§ 14. Some SC-metrics of an n-cell. Now we may show the following theorem, which contains the solution of problem II (see § 1).

THEOREM. There exists for every \( n = 2, 3, \ldots \) and \( m = 0, 1, \ldots \) a SC-metric \( \bar{\gamma}_m^3 \) of \( S^n \) such that the n-cell \( S^n \) metrized by \( \bar{\gamma}_m^3 \) has exactly \( m \) terminal points, i.e.

\[
\bar{\gamma}(S^n, \bar{\gamma}_m^3) = m.
\]

Proof. Of course, it is sufficient to define such metrics in the spaces homeomorphic to \( S^n \), i.e., in an arbitrary n-cell.

For \( n = 2 \) we put \( \bar{\gamma}_0^3 = \bar{\gamma}_1^3 = \bar{\gamma}_2^3 = \bar{\gamma}_3^3 = \bar{\gamma} \) (see 13.1, 13.3 and § 12) and \( \bar{\gamma}_i^3 \) an ordinary Euclidean metric of convex plane \( i \)-gonal figure for \( i = 3, 4, \ldots \).

Suppose the theorem is true for \( n = k \). We can diminish the diameter (e.g., to 1) of the space \( (S^k, \bar{\gamma}_m^3) \) without changing the number of terminal points, for instance by dividing every diameter by the distance. Then we may apply the results of § 11. Namely, let

\[
\bar{\gamma}_m^3 = (\bar{\gamma}_m^3)_n\text{ for } m = 0, 1, \ldots
\]

We conclude from 11.5 that

\[
\bar{\gamma}(S^{k+1}, \bar{\gamma}_m^3) = \bar{\gamma}(S^k \times S, \bar{\gamma}_m^3) = \bar{\gamma}(S^k, \bar{\gamma}_m^3) \times (1) \text{ for } m = 0, 1, \ldots
\]

It follows that

\[
\bar{\gamma}(S^{k+1}, \bar{\gamma}_m^3) = \bar{\gamma}(S^k, \bar{\gamma}_m^3) = m \text{ for } m = 0, 1, \ldots
\]

References


A generalization of the incompleteness theorem

by

A. Mostowski (Warszawa)

The aim of this paper is to prove the following generalization of the Gödel incompleteness theorem (cf. [1] and [9]):

Let a formula \( \Phi \) with one numerical free variable be called free for a system \( S \) if for every \( n \) formulas \( \Phi(d_1), \Phi(d_2), \ldots, \Phi(d_n) \) are completely independent (i.e., every conjunction formed of some of the these formulas and of the negations of the remaining ones is consistent; \( \Phi(d_i) \) denotes here the formula obtained from \( \Phi \) by substituting the \( i \)-th numeral for the variable of \( \Phi \)). We shall prove that free formulas exist for certain systems \( S \) and some of their extensions. In fact we shall prove for a class of formal systems \( S \) a slightly more general result: given a family of extensions of \( S \) satisfying certain very general assumptions, there exists a formula which is free for every extension of this family.

The following circumstance deserves perhaps mentioning and justifies to a certain extent the length of the paper. Our considerations prove the existence of free formulas not only for systems based on the usual rules of proof but also for systems based on the rule \( \omega \). Thus they furnish another illustration of the parallelism noted already in [2] between these two kinds of systems. Our discussion of systems based on the rule \( \omega \) rests on the remark due to J. R. Shoenfield that the decomposition of the \( H_\omega \) sets into constituents (cf. Kleene [5], theorem I, p. 417) can on several occasions be exploited in the same way as the recursive enumerability of the \( S^\omega_n \) sets. Thus our paper can be considered as a test of this useful heuristic principle. From a result noted at the end of the paper it follows that no similar phenomenon occurs for \( H_\omega \) sets.

In view of this remarks the author hopes that his paper in spite of its rather special subject may throw some light on a more important and broader topic, to wit the constructive analogue of the theory of projective sets.

