A sequent calculus for 1-backtracking

Stefano Berardi and Yoriyuki Yamagata

Background : LCM

Susumu Hayashi and N. Nakata (2001)

- Analogy of Constructive Mathematics (realized by recursive functions)

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- Analogy of Constructive Mathematics (realized by recursive functions)
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- Covers large parts of classical elementary mathematics

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EM_1 suffices for elementary classical mathematics.

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- We will look at this more closely ...

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$\Leftrightarrow \mathcal{E}$ has a winning strategy of $bck^1(T_A)$,

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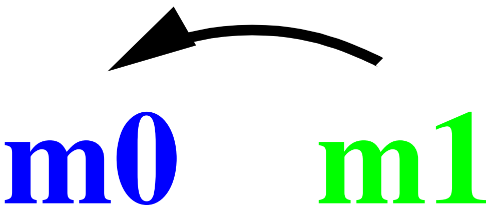
Our contribution : PA_1

$\mathcal{E}$ has a winning strategy σ of $bck^1(T_A)$,
$\Leftrightarrow$ There is a proof $\pi(\sigma)$ of A in PA_1,
where PA_1 is a sequent calculus without Exchange,
and $\pi(\sigma)$ is tree-isomorphic to σ.

backtracking play



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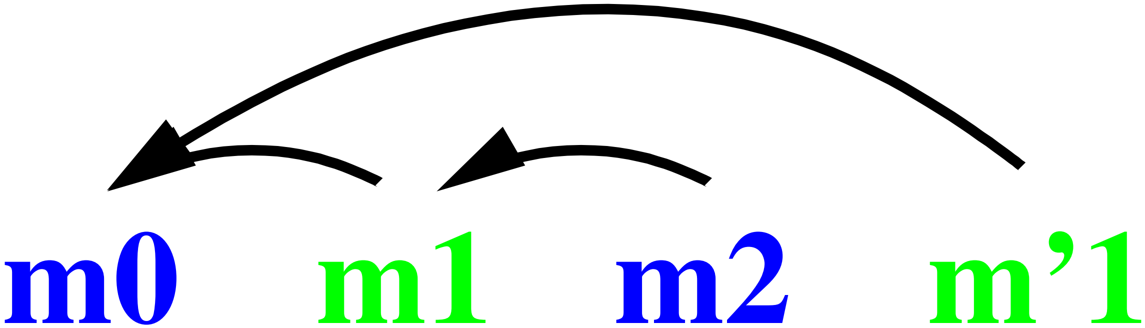

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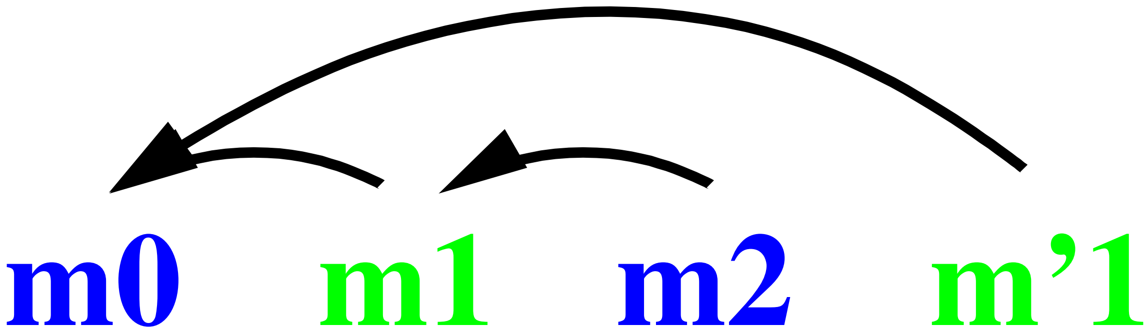
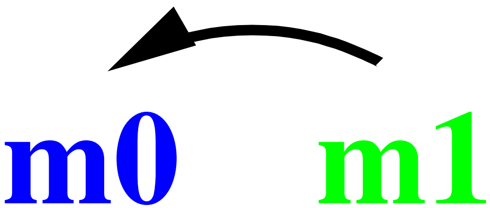


$m1, m2$ are cancelled and $\mathcal{E}$ replies $m0$.

