ULTRAPRODUCTS IN ANALYSIS

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Abstract. Basic concepts of ultraproduct and some applications in Analysis, mainly in Banach spaces theory, will be discussed.

It appears that the concept of ultraproduct, originated as a fundamental method of a model theory, is widely used as an important tool in analysis. When one studies local properties of a Banach space, for example, these constructions turned out to be useful as we will see later. In this writing we are invited to look at some basic ideas of these applications.

1. Ultrafilter and Ultralimit

Let us start with the definition of filters on a given index set.

Definition 1.1. *A filter $\mathscr{F}$ on a given index set I is a collection of nonempty subsets of I such that*

(1) $A, B \in \mathscr{F} \Longrightarrow A \cap B \in \mathscr{F}$, *and*
(2) $A \in \mathscr{F}, A \subset C \Longrightarrow C \in \mathscr{F}$.

Proposition 1.2. *Each filter on a given index set I is dominated by a maximal filter.*

Proof. Immediate consequence of Zorn's lemma. □

Definition 1.3. *A filter which is maximal is called an ultrafilter.*

We have following important characterization of an ultrafilter.

Proposition 1.4. *Let $\mathscr{F}$ be a filter on I. Then $\mathscr{F}$ is an ultrafilter if and only if for any subset $Y \subset I$, we have either $Y \in \mathscr{F}$ or $Y^c \in \mathscr{F}$.*

Proof. ($\Longrightarrow$) Since $I \in \mathscr{F}$, suppose $\emptyset \neq Y \notin \mathscr{F}$. We have to show that $Y^c \in \mathscr{F}$. Define $\mathscr{G} = \{Z \subset I : \exists A \in \mathscr{F}$ such that $A \cap Y^c \subset Z\}$. Since $Y \notin \mathscr{F}$, for any $A \in \mathscr{F}$, $A \cap Y^c \neq \emptyset$. Therefore, $\mathscr{G}$ is a collection of nonempty subsets of I. Moreover,

(1) If $Z_1, Z_2 \in \mathscr{G}$, then we can find $A_1, A_2 \in \mathscr{F}$ such that $A_1 \cap Y^c \subset Z_1$, $A_2 \cap Y^c \subset Z_2$. Now $A_1 \cap A_2 \in \mathscr{F}$ and $A_1 \cap A_2 \cap Y^c \subset Z_1 \cap Z_2$, i.e., $Z_1 \cap Z_2 \in \mathscr{G}$.
(2) If $Z_1 \subset Z_2$ and $Z_1 \in \mathscr{G}$, then it's clear that $Z_2 \in \mathscr{G}$.

So $\mathscr{G}$ is a filter. Since $A \supset A \cap Y^c$ for any $A \in \mathscr{F}$, we have $\mathscr{F} \subset \mathscr{G}$, and hence $\mathscr{F} = \mathscr{G}$ by maximality of $\mathscr{F}$. Noting that $Y^c \in \mathscr{G} = \mathscr{F}$, we are done.
($\Longleftarrow$) Suppose $\mathscr{G}$ is a filter such that $\mathscr{F} \subset \mathscr{G}$. Suppose to the contrary that $\exists Y \in \mathscr{G} \setminus \mathscr{F}$. From our assumption, $Y^c \in \mathscr{F}$ and hence $Y^c \in \mathscr{G}$. Then $Y \cap Y^c \in \mathscr{G}$, which is impossible. □

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Remark An ultrafilter $\mathscr{U}$ is called trivial (or non-free, principal, or fixed) if it is generated by a single element $i_0 \in I$, that means, $I_0 \in \mathscr{U}$ if and only if $i_0 \in I_0$. An ultrafilter which is not trivial is called free. Another notion is of importance for several constructions: an ultrafilter $\mathscr{U}$ is called countably incomplete if there is a sequence of elements of $\mathscr{U}$ satisfying

$$I_1 \supseteq I_2 \supseteq \cdots, \quad \bigcap_{k=1}^{\infty} I_k = \emptyset.$$

Some authors just say ultrafilters when mentioning countably incomplete ultrafilters, because it's the most interesting case. If I is finite, every ultrafilter is fixed. If I is infinite, then there is a free ultrafilter on I. See [2].

