi \Rightarrow \Delta\rangle\\
\equiv\text{−r:} & \langle\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi\rangle, \langle\Gamma \Rightarrow \Delta, \psi \rightarrow \varphi\rangle & / & \langle\Gamma \Rightarrow \Delta, \varphi \equiv \psi\rangle\\
\text{Cut:} & \langle\Gamma \Rightarrow \Delta, \varphi\rangle, \langle\Pi, \varphi \Rightarrow \Lambda\rangle & / & \langle\Gamma, \Pi \Rightarrow \Delta, \Lambda\rangle
\end{array}
$$

In order to adopt some calculus, we have to be persuaded that its rules are sound. We will go through the particular rules and show that from a tautological sequent one cannot infer non-tautological sequent. Soundness of the axiom and all the rules can be checked directly with respect to semantics of the sequent notion.

[2]Something similar to the notion of sequent is (though not obviously) hidden in the Z-calculus [4]. Sequents in Z-calculus are special in the sense that their succedent is always a single formula. Antecedents in Z-calculus are those oddities embraced by $\lceil$ and $\rceil$ in the leaves of the proof-tree. Here in Gentzen's calculus we deal with sequents explicitly.

[3]I have added the rules $\equiv$−l and $\equiv$−r in addition to the ones in the original text [1].

A: $/ \quad \langle \Gamma, \varphi \Rightarrow \Delta, \varphi \rangle$

It is necessary to show that if all the formulæ in antecedent are true, then at least one formula in the succedent is true. But this is obvious. If all the formulæ Γ, φ are true, then certainly at least one formula from the succedent Δ, φ must be true. Formula φ will be true for sure.

W: $\langle \Gamma \Rightarrow \Delta \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \varphi \rangle$

If

- the first sequent is tautological
- and everything in Γ (antecedent of the second sequent) is true

then when we see that something from Δ (succedent of the first sequent) must be true.

But then also something from Δ, φ must be true. Therefore if the first sequent is tautological, the second one must be tautological too.

W: $\langle \Gamma \Rightarrow \Delta \rangle \quad / \quad \langle \Gamma, \varphi \Rightarrow \Delta \rangle$

If

- the first sequent is tautological
- and everything in Γ, φ (antecedent of the second sequent) is true

then we see that the antecedent of the first sequent is fulfilled. Something from its succedent Δ must therefore be true. But this implies that also something of the second sequent's succedent (which is also Δ) must be true. Thus, whenever the first sequent is tautological, the second one must be tautological too.

$\vee$−r**:** $\langle \Gamma \Rightarrow \Delta, \varphi \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \varphi \vee \psi \rangle$

If

- the first sequent is tautological
- and everything in Γ (antecedent of the second sequent) is true

then we see that something from Δ, φ must be true. But by definition of the semantics of the disjunction connective we can boldly claim that also surely something from $\Delta, \varphi \vee \psi$ (the succedent of the second sequent) must be true. Thus, whenever the first sequent is tautological, the second one must be tautological too.

$\vee$–r: $\langle \Gamma \Rightarrow \Delta, \varphi \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \psi \vee \varphi \rangle$

If

- the first sequent is tautological
- and everything in Γ (antecedent of the second sequent) is true

then we see that something from Δ, φ must be true. But by definition of the semantics of the disjunction connective we can boldly claim that also surely something from $\Delta, \psi \vee \varphi$ (the succedent of the second sequent) must be true. Thus, whenever the first sequent is tautological, the second one must be tautological too.

&–l: $\langle \Gamma, \varphi \Rightarrow \Delta \rangle \quad / \quad \langle \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta \rangle$

If

- the first sequent is tautological
- and everything in $\Gamma, \varphi \,\&\, \psi$ (the antecedent of the second sequent) is true

then we see that also everything in Γ, φ (antecedent of the first sequent) must be true and therefore something from Δ (the succedent of the first sequent) must be true. But Δ is also succedent of the second sequent. Thus, whenever the first sequent is tautological, the second one must be tautological too.

&–l: $\langle \Gamma, \varphi \Rightarrow \Delta \rangle \quad / \quad \langle \Gamma, \psi \,\&\, \varphi \Rightarrow \Delta \rangle$

If

- the first sequent is tautological
- and everything in $\Gamma, \psi \,\&\, \varphi$ (the antecedent of the second sequent) is true

then we see that also everything in Γ, φ (antecedent of the first sequent) must be true and therefore something from Δ (the succedent of the first sequent) must be true. But Δ is also succedent of the second sequent. Thus, whenever the first sequent is tautological, the second one must be tautological too.