1-backtracking play



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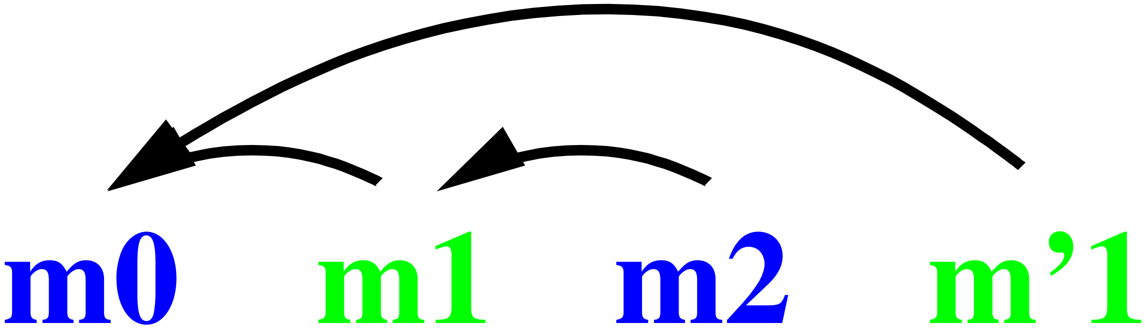

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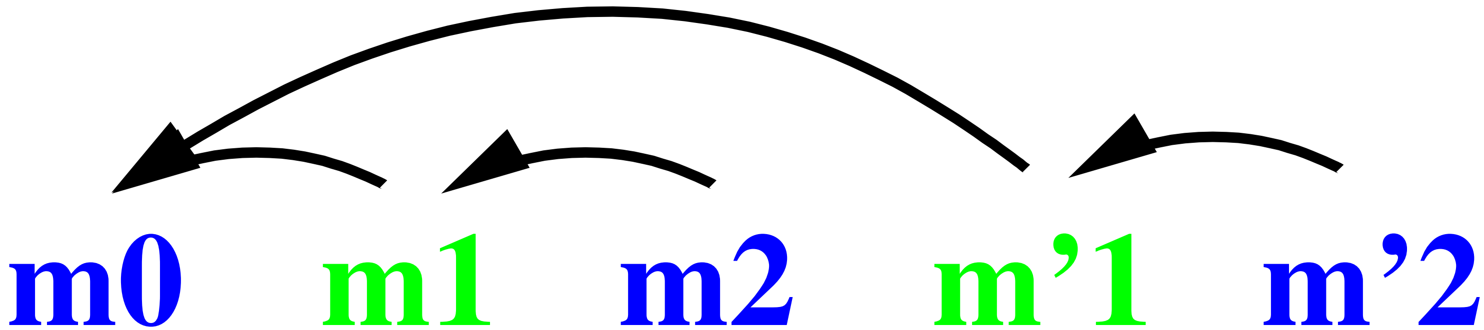

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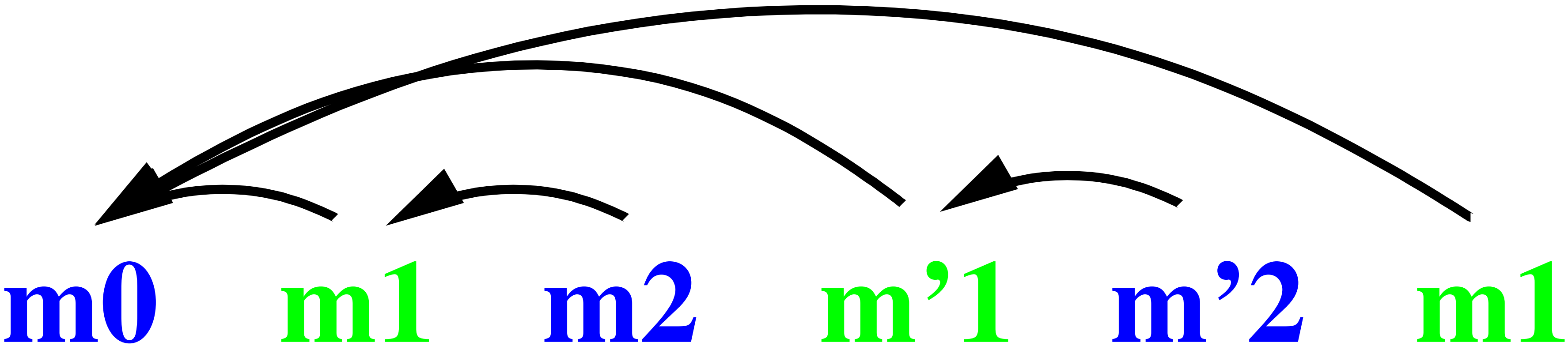