Definition 1.5. *Let $\mathscr{F}$ be a filter on I. A set of real numbers $(x_i)_{i\in I}$ is said to converge to $x \in \mathbb{R}$ with respect to $\mathscr{F}$ if for all $\epsilon > 0$, $\{i \in I : |x_i - x| < \epsilon\} \in \mathscr{F}$. In this case, x is called the ultralimit of $(x_i)_{i\in I}$, denoted $x = \lim_{\mathscr{U}} x_i$.*

Remark If such x exists in the previous definition, it is unique, since $\mathscr{F}$ consists of nonempty sets. So this definition makes sense.

The following is one of the important features of ultralimits.

Theorem 1.6. *Let $\mathscr{U}$ be an ultrafilter on I and $(x_i)_{i\in I}$ be a set of bounded real numbers, then $(x_i)_{i\in I}$ must converge.*

Proof. Suppose $|x_i| \le M$, $\forall i \in I$. Let $S_0^+ = \{i \in I : x_i \ge 0\}$, $S_0^- = \{i \in I : x_i \le 0\}$. Noting $S_0^- \supset (S_0^+)^c$, we know that at least one of these should be in $\mathscr{U}$; say S_0^+ and set $T_0 = [0, M]$. Define $S_1^+ = \{i \in I : x_i \ge \frac{M}{2}\}$, $S_1^- = \{i \in I : x_i \le \frac{M}{2}\}$. Just like before, we may assume that, say, $S_1^- \in \mathscr{U}$; set $T_1 = [0, \frac{M}{2}]$. In a like fashion, define $S_2^+ = \{i \in I : x_i \ge \frac{M}{4}\}$, $S_2^- = \{i \in I : x_i \le \frac{M}{4}\}$. Assume $S_2^+ \in \mathscr{U}$ and set $T_2 = [\frac{M}{4}, \frac{M}{2}]$. Iterate this process. By the *Principle of Nested Intervals*, we get a point $x \in \mathbb{R}$ such that $\bigcap_{k=0}^{\infty} I_k = \{x\}$. We claim that $\lim_{\mathscr{U}} x_i = x$. Fix $\epsilon > 0$ and consider the set $\{i \in I : |x_i - x| < \epsilon\}$. Choose N so that $\frac{M}{2^N} < \epsilon$. Since the length of $T_N = \frac{M}{2^N}$, we have $\bigcap_{k=0}^{N} \tilde{S}_k \subset \{i \in I : |x_i - x| < \epsilon\}$, where $\tilde{S}_k$ is one of S_k^+ or S_k^- whichever is in $\mathscr{U}$. □

Above theorem is true in more general setting; in fact, we have the following result.

Theorem 1.7. *Let K be a compact Hausdorff space. Then for each family $(x_i)_{i\in I}$ with $x_i \in K$ the limit*

$$\lim_{\mathscr{U}} x_i = x$$

exists in K. This means, there is a unique point $x \in K$ such that, for each neighborhood V of x, the set $\{i \in I : x_i \in V\}$ belongs to $\mathscr{U}$.

Proof. Refer to [2]. □

Remark Actually, a basic fact from general topology is that X is a compact Hausdorff space if and only if for every indexed family $(x_i)_{i\in I}$ in X and every ultrafilter $\mathscr{U}$ on I the ultralimit of $(x_i)_{i\in I}$ exists and is unique. See [4]

Usual rules of limits also work for ultralimits. More precisely, we have following proposition. We omit its proof since it is just a simple check of definition.

Proposition 1.8. *For ultralimits, we have that*

(1) $\lim_{\mathscr{U}} |ax_i| = |a| \cdot \lim_{\mathscr{U}} |x_i|$.
(2) $\lim_{\mathscr{U}} |x_i + y_i| \le \lim_{\mathscr{U}} |x_i| + \lim_{\mathscr{U}} |y_i|$.
(3) *If* $x_i \le y_i$ *for any* i, *then* $\lim_{\mathscr{U}} x_i \le \lim_{\mathscr{U}} y_i$.
(4) $\lim_{\mathscr{U}} (x_i + y_i) = \lim_{\mathscr{U}} x_i + \lim_{\mathscr{U}} y_i$.
(5) $\lim_{\mathscr{U}} C = C$ *for any* C *constant.*
(6) $\lim_{\mathscr{U}} x_i \cdot y_i = \lim_{\mathscr{U}} x_i \cdot \lim_{\mathscr{U}} y_i$.