&–r: $\langle \Gamma \Rightarrow \Delta, \varphi \rangle \;,\; \langle \Gamma \Rightarrow \Delta, \psi \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \varphi \,\&\, \psi \rangle$

If

- the first and the sequent are tautological
- and everything in Γ (the antecedent of the third sequent) is true.

Then we see that:

- something from Δ, φ must be true
- and something from Δ, ψ must be true.

If Δ contains true formula then also $\Delta, \varphi \,\&\, \psi$ (succedent of the third sequent) contains true formula. If Δ contains no true formula then from the above conclusion we are sure that both φ and ψ must be true. Therefore in any case $\Delta, \varphi \,\&\, \psi$ contains at least one true formula. Thus, whenever the first two sequents are tautological, the third one is tautological also.

$\neg$−l: $\quad \langle \Gamma \Rightarrow \Delta, \varphi \rangle \quad / \quad \langle \Gamma, \neg\varphi \Rightarrow \Delta \rangle$

If

- the first sequent is tautological
- and everything in $\Gamma, \neg\varphi$ (the antecedent of the second sequent) is true

then we see that also everything in Γ (the antecedent of the first sequent) must be true. This means that something from Δ, φ (the succedent of the second sequent) must be true. But we know that it cannot be φ because $\neg\varphi$ is true. So we know that if some formula from Δ, φ is to be true, it is in Δ. But this is exactly the succedent of the second sequent so whenever the first sequent is tautological, the second one shall be tautological too.

$\neg$−r: $\quad \langle \Gamma, \varphi \Rightarrow \Delta \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \neg\varphi \rangle$

If

- the first sequent is tautological
- and everything in Γ is true

then there are two possibilities. We do not know if φ is true or false. But that does not matter because in either case some formula in the succedent $\Delta, \neg\varphi$ of the second sequent must be true. If φ is true then we know that something in Δ (succedent of the first sequent) is true. This also implies that something in $\Delta, \neg\varphi$ must be true.

If φ is false then $\neg\varphi$ is true and then again something in $\Delta, \neg\varphi$ (succedent of the second sequent) must be true. Thus whenever the first sequent is tautological, the second sequent must be tautological too.

$\rightarrow$−r: $\quad \langle \Gamma, \varphi \Rightarrow \Delta, \psi \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi \rangle$

If

- the first sequent is true

- and everything in Γ is true

then there are two possibilities. Formula φ is either true or false. If it is false, then the succedent of the second sequent contains true formula, the $\varphi \rightarrow \psi$ must be true.

If φ is it is true then there are again two options. Either there is some true formula in Δ, or there is no true formula in Δ. In the first case (there is some true formula in Δ) the succedent of the second sequent is true. In the second case (there is no true formula in Δ) we know that ψ must be true because the first sequent is tautological and all formulæ in its antecedent are true. But if both φ and ψ are true, then $\varphi \rightarrow \psi$ is also true and therefore succedent of the second sequent contains true formula also in this case.

So, whenever the first sequent is tautological, the second sequent must be tautological too.

$\rightarrow$**–l:** $\quad \langle \Gamma \Rightarrow \Delta, \varphi \rangle \;\; , \;\; \langle \Pi, \psi \Rightarrow \Lambda \rangle \quad / \quad \langle \Gamma, \Pi, \varphi \rightarrow \psi \Rightarrow \Delta, \Lambda \rangle$

If

- The first two sequents are true
- The everything in the antecedent $\Gamma, \Pi, \varphi \rightarrow \psi$ of the third sequent is true

then if we take into consideration that $\varphi \rightarrow \psi$ is true there are again two possibilities. Formula φ is either false or ψ is true. In the first case (if φ is false) we know (according to the first sequent whose antecedent is fulfilled) that something from Δ must be true. So in this case we are sure that something from the succedent of the third sequent will be true. In the second case (if ψ is true) we known (according to the second sequent whose antecedent is fulfilled) that something from Λ must be true. So also in this case we are sure that something from the succedent of the third sequent will be true.

