

# Ł – decidability of some invariant systems.

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# Outline

- Definition of the invariant system
- System  $W$
- Sobociński's system
- Remarks

# Invariant systems

- System  $(X, R)$  is an invariant system if:
- $Sb(X) = X$ , and
- $R \subseteq \underline{\underline{Struct}}$ ,

where the class of structural rules is:

$$r \in \underline{\underline{Struct}} \Leftrightarrow \forall \Pi \subseteq S \forall \alpha \in S \forall e: At \rightarrow S$$

$$[(\Pi, \alpha) \in r \Rightarrow (h^e(\Pi), h^e(\alpha)) \in r]$$

# $\perp$ -decidable systems

- The system based on a set of rejection rules and a set of rejected axioms is called  **$\perp$ -decidable** if and only if the following conditions hold:

$$(I) T \cap T^* = \emptyset$$

$$(II) T \cup T^* = S$$

- where  $T$  is the set of all theses,
- $T^*$  is the set of all rejected formulas, and
- $S$  is the set of all formulas

# System W

- The system W is defined by the following matrix:

$$M_w = \left( \left\{ 0, \frac{1}{2}, 1 \right\}, \{1\}, \{c, k, a, \neg\} \right)$$

- where:

$$k(x, y) = \begin{cases} \min(x, y), & \text{if } x \neq \frac{1}{2} \text{ and } y \neq \frac{1}{2} \\ \frac{1}{2}, & \text{else} \end{cases}$$

$$\neg x = 1 - x$$

$$c(x, y) = \begin{cases} 0, & \text{if } x=1 \text{ and } (y=0 \text{ or } y=\frac{1}{2}) \\ 1, & \text{else} \end{cases}$$

$$a(x, y) = \begin{cases} \max(x, y), & \text{if } x \neq \frac{1}{2} \text{ and } y \neq \frac{1}{2} \\ \frac{1}{2}, & \text{else} \end{cases}$$

# System W – cont.

- Consider the following functors:

$$F_0(p, q) = CCpqCApqKpq$$

$$F_{\frac{1}{2}}(p, q) = ACCNpApqApq$$

$$F_1(p, q) = F_0(q, p).$$

# System W – cont.

- The following functions correspond to functors in the matrix  $M_W$ :

$$f_0(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ and } y = 1 \\ 1, & \text{if } x \neq 0 \text{ or } y \neq 1 \end{cases}$$

$$f_{\frac{1}{2}}(x, y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = \frac{1}{2} \\ 1, & \text{if } x \neq 1 \text{ or } y \neq \frac{1}{2} \end{cases}$$

$$f_1(x, y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0 \\ 1, & \text{if } x \neq 1 \text{ or } y \neq 0 \end{cases}$$

# Rejected axioms

- The formulas:

$$F_0(p, q), F_{\frac{1}{2}}(s, r), F_1(t, u)$$

- or generalized disjunctions that can be build from the above formulas.



# System $W$ – cont.

- The main theorem:

Let  $\alpha$  be an arbitrary formula of the system  $\mathbf{W}$ . If  $\alpha \notin T$ , then  $\alpha \in T^*$ .

# Sobociński's system

- The implicational-negational system of Sobociński is given by the following matrix:

$$M_S^{c,n} = (\{0,1,2,\dots,k-1\}, \{1,2,\dots,k-1\}, \{c,n\})$$

- where  $k \geq 3$ , and for  $x, y \in \{0,1,\dots,k-1\}$

$$c(x, y) = \begin{cases} y, & \text{if } x \neq y \\ k-1, & \text{if } x = y \end{cases} \quad n(x) = \begin{cases} x+1, & \text{if } x < k-1 \\ 0, & \text{if } x = k-1 \end{cases}$$

# Sobociński's system –cont.

- Consider the following functors:

$$G_0(p, q) = CpNCqq,$$

$$G_1(p, q) = CpN^2Cqq, \dots,$$

$$G_{k-2}(p, q) = CpN^{k-1}Cqq$$

- where for  $l \geq 1$ :

$$\begin{cases} N^1\alpha = N\alpha \\ N^{l+1}\alpha = NN^l\alpha \end{cases}$$

# Sobociński's system – cont.

- The following functions correspond to functors in the matrix  $M_S^{c,n}$ :

- $g_0(x, y) = \begin{cases} 0, & \text{if } x \neq 0 \\ k-1, & \text{if } x = 0 \end{cases}$

- $g_1(x, y) = \begin{cases} 1, & \text{if } x \neq 1 \\ k-1, & \text{if } x = 1 \end{cases}$

- .....

- $g_{k-2}(x, y) = \begin{cases} k-2, & \text{if } x \neq k-2 \\ k-1, & \text{if } x = k-2 \end{cases}$

# Sobociński's system – cont.

- The negations of the functions defined on the previous slides:

- $$n(g_0(x, y)) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- $$n(g_1(x, y)) = \begin{cases} 2, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$$

- .....

- $$n(g_{k-2}(x, y)) = \begin{cases} k-1, & \text{if } x \neq k-2 \\ 0, & \text{if } x = k-2 \end{cases}$$

# Sobociński's system – cont.

- Now we can define the following functors:

- $$F_{k-1}(p, q) = CNG_0(q, p)CNG_1(q, p)...$$
$$...CNG_{k-3}(q, p)CNG_{k-2}(q, p)NG_{k-2}(p, q)$$

- $$F_{k-2}(p, q) = CF_{k-1}(p, q)CNG_0(q, p)CNG_1(q, p)...$$
$$...CNG_{k-3}(q, p)NG_{k-2}(p, q)$$

- and in general for  $0 \leq l < k$ :

- $$F_l(p, q) = CF_{l+1}(p, q) ... CF_{l-2}CF_{l-1}(p, q)CNG_0(q, p)...$$
$$...CNG_{l-1}(q, p)NG_{k-2}(p, q)$$

# Sobociński's system – cont.

- The following functions correspond to functors in the matrix  $M_S^{c,n}$ :

$$f_l(x, y) = \begin{cases} 0, & \text{if } x = k - 2 \text{ and } y = l \\ k - 1, & \text{if } x \neq k - 2 \text{ or } y \neq l \end{cases}$$

# Sobociński's system – cont.

- Now consider the following functor:

$$A(p, q) = CN^2 CppCCqpNG_0(p, q).$$

- The following function corresponds to the functor in the matrix  $M_S^{c,n}$ :

$$a(x, y) = \max\{x, y\} \text{ for } x, y \in \{0, k-1\}$$



# Rejected axioms

- The formulas:

- $F_i(p, q)$  or

- $A(A(F_i(r, p), F_j(q, s)), \dots), F_t(u, v))$

# Sobociński's System – cont.

- The main theorem:

Let  $\alpha$  be an arbitrary formula of the Sobociński system.

If  $\alpha \notin T$ , then  $\alpha \in T^*$ .