1. We consider a consistent theory \( T \) with standard formalization and infinite sequence \( A_0, A_1, \ldots \) of its terms without free variables. The Gödel number of a formula \( \Phi \) will be denoted by \( \Gamma(\Phi) \). A \( k \)-ary relation \( R \) (i.e., a subset of \( N^k = N_0 \times \cdots \times N_k \) where \( N_0 \) is the set of integers \( \geq 0 \))
is weakly representable in $T$ if there is a formula $\Phi$ with $k$ free variables such that

(1) $$(n_1, ..., n_k) \in R \iff \Phi(a_{n_1}, ..., a_{n_k})$$

$R$ is strongly representable in $T$ if besides (1) the equivalence

(2) $$(n_1, ..., n_k) \in R \iff \neg \Phi(a_{n_1}, ..., a_{n_k})$$

is true for arbitrary $n_1, ..., n_k$.

A function $f : N^k \to N$ is representable in $T$ if there is a formula $\Phi$ with $k+1$ free variables such that

$$\neg \Phi(a_{n_1}, ..., a_{n_k}, x) \iff x = a_{f(n_1, ..., n_k)}.$$ 

A relation $R$ is weakly or strongly representable relatively to a set $Q \subseteq N$ if (1) and (2) hold for arbitrary $n_1, n_2, ..., n_k$ in $Q$.

We shall assume

I. There are primitive recursive functions $Neg, Imp, Con, Al, Sh, En, \delta$ such that $Neg(\Gamma) = \neg \Gamma$, $\neg Imp(\neg \Gamma) = \Gamma \lor \neg \Gamma$, $\neg Con(\neg \Gamma) = \Gamma \lor \neg \Gamma$, $\neg Al(\neg \Gamma) = \Gamma \lor \neg \Gamma$, $\neg Sh(\neg \Gamma, \Delta, \neg \Gamma) = \neg \Delta$, $\neg En(\neg \Gamma, \Delta, \neg \Gamma, \Delta)$.

We shall also assume that there is a set $P$, called the set of proofs of $T$, and a quaternary relation $\xleftarrow{\lambda} \subseteq T$, and the following conditions:

II. The relation $(m, n) \in (p, q)$ is reflexive, transitive, and well-founded in $P \times N$.

IV. There is a formula $\Pi(x)$ which weakly represents $P$ in $T$ and a formula $M(x, y; t, u; v)$ with the free variables indicated which strongly represents $\xleftarrow{\lambda}$ in $T$ relatively to $P \times N \times P \times N$. Furthermore, these formulas satisfy the conditions:

(i) $\Pi(x) \& \Pi(\Delta) \& M(x, y; t, u; v) \& M(x, z; y, u; v) \iff y = z$;

(ii) if $\langle p, q \rangle \in \Pi(x)$, then $\Pi(x) \& M(x, y; t, u; v) \iff \exists \langle p, q \rangle$;

(iii) if $\langle p, q \rangle \in \Pi(x)$, then $\Pi(x) \& M(x, y; t, u; v) \iff \exists \langle p, q \rangle$.

Let $(A_1, j) = 0, 1, 2, ..., \infty$ be a family of sets each of which consists of formulas of $T$. We shall say that $(A_1)$ is a representable family of consistent extensions of $T$ if, for each $j$,

(a) $\neg \Phi$ implies $\Phi \in A_1$;

(b) $\Phi \in A_1$ implies $\neg \Phi \in A_1$;

(c) $\Phi \in A_1$ and $\Phi \centernot{= \boxtimes} V \in A_1$ imply $\Phi \in A_1$;

(d) $(a) \Phi \in A_1$ implies $\Phi(\Delta_1, j) \in A_1$ for $k, n \in N_0$;

(e) there is a ternary relation $C$ such that $\Phi \in A_1 = (Ep)(p \in P) \& C(p, j, \Gamma)$.