$\equiv$**–l:** $\quad \langle \Gamma, \varphi \rightarrow \psi, \psi \rightarrow \varphi \Rightarrow \Delta \rangle \quad / \quad \langle \Gamma, \varphi \equiv \psi \Rightarrow \Delta \rangle$

If

- The first sequent is tautological
- and everything in the antecedent of the second sequent is true

then also everything in the antecedent of the first sequent must be true and thus something in Δ (succedent of the first sequent) must be true. But this is also succedent of the second sequent. Thus, whenever the first sequent is tautological, the second one is also tautological.

$\equiv$**–r:** $\quad \langle \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi \rangle \;\; , \;\; \langle \Gamma \Rightarrow \Delta, \psi \rightarrow \varphi \rangle \quad / \quad \langle \Gamma \Rightarrow \Delta, \varphi \equiv \psi \rangle$

If

- The first two sequents are tautological

- and everything in Γ (antecedent of the third sequent) is true

then we see that all the formulæ in the antecedents of the first two sequents are fulfilled. This means that something from their respective succedents must be true. There are two possibilities. Either something from Δ is true or both $\varphi \to \psi$ and $\psi \to \varphi$ are true. Either way, something from the succedent of the third sequent will be true. So whenever the first two sequents are tautological, the third one must be tautological too.

Cut: $\langle \Gamma \Rightarrow \Delta, \varphi \rangle, \langle \Pi, \varphi \Rightarrow \Lambda \rangle \;\; / \;\; \langle \Gamma, \Pi \Rightarrow \Delta, \Lambda \rangle$

If

- the first two sequents are tautological
- and everything in Γ, Π is true

then there are two possibilities. Formula φ is either true or false. If φ is true then from the second sequent follows that something in Λ must be true. If φ is false then from the first sequent follows that something from Δ must be true. In any case (regardless of the value of φ) something in Δ, Λ (succedent of the third sequent) will be true. Thus, whenever the first two sequents are true, the third sequent must be also true.

3 Initial Experiments

Gentzen's Calculus can be viewed as a language for expressing proofs of sequents[4]. This language has strict rules which define what sentences belong to this language and which ones don't. Some of these sentences are also in this document—all of those weird fraction-like oddities are simply sentences of this language with strict rules however weird they might look.

So, here are some of my attempts to create proof-trees of various sequents I decided to prove.

The proof of the $\langle \varphi \,\&\, \psi \Rightarrow \psi \,\&\, \varphi \rangle$. Let's call it $\&-$comm.

$$\dfrac{\dfrac{\langle \psi \Rightarrow \psi \rangle}{\langle \varphi \,\&\, \psi \Rightarrow \psi \rangle}\ \&-\mathrm{l} \qquad \dfrac{\langle \varphi \Rightarrow \varphi \rangle}{\langle \varphi \,\&\, \psi \Rightarrow \psi \rangle}\ \&-\mathrm{l}}{\langle \varphi \,\&\, \psi \Rightarrow \psi \,\&\, \varphi \rangle}\ \&-\mathrm{r}$$

The proof of the $\langle \varphi \vee \psi \Rightarrow \psi \vee \varphi \rangle$. Let's call it $\vee-$comm.

$$\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi \rangle}{\langle \varphi \Rightarrow \psi \vee \varphi \rangle}\ \vee-\mathrm{r} \qquad \dfrac{\langle \psi \Rightarrow \psi \rangle}{\langle \psi \Rightarrow \psi \vee \varphi \rangle}\ \vee-\mathrm{r}}{\langle \varphi \vee \psi \Rightarrow \psi \vee \varphi \rangle}\ \vee-\mathrm{l}$$

[4]Here I deal only with sequents which contain solely propositions. Although there exist also extension toward predicate calculus.

Below I show how it is possible to infer $\langle \Gamma, \psi \,\&\, \varphi \Rightarrow \Delta \rangle$ from $\langle \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta \rangle$. This is not a proof[5]. But could be part of some proof.

$$\dfrac{\dfrac{\vdots}{\langle \psi \,\&\, \varphi \Rightarrow \varphi \,\&\, \psi \rangle}\,\&\!-\mathrm{comm} \qquad \langle \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta \rangle}{\langle \Gamma, \psi \,\&\, \varphi \Rightarrow \Delta \rangle}\,\mathrm{Cut}$$

You can see that this proof uses the already proved sequent— $\&$*–comm*. It would be now "boring" to finish it because it is clear what shall be put there. Such a habit is advantageous because it is possible to name interesting tautological sequents and *reuse* them elsewhere, but care must be taken. It can be viewed as an "acronym" of sub-proof-tree. It is possible to expand all such acronyms, the resulting proof-tree should obey the rules of the Gentzen's calculus and should be finite. Only then one can consider the root sequent to be inferred in Gentzen's calculus and therefore tautological.