Remark (3)+(4) gives a proof of (2).

We close this section with a proposition about the product of ultrafilters.

Proposition 1.9. *Let* $\mathscr{U}, \mathscr{V}$ *be ultrafilters on index sets* I, J, *respectively. Then* $\mathscr{U} \times \mathscr{V}$ *is an ultrafilter on* $I \times J$ *where* $\mathscr{U} \times \mathscr{V}$ *is defined by:*

$$X \in \mathscr{U} \times \mathscr{V} \text{ if and only if } \{j : \{i : (i,j) \in X\} \in \mathscr{U}\} \in \mathscr{V}.$$

Proof. First we show that $\mathscr{U} \times \mathscr{V}$ is a filter on $I \times J$. Note that $\mathscr{U} \times \mathscr{V}$ consists of nonempty sets.

(1) Suppose $A, B \in \mathscr{U} \times \mathscr{V}$. We want to show that $\{j : \{i : (i,j) \in A \cap B\} \in \mathscr{U}\} \in \mathscr{V}$. Let $\tilde{A} = \{j : \{i : (i,j) \in A\} \in \mathscr{U}\}$ and $\tilde{B} = \{j : \{i : (i,j) \in B\} \in \mathscr{U}\}$. Since $\tilde{A}, \tilde{B} \in \mathscr{V}$, $\tilde{A} \cap \tilde{B} \in \mathscr{V}$. Choose $j_0 \in \tilde{A} \cap \tilde{B}$ and let $A_{j_0} = \{i : (i, j_0) \in A\}$, $B_{j_0} = \{i : (i, j_0) \in B\}$. Now $A_{j_0}, B_{j_0} \in \mathscr{U}$ so $A_{j_0} \cap B_{j_0} = \{i : (i, j_0) \in A \cap B\} \in \mathscr{U}$. This shows that $\tilde{A} \cap \tilde{B} \subset \{j : \{i : (i,j) \in A \cap B\} \in \mathscr{U}\}$ and hence $\{j : \{i : (i,j) \in A \cap B\} \in \mathscr{U}\} \in \mathscr{V}$.
(2) Suppose $A \in \mathscr{U} \times \mathscr{V}$ and $A \subset C$. Choose $j_0 \in \{j : \{i : (i,j) \in A\} \in \mathscr{U}\}$. Since $A \subset C$, $j_0 \in \{j : \{i : (i,j) \in C\} \in \mathscr{U}\}$. Thus $\{j : \{i : (i,j) \in A\} \in \mathscr{U}\} (\in \mathscr{V}) \subset \{j : \{i : (i,j) \in C\} \in \mathscr{U}\}$, which shows $C \in \mathscr{V}$.

To show that $\mathscr{U} \times \mathscr{V}$ is maximal, pick any $X \subset I \times J$ and suppose $X \notin \mathscr{U} \times \mathscr{V}$. By maximality of $\mathscr{V}$, $\{j : \{i : (i,j) \in X\} \notin \mathscr{U}\} \in \mathscr{V}$. Choose $j_0 \in \{j : \{i : (i,j) \in X\} \notin \mathscr{U}\}$ and consider the set $\{i : (i, j_0) \in X\}$. By maximality of $\mathscr{U}$, we have $\{i : (i, j_0) \in X^c\} \in \mathscr{U}$, which shows that $\{j : \{i : (i,j) \in X\} \notin \mathscr{U}\} (\in \mathscr{V}) \subset \{j : \{i : (i,j) \in X^c\} \in \mathscr{U}\}$. Thus $\{j : \{i : (i,j) \in X^c\} \in \mathscr{U}\} \in \mathscr{V}$. □

2. Ultraproducts of Banach Spaces

Let $(E_i)_{i \in I}$ be a family of Banach spaces. Consider the space $\ell_\infty(I, E_i)$ of families $(x_i)_{i \in I}$ with $x_i \in E_i (i \in I)$ and

$$\|(x_i)\|_\infty = \sup_{i \in I} \|x_i\| < \infty.$$

Then $\ell_\infty(I, E_i)$ is a Banach space, which is too big in some sense. Let $N_{\mathscr{U}}$ be the subset of all those families $(x_i)_{i \in I} \in \ell_\infty(I, E_i)$ with $\lim_{\mathscr{U}} \|x_i\| = 0$. One can easily verify that $N_{\mathscr{U}}$ is a closed linear subspace of $\ell_\infty(I, E_i)$. So we are ready to give

Definition 2.1. *The* ultraproduct $(E_i)_{\mathscr{U}}$ *of the family of Banach spaces* $(E_i)_{i \in I}$ *with respect to the ultrafilter* $\mathscr{U}$ *is the quotient space*

$$\ell_\infty(I, E_i)/N_{\mathscr{U}},$$

equipped with the canonical quotient norm.