This is allowed

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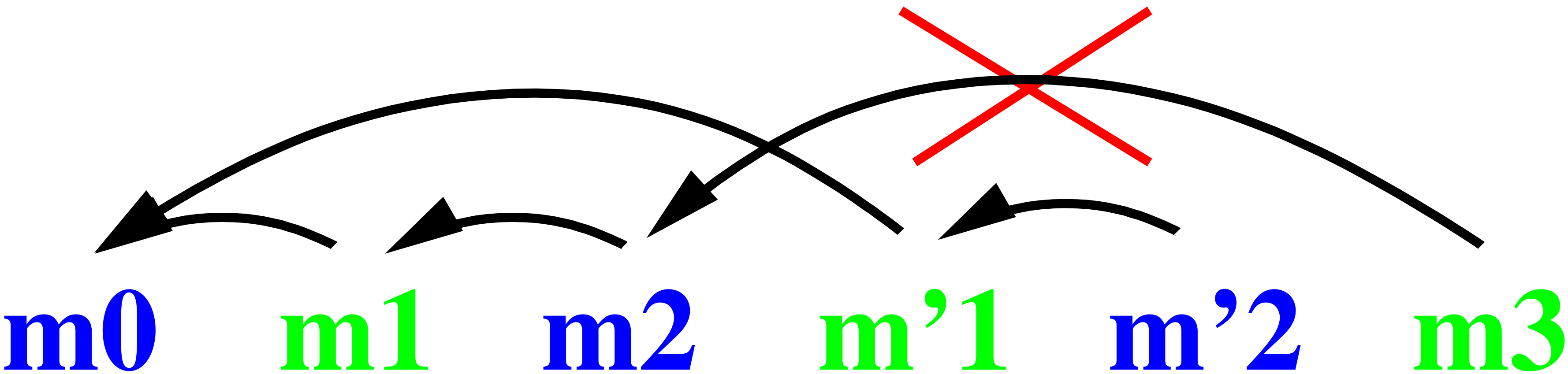

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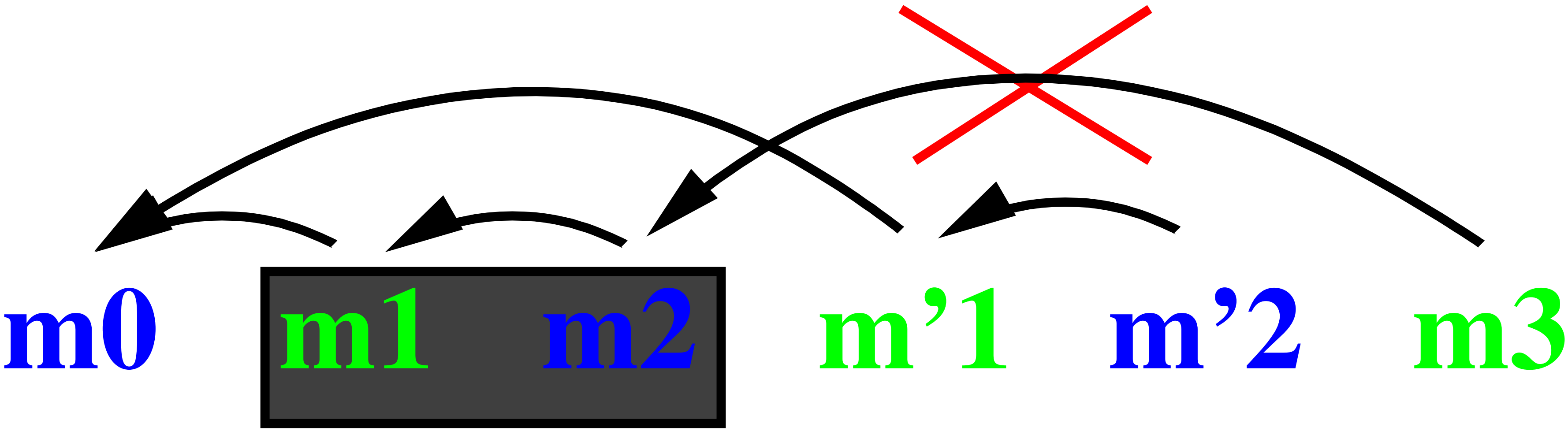
1-backtracking play