Similarly, it is possible to infer $\langle \Gamma \Rightarrow \psi \,\&\, \varphi, \Delta \rangle$ from $\langle \Gamma \Rightarrow \varphi \,\&\, \psi, \Delta \rangle$

$$\dfrac{\langle \Gamma \Rightarrow \varphi \,\&\, \psi, \Delta \rangle \qquad \dfrac{\vdots}{\langle \varphi \,\&\, \psi \Rightarrow \psi \,\&\, \varphi \rangle}\,\&\!-\mathrm{comm}}{\langle \Gamma \Rightarrow \psi \,\&\, \varphi, \Delta \rangle}\,\mathrm{Cut}$$

A proof of the sequent $\langle \varphi \Rightarrow \neg\neg\varphi \rangle$:

$$\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi \rangle}{\langle \varphi, \neg\varphi \Rightarrow \rangle}\,\neg\!-\!\mathrm{l}}{\langle \varphi \Rightarrow \neg\neg\varphi \rangle}\,\neg\!-\!\mathrm{r}$$

A proof of the sequent $\langle \neg\neg\varphi \Rightarrow \varphi \rangle$:

$$\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi \rangle}{\langle \Rightarrow \varphi, \neg\varphi \rangle}\,\neg\!-\!\mathrm{r}}{\langle \neg\neg\varphi \Rightarrow \varphi \rangle}\,\neg\!-\!\mathrm{l}$$

Here we show how it is possible to infer $\langle \Gamma \Rightarrow \neg\neg\varphi, \Delta \rangle$ from $\langle \Gamma \Rightarrow \varphi, \Delta \rangle$. In the proof we use already proved $\langle \varphi \Rightarrow \neg\neg\varphi \rangle$. Its proof is not repeated below:

$$\dfrac{\langle \Gamma \Rightarrow \varphi, \Delta \rangle \qquad \dfrac{\vdots}{\langle \varphi \Rightarrow \neg\neg\varphi \rangle}}{\langle \Gamma \Rightarrow \neg\neg\varphi, \Delta \rangle}\,\mathrm{Cut}$$

Here we show how it is possible to infer $\langle \Gamma \Rightarrow \varphi, \Delta \rangle$ from $\langle \Gamma \Rightarrow \neg\neg\varphi, \Delta \rangle$. In the proof we used already proved $\langle \neg\neg\varphi \Rightarrow \varphi \rangle$. Its proof is not repeated below:

[5] The sequent $\langle \Gamma, \varphi \,\&\, \psi \Rightarrow \Delta \rangle$ need not to be an axiom.

$$\dfrac{\langle \Gamma \Rightarrow \neg\neg\varphi, \Delta\rangle \qquad \dfrac{\vdots}{\langle \neg\neg\varphi \Rightarrow \varphi\rangle}}{\langle \Gamma \Rightarrow \varphi, \Delta\rangle}\,\text{Cut}$$

Here we show how it is possible to infer $\langle \Gamma, \neg\neg\varphi \Rightarrow \Delta\rangle$ from $\langle \Gamma, \varphi \Rightarrow \Delta\rangle$. In the proof we used already proved $\langle \neg\neg\varphi \Rightarrow \varphi\rangle$. Its proof is not repeated below:

$$\dfrac{\dfrac{\vdots}{\langle \neg\neg\varphi \Rightarrow \varphi\rangle} \qquad \langle \Gamma, \varphi \Rightarrow \Delta\rangle}{\langle \Gamma, \neg\neg\varphi \Rightarrow \Delta\rangle}\,\text{Cut}$$