Remark Often the ultraproduct $(E_i)_{\mathscr{U}}$ is also denoted by $\prod_{i\in I} E_i/\mathscr{U}$.

Given an element $(x_i) \in \ell_\infty(I, E_i)$, the corresponding equivalence class in $(E_i)_{\mathscr{U}}$ will be denoted by $(x_i)_{\mathscr{U}}$. If all the spaces E_i are identical with a certain E, then we speak of an ultrapower, denoted by $(E)_{\mathscr{U}}$ or $E^I/\mathscr{U}$. There is a canonical (or diagonal) *isometric* embedding $\mathcal{J}$ of E into its ultrapower $(E)_{\mathscr{U}}$, which is defined by $\mathcal{J}x = (x_i)_{\mathscr{U}}$, where $x_i \equiv x$. This embedding is generally not surjective; it is, however, when the ultrafilter $\mathscr{U}$ is principal or the space E is finite dimensional. See [4]. It is worth mentioning that the quotient norm of $(x_i)_{\mathscr{U}}$ can be computed by the following formula:

Proposition 2.2. *Using the notations above, we have*

$$\|(x_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|x_i\|.$$

Proof. ($\geq$) First of all,

$$\lim_{\mathscr{U}} \|x_i + y_i\| \leq \|x_i + y_i\|_\infty \triangleq \|(x_i) + (y_i)\|_\infty. \tag{2.1}$$

If $(y_i) \in N_{\mathscr{U}}$, then

$$\lim_{\mathscr{U}} \|x_i + y_i\| \leq \lim_{\mathscr{U}} \|x_i\| + \underbrace{\lim_{\mathscr{U}} \|y_i\|}_{\substack{\| \\ 0}} = \lim_{\mathscr{U}} \|x_i\| \leq \lim_{\mathscr{U}} \|x_i + y_i\| + \underbrace{\lim_{\mathscr{U}} \| - y_i\|}_{\substack{\| \\ 0}} = \lim_{\mathscr{U}} \|x_i + y_i\|,$$

that is, $\lim_{\mathscr{U}} \|x_i + y_i\| = \lim_{\mathscr{U}} \|x_i\|$. Taking infimum over $(y_i) \in N_{\mathscr{U}}$ in (2.1), we get

$$\|(x_i)_{\mathscr{U}}\| \geq \lim_{\mathscr{U}} \|x_i + y_i\| = \lim_{\mathscr{U}} \|x_i\|.$$

($\leq$) Let $L \triangleq \lim_{\mathscr{U}} \|x_i\|$ and $\delta > 0$ be given. Let $J \triangleq \{i \in I : |\|x_i\| - L| < \delta\}$, then $J \in \mathscr{U}$ and

$$\|x_i\| < L + \delta \tag{2.2}$$

for all $i \in J$. Define $(x_i^J)_{i\in I}$ by

$$x_i^J = \begin{cases} x_i, & i \in J, \\ 0, & \textit{otherwise}. \end{cases}$$

Now we claim that

$$\underbrace{\|(x_i)_{\mathscr{U}}\|}_{\substack{\triangleleft\| \\ \inf_{(k_i)\in N_{\mathscr{U}}} \|(x_i)+(k_i)\|_\infty}} \leq \underbrace{\|(x_i^J)_{\mathscr{U}}\|}_{\substack{\triangleleft\| \\ \inf_{(\ell_i)\in N_{\mathscr{U}}} \|(x_i^J)+(\ell_i)\|_\infty}}. \tag{2.3}$$

Indeed, $(x_i^J) + (\ell_i) = (x_i) + (\ell_i) - (m_i)$, where

$$m_i = \begin{cases} 0, & i \in J \\ x_i, & \textit{otherwise}. \end{cases}$$

Since $J \subset \{i \in I : \|m_i\| < \epsilon\}$ for every $\epsilon > 0$, this shows that $\lim_{\mathscr{U}} \|m_i\| = 0$ and hence we obtain the inequality (2.3). Thus, $\|(x_i)_{\mathscr{U}}\| \leq \|(x_i^J)_{\mathscr{U}}\| \leq \sup_i \|x_i^J\| \overset{(2.2)}{\leq} L+\delta$. Letting $\delta \to 0$, we finally have $\|(x_i)_{\mathscr{U}}\| \leq L = \lim_{\mathscr{U}} \|x_i\|$. □

Remark Actually, in (2.3), $\|(x_i)_{\mathscr{U}}\| = \|(x_i^J)_{\mathscr{U}}\|$.