inaccessible after $m'1$

1-excluded middle

1-excluded middle EM_1 is a schema

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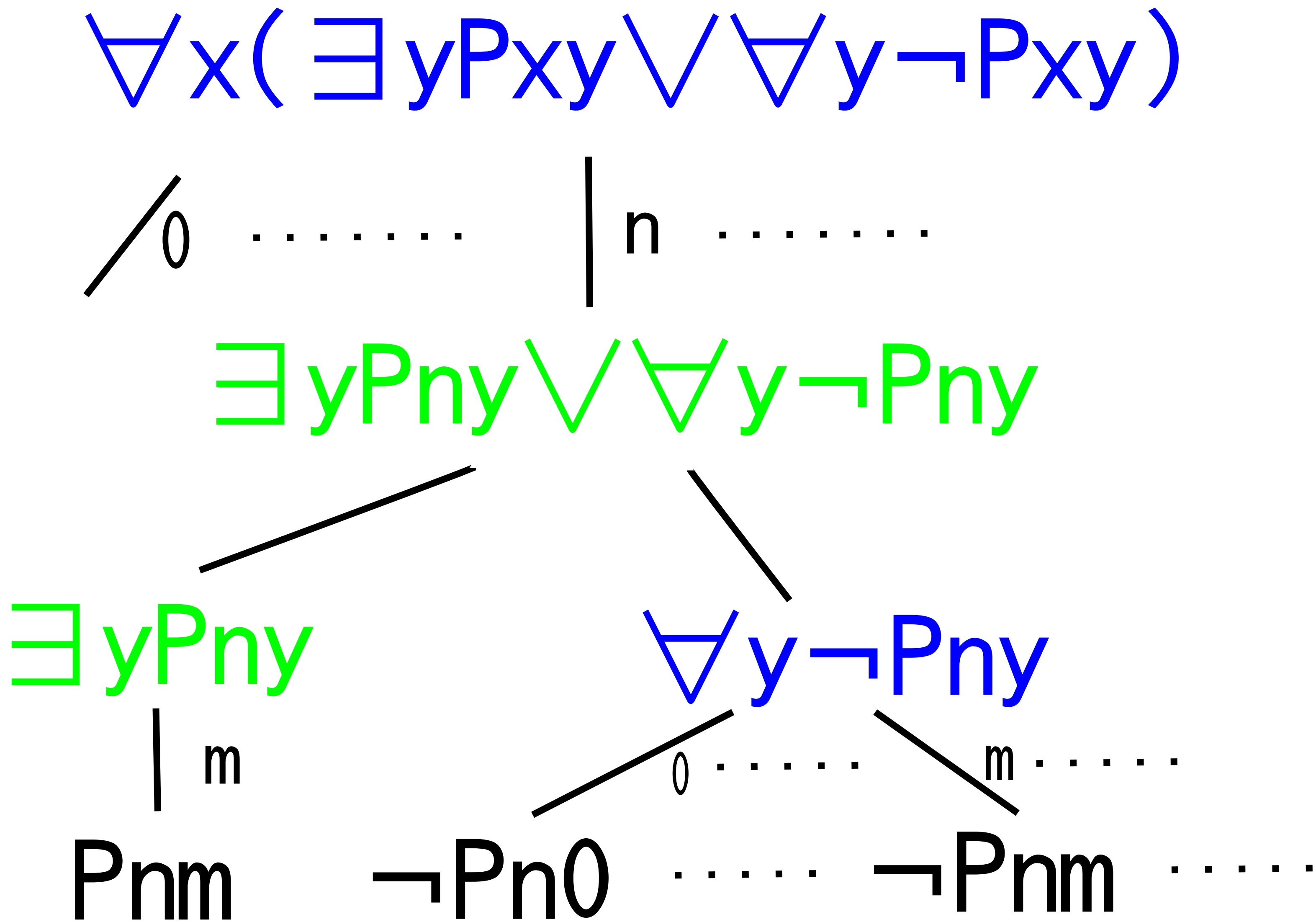
$$\forall x(\exists y Pxy \vee \forall y \neg Pxy)$$

P : decidable

Tarski game G of EM_1



Tarski game G of EM_1(2)