Here we show how it is possible to infer $\langle \Gamma, \varphi \Rightarrow \Delta\rangle$ from $\langle \Gamma, \neg\neg\varphi \Rightarrow \Delta\rangle$. In the proof we used already proved $\langle \varphi \Rightarrow \neg\neg\varphi\rangle$. Its proof is not repeated below:

$$\dfrac{\dfrac{\vdots}{\langle \varphi \Rightarrow \neg\neg\varphi\rangle} \qquad \langle \Gamma, \neg\neg\varphi \Rightarrow \Delta\rangle}{\langle \Gamma, \varphi \Rightarrow \Delta\rangle}\,\text{Cut}$$

Proofs of various sequents resembling DeMorgan rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi\rangle}{\langle \varphi \Rightarrow \varphi \vee \psi\rangle}\,\vee{-}\mathrm{r}}{\langle \varphi, \neg(\varphi \vee \psi) \Rightarrow\rangle}\,\neg{-}\mathrm{l}}{\langle \neg(\varphi \vee \psi) \Rightarrow \neg\varphi\rangle}\,\neg{-}\mathrm{r} \qquad \dfrac{\dfrac{\dfrac{\langle \psi \Rightarrow \psi\rangle}{\langle \psi \Rightarrow \varphi \vee \psi\rangle}\,\vee{-}\mathrm{r}}{\langle \psi, \neg(\varphi \vee \psi) \Rightarrow\rangle}\,\neg{-}\mathrm{l}}{\langle \neg(\varphi \vee \psi) \Rightarrow \neg\psi\rangle}\,\neg{-}\mathrm{r}}{\langle \neg(\varphi \vee \psi) \Rightarrow \neg\varphi \,\&\, \neg\psi\rangle}\,\&{-}\mathrm{r}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi\rangle}{\langle \varphi, \neg\varphi \Rightarrow\rangle}\,\neg{-}\mathrm{l}}{\langle \varphi, \neg\varphi \,\&\, \neg\psi \Rightarrow\rangle}\,\&{-}\mathrm{l}}{\langle \varphi \Rightarrow \neg(\neg\varphi \,\&\, \neg\psi)\rangle}\,\neg{-}\mathrm{r} \qquad \dfrac{\dfrac{\dfrac{\langle \psi \Rightarrow \psi\rangle}{\langle \psi, \neg\psi \Rightarrow\rangle}\,\neg{-}\mathrm{l}}{\langle \psi, \neg\varphi \,\&\, \neg\psi \Rightarrow\rangle}\,\&{-}\mathrm{l}}{\langle \psi \Rightarrow \neg(\neg\varphi \,\&\, \neg\psi)\rangle}\,\neg{-}\mathrm{r}}{\langle \varphi \vee \psi \Rightarrow \neg(\neg\varphi \,\&\, \neg\psi)\rangle}\,\vee{-}\mathrm{l}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi\rangle}{\langle \varphi, \neg\varphi \Rightarrow\rangle}\,\neg{-}\mathrm{l}}{\langle \varphi, \neg\varphi \,\&\, \neg\psi \Rightarrow\rangle}\,\&{-}\mathrm{l}}{\langle \varphi \Rightarrow \neg(\neg\varphi \,\&\, \neg\psi)\rangle}\,\neg{-}\mathrm{r} \qquad \dfrac{\dfrac{\dfrac{\langle \psi \Rightarrow \psi\rangle}{\langle \psi, \neg\psi \Rightarrow\rangle}\,\neg{-}\mathrm{l}}{\langle \psi, \neg\varphi \,\&\, \neg\psi \Rightarrow\rangle}\,\&{-}\mathrm{l}}{\langle \psi \Rightarrow \neg(\neg\varphi \,\&\, \neg\psi)\rangle}\,\neg{-}\mathrm{r}}{\langle \varphi \,\&\, \psi \Rightarrow \neg(\neg\varphi \,\&\, \neg\psi)\rangle}\,\&{-}\mathrm{l}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle \varphi \Rightarrow \varphi \rangle}{\langle \varphi \Rightarrow \varphi \,\&\, \psi \rangle}\ \&\text{–r}}{\langle \Rightarrow \neg\varphi, \varphi \,\&\, \psi \rangle}\ \neg\text{–r}}{\langle \neg(\varphi \,\&\, \psi) \Rightarrow \neg\varphi \rangle}\ \neg\text{–l} \qquad \dfrac{\dfrac{\dfrac{\langle \psi \Rightarrow \psi \rangle}{\langle \psi \Rightarrow \varphi \,\&\, \psi \rangle}\ \&\text{–r}}{\langle \Rightarrow \neg\psi, \varphi \,\&\, \psi \rangle}\ \neg\text{–r}}{\langle \neg(\varphi \,\&\, \psi) \Rightarrow \neg\psi \rangle}\ \neg\text{–l}}{\langle \neg(\varphi \,\&\, \psi) \Rightarrow \neg\varphi \vee \neg\psi \rangle}\ \vee\text{–r}$$