(f) there are formulas $\Gamma(x, y, z), \Gamma(\Delta, y, z)$ with the free variables indicated which strongly represent relatively to $P \times N_0$ the relations $C(p, j, n)$ and $C^*(p, j, n) = C[p, j, \neg \Phi(n)]$;

(g) $\neg \Pi(x) \& \Pi(\Delta, y) \& M(x, y; t, u; v) \& M(x, y; t, u; v) \iff \Gamma(x, y, z) \equiv \Gamma(\Delta, y, z)$;

$h \Gamma(x, y, z) \equiv \Gamma(\Delta, y, z)$.

We shall denote by $\ll$ the relation $(m, n) \ll (p, q)$ and $(m, n) \not\ll (p, q)$ and by $\bar{M}(x, y; t, u; v)$ the formula $\bar{M}(x, y; t, u; v) \iff \exists \langle p, q \rangle$.

$T'$ strongly represents $\xleftarrow{\lambda}$ relatively to the set $P \times N_0 \times P \times N_0$.

2. In this section we generalize Rosser's proof [9] and obtain

THEOREM 1. If $(A_1)$ is a representable family of consistent extensions of $T$, then there is a closed formula $\Theta$ such that for any $j$, neither $\Theta$ nor $\neg \Theta$ is in $A_j$.

Proof. Let $\sigma(\neg \Phi) = \text{Subst}(\neg \Phi[I, \bar{M}])$ and let $\Sigma(x, y) = \text{Subst}(\neg \Phi)[x, y]$ be a formula which strongly represents $\sigma$ in $T$. Consider the relation $R(I, m, n)$ defined thus:

$$C(I, m, n) \equiv (Ep)(p \in P) \& (p, q) \equiv (I, m) \& C^*(p, q, n)$$

and the formula $\Phi(u, v; y)$:

(3) $\Pi(u, v; y) \equiv (Es, s, t)[H(u) \& M(s, t; u, v) \& \Pi(s, t, y)]$.

We shall show that $\Phi$ strongly represents $\sigma$ relatively to $P \times N_0$. Indeed, if $1 \in P$ and $R(I, m, n)$, then either non-$C(I, m, n)$ or there are $p, q$ such that $p \in P$ and $(p, q) \equiv (I, m)$ and $C(p, q, n)$. In the former case $\neg \Pi(I, m, n)$ is $\Phi(x, y)$, and in the latter $\Pi(I, m, n)$.

Next assume that $I \in P$ and non-$R(I, m, n)$. It follows that $C(I, m, n)$ and non-$C(p, q, n)$ for every pair $(p, q)$ such that $p \in P$ and $(p, q) \equiv (I, m)$. Using (2) and (4) we infer that $\neg \Pi(I, m, n)$ and $\Pi(I, m, n)$.

Thus we have proved

(4) $1 \in P \equiv (R(I, m, n) = \neg \Phi[I, \bar{M}, \bar{M}])$;

(5) $1 \in P \equiv (R(I, m, n) = \neg \Phi[I, \bar{M}, \bar{M}])$.

Let $\sigma$ be the formula $(u, v; z)[H(u) \& M(s, t; u, v) \& \Phi(u, v; y)]$ and $\Theta$ the formula $\text{Subst}(\neg \Phi[I, \bar{M}])$. Hence $\neg \Theta[I, \bar{M}, \bar{M}]$. $\Sigma[I, \bar{M}, \bar{M}]$ and we obtain

(6) $\neg \Theta[u, v; \Pi(u, v; y)]$.

Using (3) we obtain by elementary logical transformations

(7) $\neg \Theta[u, v; \Pi(u, v; y)]$.