There is no recursive winning strategy for G,

Tarski game G of $EM_1(2)$

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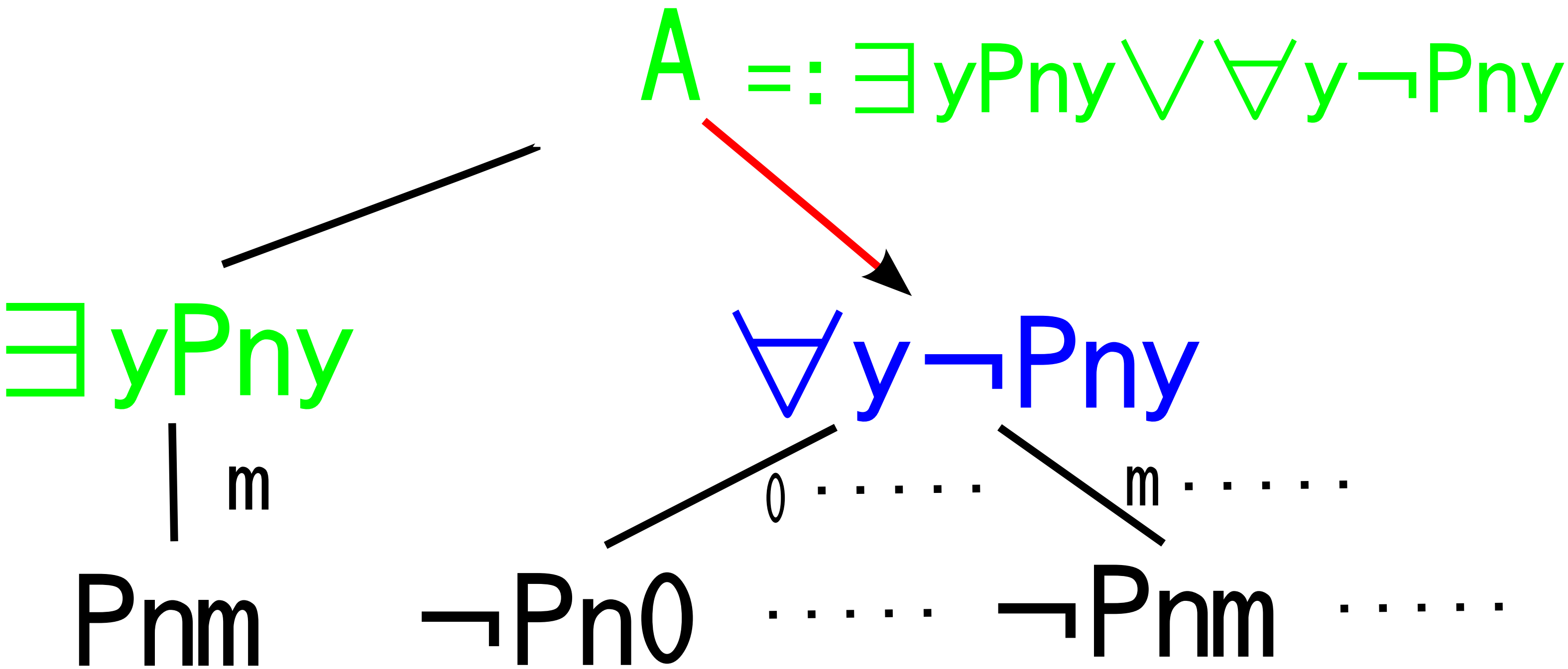
because $\mathcal{E}$ must choose $\exists y P x y$ or $\forall y \neg P x y$