4 Completeness

Let $\langle \Gamma \Rightarrow \Delta \rangle$ be a tautological sequent[6]. We have to prove that its is provable in Gentzen's calculus. This can be shown by induction with respect to number of logical connectives in $\Gamma \cup \Delta$.

4.1 Step 1

Here we will prove that all tautological sequents containing solely atomic formulæ are provable:

Suppose that in set $\Gamma \cup \Delta$ are not logical connectives, i.e. there are only atomic formulæ[7]. According to the assumption, this sequent is tautological. This implies that $\Gamma \cap \Delta = \emptyset$ is false. If it were true, we would be able to find such an evaluation of formulæ under which everything in the antecedent would be true and nothing in the succedent would be true. So, clearly, for every tautological sequent composed solely from atomic formulæ $\Gamma \cap \Delta \neq \emptyset$ holds. But then, we clearly see that all such sequents are provable since they are initial sequents—see the rule (or axiom) A of Gentzen's calculus.

4.2 Step $n \to n-1$

Now we will prove that if any tautological sequent with $n-1$ logical connectives is provable in Gentzen's calculus, then also sequent with n logical connectives is provable in this calculus:

Suppose that set $\Gamma \cup \Delta$ contains also non-atomic formulæ. Let's choose arbitrarily any of them and let's denominate it χ. Either $\chi \in \Gamma$ or $\chi \in \Delta$. We will explore both possibilities.

Suppose that $\chi \in \Gamma$. Let's denominate $\Pi = \Gamma \setminus \{\chi\}$. That is $\Pi, \chi = \Gamma$. Formula χ might be in one of the following forms: $\neg\varphi$, $\varphi \,\&\, \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ or $\varphi \equiv \psi$. With regard to the form of χ write above $\langle \Gamma \Rightarrow \Delta \rangle$ one (in case of $\neg$, $\&$ or $\equiv$) or two sequents (in case of $\vee$ or $\to$) according to the following recipe:

$$\dfrac{\begin{matrix}\vdots\\ \langle \Pi \Rightarrow \Delta, \varphi \rangle\end{matrix}}{\langle \Pi, \neg\varphi \Rightarrow \Delta \rangle}\ \neg\text{–l}$$

[6] The completeness proof is adopted from [1]. Here is its complete variant. We have extended the original proof in order to be able to deal with $\equiv$ logical connective.

[7] i.e. without logical connectives

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \langle \Pi, \varphi, \psi \Rightarrow \Delta \rangle\end{array}}{\langle \Pi, \varphi, \varphi \,\&\, \psi \Rightarrow \Delta \rangle}\ \&{-}\mathrm{l}}{\langle \Pi, \varphi \,\&\, \varphi, \varphi \,\&\, \psi \Rightarrow \Delta \rangle}\ \&{-}\mathrm{l}}{\langle \Pi, \varphi \,\&\, \psi \Rightarrow \Delta \rangle}$$

$$\dfrac{\begin{array}{c}\vdots\\ \langle \Pi, \varphi \Rightarrow \Delta \rangle\end{array} \qquad\qquad \begin{array}{c}\vdots\\ \langle \Pi, \psi \Rightarrow \Delta \rangle\end{array}}{\langle \Pi, \varphi \vee \psi \Rightarrow \Delta \rangle}\ \vee{-}\mathrm{l}$$