$\equiv (Es, s, t)\Pi(s, t; u, v) \& \Pi(s, t, y)$,
This proves that

\[ \vdash \sim \Theta = \Theta(u, v)[M(u, v; A_{p_1}, A_{j_1}) \land \Gamma(u, v)] , \]

and so the right-hand side of this equivalence belongs to \( A_H \). Using IV (iii) we infer that it is not for every pair \((p, j)\) with \( p \in P \) and \((p, j) \prec (p_1, j_1)\) that the formula \( \vdash \sim \Gamma(\Theta(u, v)) \) holds. Hence there is a pair \((p_2, j_2)\) such that \( p_2 \in P \) and \((p_2, j_2) \prec (p_1, j_1)\) and

\[ \vdash \Gamma(\Theta(u, v)) \cap M(s, t; A_{p_2}, A_{j_2}) \cap \Gamma(\Theta(u, v)) . \]

This is a contradiction since, by (12), \((p_2, j_2) \prec (p_1, j_1)\), and hence, by (11), \( \vdash \Gamma(\Theta(u, v)) \cap M(s, t; A_{p_2}, A_{j_2}) \cap \Gamma(\Theta(u, v)) \). Let us now assume that \( \sim \Theta \in A_{J_1} \) for some \( j_1 \), i.e., that \( C(p_1, j_1, \sim \Theta) \) for some pairs \((p_1, j_1)\) with \( p_1 \in P \). This gives us formula (13) and we can argue as above and infer that there is a pair \((p_2, j_2)\) with \( p_2 \in P \) such that (14). This proves that \( \vdash \sim \Gamma(\Theta(u, v)) \cap M(s, t; A_{p_2}, A_{j_2}) \cap \Gamma(\Theta(u, v)) \), which contradicts our former result.

Theorem 1 is thus proved. From its proof we also obtain

**Theorem 1**. If \( \Gamma \) and \( \Gamma^* \) are arbitrary formulas satisfying (1) and (g), then the formula \( \Theta \) defined by (3) and (6) is undecidable in any \( A_{J_1} \).

3. In this section we shall add two more assumptions to our assumptions 1-IV concerning the theory \( T \):

V. For every primitive recursive function \( f(x_1, \ldots, x_n) \) there is a formula \( a(x_1, \ldots, x_n; y) \) which strongly represents \( f \) and satisfies the condition

\[ \vdash \sim \Theta(x_1, \ldots, x_n; y) \cap \sim a(x_1, \ldots, x_n; y) \cap \sim a(x_1, \ldots, x_n; y) \].

VI. For every \( k > 0 \) there is a formula \( H^k(x_1, \ldots, x_k) \) with the variables indicated such that:

(i) there is a primitive recursive function \( f \) for which

\[ \vdash H^k(\xi(f, 0, \ldots, 0), \xi(f, k, \ldots, 0), \ldots, \xi(f, k^2, \ldots, 0)) \cap H^k(\xi(f, 0, \ldots, 0), \xi(f, k, \ldots, 0), \ldots, \xi(f, k^2, \ldots, 0)) \],

(ii) there is a primitive recursive function \( r(j) \) such that

\[ \vdash H^k(\xi(r, 0), \xi(r, 1), \ldots, \xi(r, k)) \]

\[ = (\xi(r, 0), \xi(r, 1), \ldots, \xi(r, k)) \]

\[ \vdash H^k(\xi(\xi(r, 0), \xi(r, 1), \ldots, \xi(r, k)), \xi(\xi(r, 0), \xi(r, 1), \ldots, \xi(r, k)), \ldots, \xi(\xi(r, 0), \xi(r, 1), \ldots, \xi(r, k))) \],

(iii) there is a primitive recursive function \( \xi(j, m, n) \) such that, if \( k \geq 2 \),

\[ \vdash H^k(\xi(j, m, n), \xi(j, m, n), \ldots, \xi(j, m, n)) \]

\[ = H^k(\xi(j, m, n), \xi(j, m, n), \ldots, \xi(j, m, n)) \].