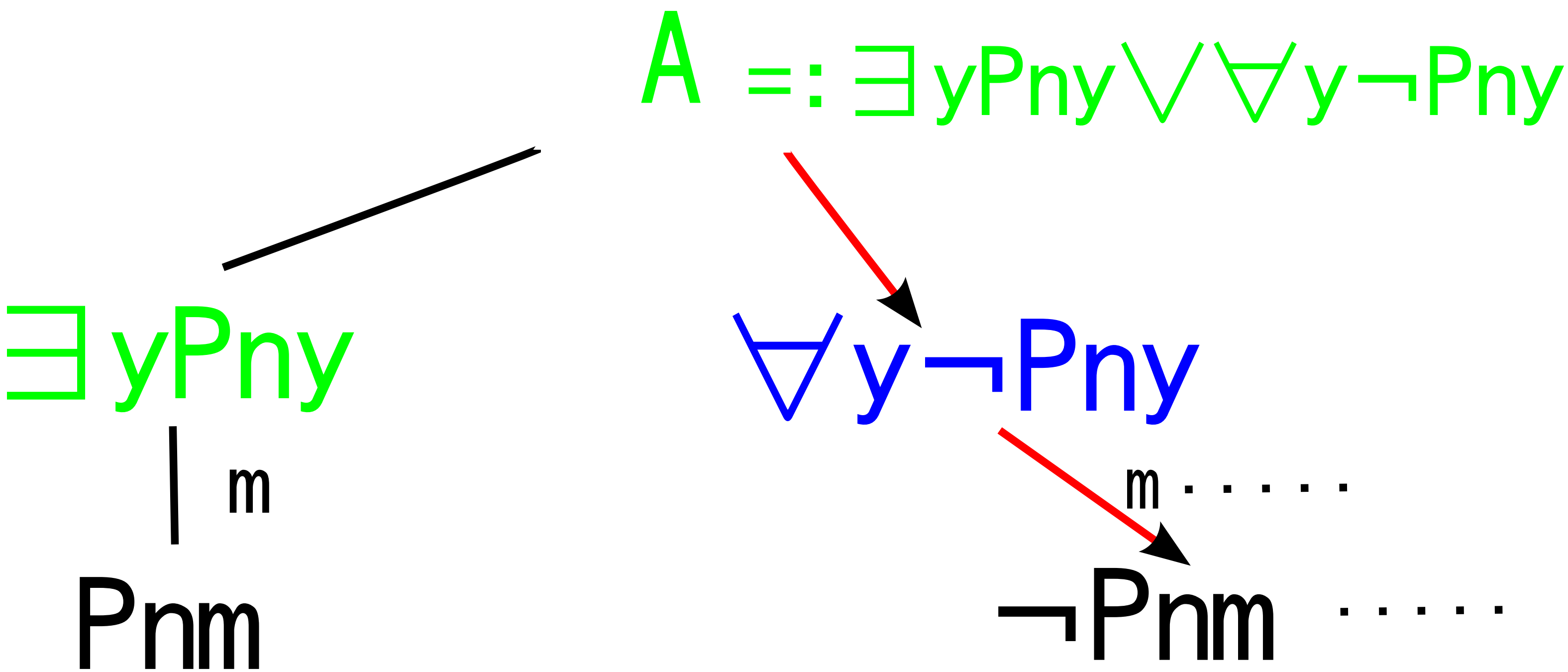
Tarski game G of $EM_1(3)$

But there is a recursive winning strategy for 1-bck play of G

1-bck play of G



1-bck play of G



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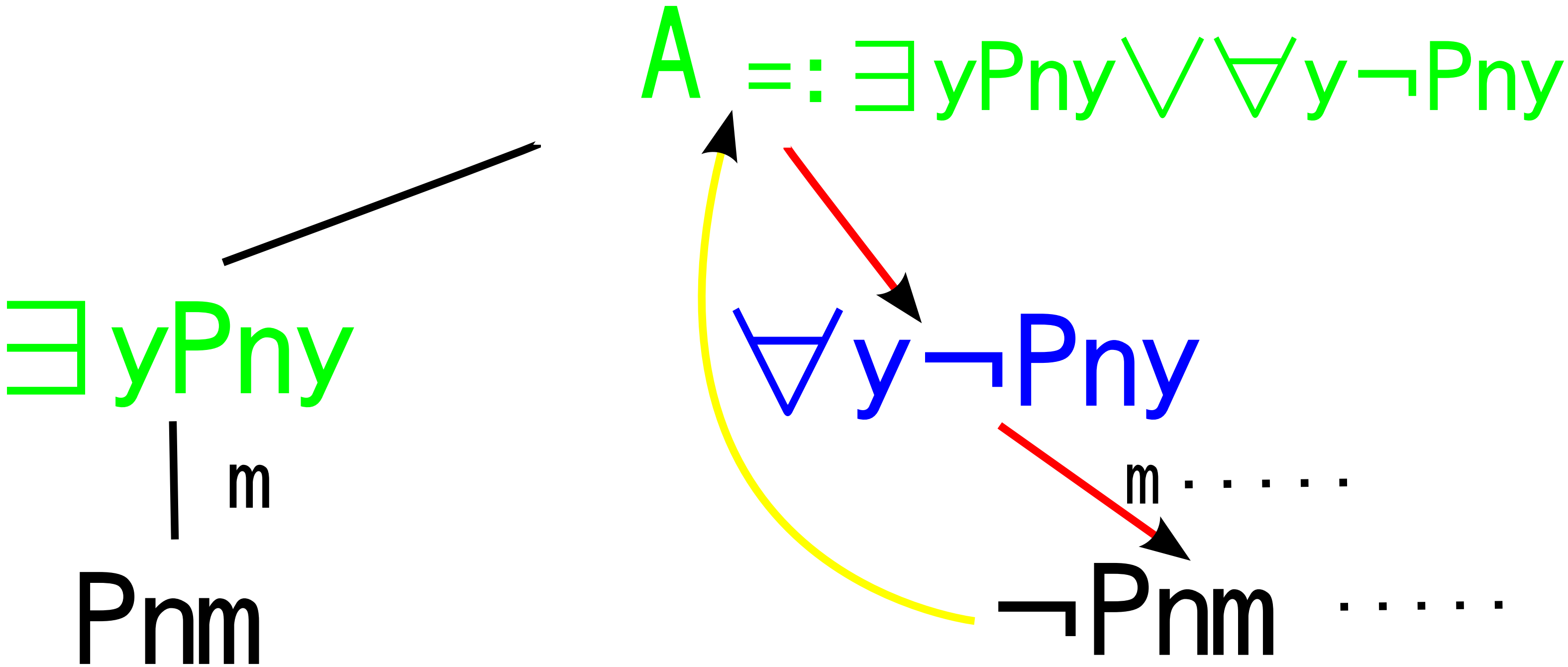