$$\dfrac{\begin{array}{c}\vdots\\ \langle \Pi \psi \Rightarrow \Delta \rangle\end{array} \quad \begin{array}{c}\vdots\\ \langle \Pi \Rightarrow \Delta, \varphi \rangle\end{array}}{\langle \Pi, \varphi \to \psi \Rightarrow \Delta \rangle}\ {\to}{-}\mathrm{l}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \langle \Pi, \psi, \varphi \Rightarrow \Delta \rangle\end{array} \quad \begin{array}{c}\vdots\\ \langle \Pi, \varphi \Rightarrow \Delta, \varphi \rangle\end{array}}{\langle \Pi, \varphi \to \psi, \varphi \Rightarrow \Delta \rangle}\ {\to}{-}\mathrm{l} \qquad \dfrac{\begin{array}{c}\vdots\\ \langle \Pi, \psi \Rightarrow \Delta, \psi \rangle\end{array} \quad \begin{array}{c}\vdots\\ \langle \Pi \Rightarrow \Delta, \varphi, \psi \rangle\end{array}}{\langle \Pi, \varphi \to \psi, \varphi \Rightarrow \Delta, \psi \rangle}\ {\to}{-}\mathrm{l}}{\langle \Pi, \varphi \to \psi, \psi \to \varphi \Rightarrow \Delta \rangle}\ {\to}{-}\mathrm{l}}{\langle \Pi, \varphi \equiv \psi \Rightarrow \Delta \rangle}\ {\equiv}{-}\mathrm{l}$$

If $\chi \in \Delta$ then let's denominate $\Lambda = \Delta \setminus \{\chi\}$. That is $\Lambda, \chi = \Delta$. Again, according to whether χ is in the form $\neg\varphi$, $\varphi \,\&\, \psi$, $\varphi \to \psi$ or $\varphi \equiv \psi$ we can continue to write the proof-tree with the following recipes:

$$\dfrac{\begin{array}{c}\vdots\\ \langle \Gamma, \varphi \Rightarrow \Lambda \rangle\end{array}}{\langle \Gamma \Rightarrow \Lambda, \neg\varphi \rangle}\ \neg{-}\mathrm{r}$$

$$\dfrac{\begin{array}{c}\vdots\\ \langle \Gamma \Rightarrow \Lambda, \varphi \rangle\end{array} \qquad\qquad \begin{array}{c}\vdots\\ \langle \Gamma \Rightarrow \Lambda, \psi \rangle\end{array}}{\langle \Gamma \Rightarrow \Lambda, \varphi \,\&\, \psi \rangle}\ \&{-}\mathrm{r}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \langle \Gamma \Rightarrow \Lambda, \varphi, \psi \rangle\end{array}}{\langle \Gamma \Rightarrow \Lambda, \varphi \vee \psi, \psi \rangle}\ \vee{-}\mathrm{r}}{\langle \Gamma \Rightarrow \Lambda, \varphi \vee \psi, \varphi \vee \psi \rangle}\ \vee{-}\mathrm{r}}{\langle \Gamma \Rightarrow \Lambda, \varphi \vee \psi \rangle}$$

$$\dfrac{\begin{array}{c}\vdots\\ \langle\Gamma,\psi\Rightarrow\Lambda,\psi\rangle\end{array}}{\langle\Gamma\Rightarrow\Lambda,\varphi\to\psi\rangle}\to\text{-r}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \langle\Gamma,\varphi\Rightarrow\Lambda,\psi\rangle\end{array}}{\langle\Gamma\Rightarrow\Lambda,\varphi\to\psi\rangle}\to\text{-r}\qquad\dfrac{\begin{array}{c}\vdots\\ \langle\Gamma,\psi\Rightarrow\Lambda,\varphi\rangle\end{array}}{\langle\Gamma\Rightarrow\Lambda,\varphi\to\psi\rangle}\to\text{-r}}{\langle\Gamma\Rightarrow\Lambda,\varphi\equiv\psi\rangle}\equiv\text{-r}$$

So, when we unfold the proof-tree according to the above hints, we get new leaves which will have exactly one less logical connective than the original sequent. Thus, from the induction assumption that the tautological sequent with $n-1$ connectives is provable in Gentzen's calculus follows that also sequent tautological sequent with n connectives is provable in Gentzen's calculus.

Because each sequent has only finite number of the logical connective, the resulting proof-tree will have finite height.

What happens when we try to prove a sequent which is not tautological? Sooner or later we will come to a sequent containing solely atomic formulæ which cannot be proved. Such as $\langle\varphi,\chi\Rightarrow\psi\rangle$. If this happens, we know that the original sequent is not tautological[8].

References

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[8] This is practical result, be it should be made clear that this happens whatever course of proof-tree we try to build.