**Proof:** This function need not be recursive in \( k \) although in the examples which we shall discuss later this is actually the case.
Let \( \{ A_j \} \) be a representable family of consistent extensions of \( T \) and \( \mathfrak{h} \) an integer such that for every \( j \leq \mathfrak{h}(h) \) the integer \( \mathfrak{h}(h) \) is the Gödel number of a closed formula \( F_h, j \). Let \( F_h, j = F_h, j \) for \( j > \mathfrak{h}(h) \) and let \( \mathcal{A}_{n}^{\mathfrak{h}} = (\forall \mathfrak{a} \in \mathcal{A}) \) if \( \mathfrak{a} \in \mathcal{A} \) where \( \mathcal{K}, L \) are functions inverse to the pairing function \( \mathcal{J}(m, n) = \frac{1}{2}(m + n)(m + n - 1) + n \).

**Lemma 1.** If neither \( F_h, j \) nor \( \neg F_h, j \) belong to \( A_k \) for \( j = 0, 1, 2, ... \), then \( \mathcal{A}_{n}^{\mathfrak{h}} \) is a representable family of consistent extensions of \( T \).

Proof. Conditions (a)-(d) are obvious. To prove (e) we denote by \( C_k \) the relation

\[
\mathcal{C}(p, L(j), (h), (h)) \cup (K(j) \leq \mathfrak{h}(h))
\]

and easily verify that (e) is satisfied. Finally, to prove (f) and (g) we denote by \( F_k \) the formula

\[
\mathcal{F}(A, B, D) \cup \mathcal{G}(y, z) \cup \mathcal{H}(y, z) \cup \mathcal{I}(y, z) \cup \mathcal{J}(y, z)
\]

and easily verify that (f) is satisfied. Finally, to prove (g) and (h) we denote by \( F_k \) the relation

\[
\mathcal{F}(A, B, D) \cup \mathcal{G}(y, z) \cup \mathcal{H}(y, z) \cup \mathcal{I}(y, z) \cup \mathcal{J}(y, z)
\]

and easily verify that (h) is satisfied.

**Theorem 2.** If \( \{ A_j \} \) is a representable family of consistent extensions of \( T \) satisfying the assumptions of Lemma 2, then there is a formula free for this family.

Proof. Let \( \Phi \) be the formula \( \{(x, z) = \} \) and let \( \delta, \eta, \varepsilon \) be primitive recursive functions defined thus:

\[
\delta(0) = \mathcal{G}(0), \quad \delta(0, 0) = \delta(0), \quad \delta(0, s) = 0 \quad \text{for} \quad s > 0, \quad \varepsilon(0) = 2^{\text{max}},
\]

\[
\delta(k + 1) = \mathcal{G}(u, v) \cup \mathcal{H}(u, v), \quad \delta(k, k) = \mathcal{G}(u, v) \cup \mathcal{H}(u, v),
\]

\[
\eta(k, j) = \begin{cases} \varepsilon(k, j) \text{Con} \delta(k + 1) & \text{for} \quad 0 \leq j < 2^k \varepsilon(k, j + 2^k) \text{Con} \delta(k + 1) & \text{for} \quad 2^k \leq j < 2^{k + 1} \varepsilon(k, j - 2^k) \text{Con} \delta(k + 1) & \text{for} \quad j > 2^{k + 1} \end{cases}
\]

\[
\varepsilon(k + 1) = \prod_{u, v < 2^{k + 1}} p_{u, v}^{\varepsilon(k, j, s)}
\]
We prove by induction on $k$ that for every $k \geq 0$ and every $j$ with $0 \leq j < 2^k$ there are formulas $\Omega_k$ and $\Sigma_{k,j}$ such that $\delta(k) = \Sigma_{k,j} \land \eta(k, j) = \overline{\Sigma_{k,j}} \land \{ \{k\} \} = \overline{\Sigma_{k,j}}$. Moreover, neither $\Sigma_{k,j}$ nor $\overline{\Sigma_{k,j}}$ belongs to $\Delta_k$. For $k = 0$ it is sufficient to take $\Omega_0 = \Sigma_{0,0}$, and for $k > 0$ $\Omega_k = \Omega_{k-1} \land \Sigma_{k,j} \land \Omega_k$ for $0 \leq j < 2^{k-1}$, $\Sigma_{k,j} = \Sigma_{k-1,j} \land \Sigma_{k,j}$ for $2^{k-1} \leq j < 2^k$. Using Lemma 2 we immediately see that neither $\Sigma_{k,j}$ nor $\overline{\Sigma_{k,j}}$ belongs to $\Delta_k$. In order to accomplish the proof it is therefore sufficient to construct a formula $\gamma(x)$ such that