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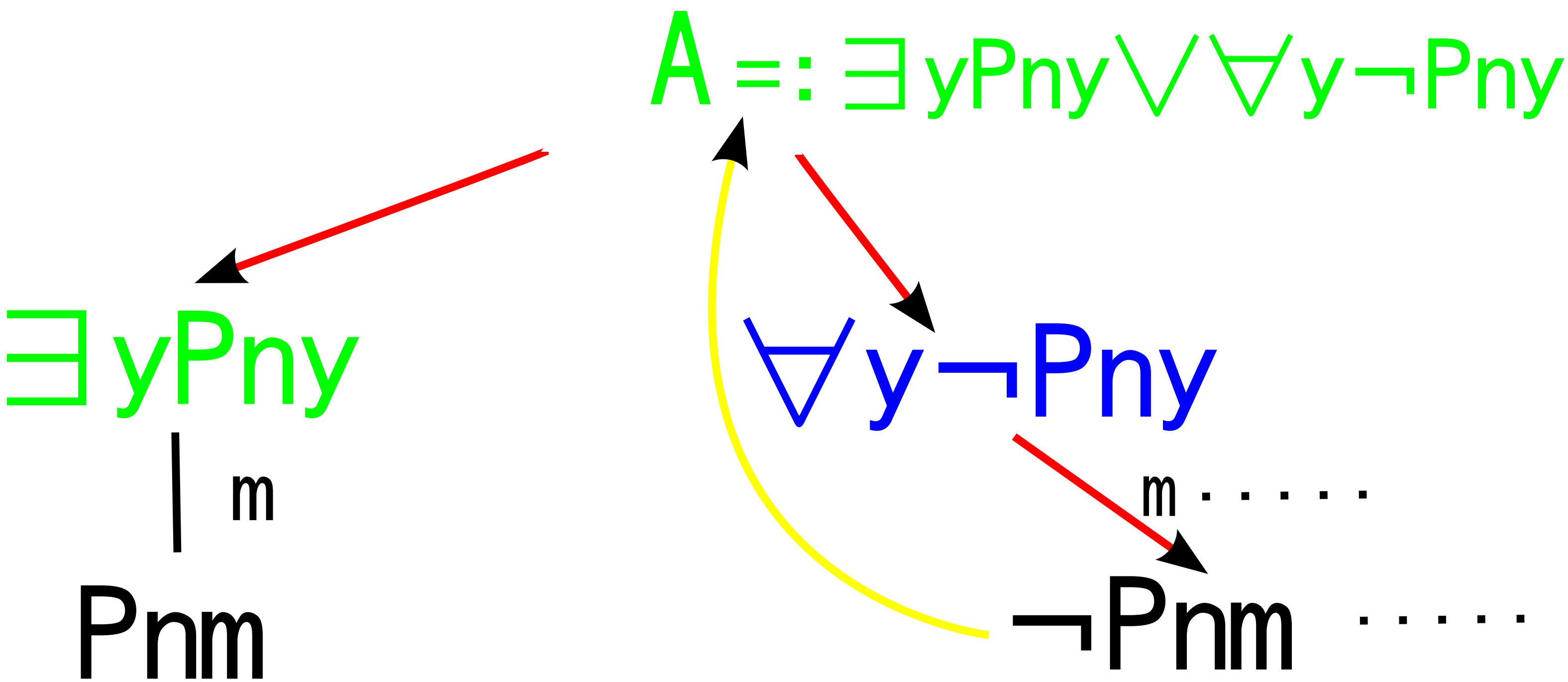