There is another important notion: the ultraproduct of operators.

Definition 2.3. *Let $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ be families of Banach spaces indexed by the same set I. For each $i \in I$ let $T_i \in L(E_i, F_i)$ be given. In addition, suppose that $\sup_{i\in I} \|T_i\| < \infty$. The operator from $(E_i)_{\mathscr{U}}$ into $(F_i)_{\mathscr{U}}$ defined by the following rule*

$$(x_i)_{\mathscr{U}} \longmapsto (T_i x_i)_{\mathscr{U}}$$

is called the ultraproduct of the family of operators $(T_i)_{i\in I}$ with respect to the ultrafilter $\mathscr{U}$, and is denoted by $(T_i)_{\mathscr{U}}$.

Remark We have to mention that the operator $(T_i)_{\mathscr{U}}$ is well-defined. Indeed $\|T_i x_i\| \leq \|T_i\| \cdot \|x_i\|$, so $\sup_{i\in I} \|T_i x_i\| < \infty$ and

$$0 \leq \lim_{\mathscr{U}} \|T_i x_i\| \leq \lim_{\mathscr{U}} \|T_i\| \cdot \|x_i\| = \lim_{\mathscr{U}} \|T_i\| \cdot \lim_{\mathscr{U}} \|x_i\|.$$

This implies that we have $\lim_{\mathscr{U}} \|T_i x_i\| = 0$ whenever $\lim_{\mathscr{U}} \|x_i\| = 0$.

How do we compute the norm of $\|(T_i)_{\mathscr{U}}\|$? Next proposition answers this question.

Proposition 2.4. $\|(T_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|T_i\|$.

Proof. ($\leq$) $\|(T_i x_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|T_i x_i\| \leq \lim_{\mathscr{U}} \|T_i\| \cdot \lim_{\mathscr{U}} \|x_i\| = \lim_{\mathscr{U}} \|T_i\| \cdot \|(x_i)_{\mathscr{U}}\|$.
($\geq$) Let $\epsilon > 0$ be given. For each $i \in I$, there exists $\tilde{x}_i$ such that $\|\tilde{x}_i\| = 1$ and $\|T_i \tilde{x}_i\| \geq \frac{\|T_i\|}{1+\epsilon}$. Now $\|(T_i)_{\mathscr{U}}\| \geq \|(T_i \tilde{x}_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|T_i \tilde{x}_i\| \geq \frac{1}{1+\epsilon} \lim_{\mathscr{U}} \|(T_i)_{\mathscr{U}}\|$. Taking $\epsilon \to 0$, we get the result. □

It is quite natural to expect that the ultraproduct of Banach spaces preserves many kinds of structures. We, without proofs, illustrate some of these results. See [3].

Proposition 2.5. *We have the following stabilities:*

1. *The class of Banach algebras is stable under ultraproducts.*
2. *The class of C^*-algebras is stable under ultraproducts.*
3. *The class of Banach lattices is stable under ultraproducts.*

Combined with classical representation theorem (Gelfand's representation theorem, for example. See [1], [6].), the results above yield

Theorem 2.6. *We have following two representations:*

1. *Let K_i $(i \in I)$ be compact Hausdorff spaces. Then there is a compact Hausdorff space K such that the ultraproduct $(C(K_i))_{\mathscr{U}}$ is linearly isometric to $C(K)$. This isometry preserves the multiplicative and the lattice structure.*
2. *Let $1 \leq p < \infty$ and let μ_i $(i \in I)$ be arbitrary (σ-additive) measures. Then $(L_p(\mu_i))_{\mathscr{U}}$ is order isometric to $L_p(\nu)$ for a certain measure ν.*

Proof. Refer to [3]. □

To be continued...

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