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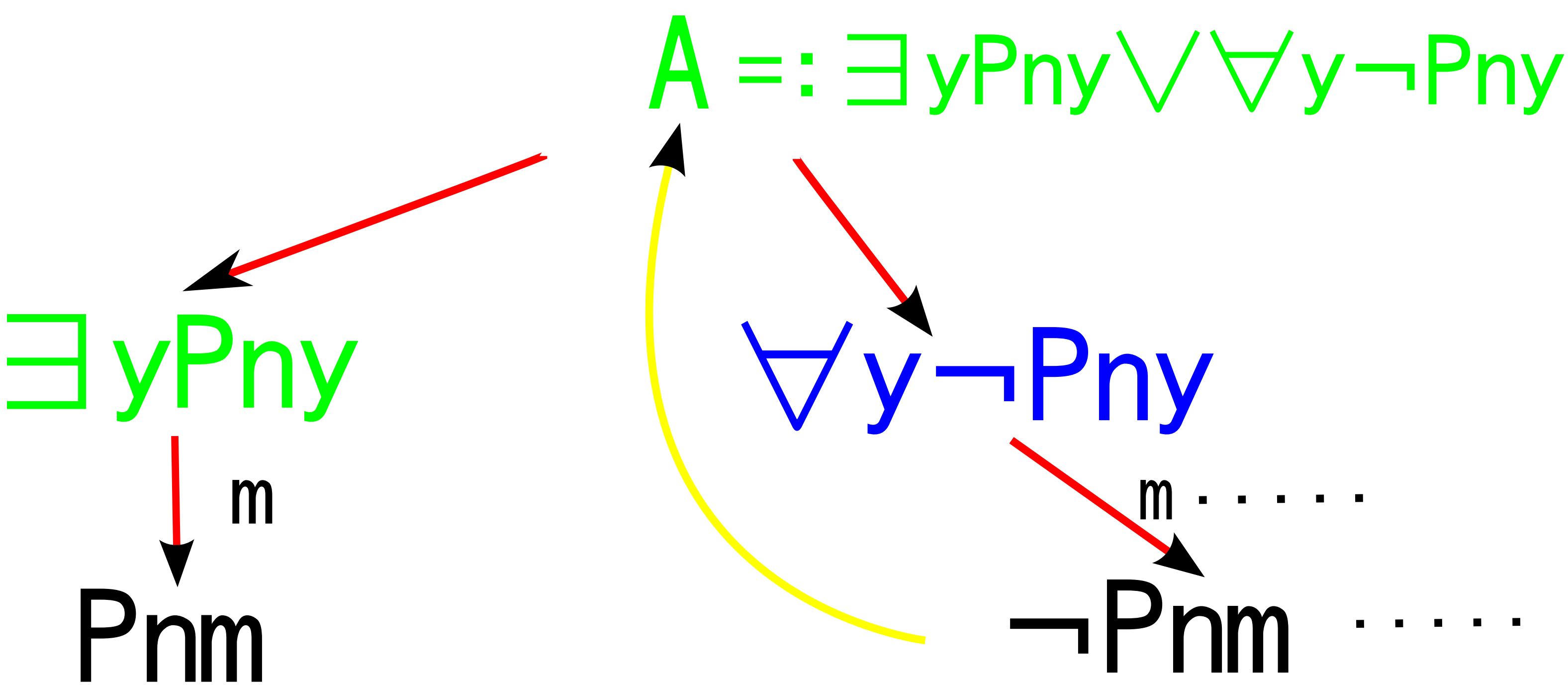

PA_1

$$\frac{}{\Gamma, p}$$

$$\frac{\Gamma, A(0) \qquad \ldots \qquad \Gamma, A(n) \qquad \ldots}{\Gamma, \forall x A(x), \Delta} \; \forall$$

$$\frac{\Gamma, \exists x A(x), A(n)}{\Gamma, \exists x A(x), \Delta} \; \exists$$

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- ω-rules

A proof of EM_1

$$
\dots \quad
\cfrac{\cfrac{\cfrac{\cfrac{}{A, \exists y Pny, Pnm}\ \text{true}}{A, \exists y Pny}\ \exists}{A, \neg Pnm}\ \vee}{A, \forall y \neg Pny}\ \forall
\quad \dots
$$

$$
\frac{A, \forall y \neg Pny}{\exists y Pny \vee \forall y \neg Pny}\ \vee
$$

Interpretation of sequent

$$B_1, B_2, \ldots, B_n, C$$

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$B_1, \ldots, B_n$: possible positions to backtrack

Isomorphic Theorem

There is a tree isomorphism between

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There is a tree isomorphism between

- a proof tree of formula A
- a winning strategy (as a tree of move) of $bck^1(T_A)$

Conclusion

- We introduce a proof system PA_1, an ω-logic without Exchange
- We show a proof of formula A in PA_1 and a winning strategies of $bck^1(T_A)$ has a tree-isomorphism

The End