$$\vdash \gamma(\Delta_k) = \theta(\psi) \quad \text{for every } k > 0.$$  

To obtain such a formula let us denote by $G(x, y)$ a formula which strongly represents the function $\delta(x)$ and take as $\gamma(x)$ the formula

$$\gamma(x) \equiv [G(x, y) \land \Pi(x) \supset H(x, \sigma(x), u, v)].$$

We then have $\vdash \gamma(\Delta_k) = \psi(\gamma, v)(\gamma) \supset H(x, \sigma(x), u, v)$, and hence by Lemma 2 we obtain (15). Theorem 2 is thus proved.

4. In the present and in the next sections we shall give examples of theories and families of their extensions to which the foregoing theory is applicable.

Denoting by $J(m, n)$ the pairing function $\frac{1}{2}(m+n)(m+n-1)+n$ we put $J(m_1, m_2, \ldots, m_n) = J(J(m_1, m_2, \ldots, m_{n-1}), m_n)$. For every $k, m (k \geq 1)$ there are uniquely determined integers $m_j = K(m_j)$ such that $m = J(m_1, m_2, \ldots, m_n)$.

Let $R^*$ be a theory which differs from $R$ (cf. [11], p. 53) by containing the three new operation symbols $\iota, \kappa, \lambda, \lambda$ and axioms

$$(\Omega_0) \quad \iota(x, y) = \psi(\iota, y) \equiv [(x=y=0) \lor (x=0) \land \psi(x\neq y)] \land \iota, \kappa, \lambda \in \{ x \neq y \} \land \iota, \kappa, \lambda \in \{ x \neq y \} \land \iota, \kappa, \lambda \in \{ x \neq y \} \land \iota, \kappa, \lambda \in \{ x \neq y \} \land \lambda.$$  

$$(\Omega_1) \quad \lambda(x, y) = x,$$

$$(\Omega_2) \quad \lambda(x, y) = x,$$

$$(\Omega_3) \quad \lambda(x, y) = y,$$

$$(\Omega_4) \quad \lambda(x, y) = \lambda(x, y) \land (y < x) \supset (x = y).$$

$$(\Omega_0) \quad \lambda(x, y) \land (y < x) \supset (x = y).$$

**Lemma 3.** The following formulas are provable in $R^*$

$$z = \kappa(\lambda, \lambda) = z = \lambda(\lambda, \lambda),$$

$$z = \lambda(\lambda, \lambda) = z = \lambda(\lambda, \lambda).$$

Indeed, writing the right-hand side of $\Omega_0$ as $(E, v)(x, \psi, x, y, z)$ we easily infer that the formula

$$z = \lambda(\lambda, \lambda) = z = \lambda(\lambda, \lambda)$$

is provable in $R$. This shows that the first equivalence is provable. The second is not provable from $\Omega_0$ or $\Omega_0$.

**Lemma 4.** Theory $R^*$ and every theory $T$ (with standard formalization) in which $R^*$ is interpretable satisfy conditions I, II, III.

Proof. I is obvious and I is implied by $V$, whereas it remains to prove II. Let $f(n_1, \ldots, n_k)$ be a primitive recursive function and $\Phi(x_1, \ldots, x_k, y)$ a formula which strongly represents $f$. The existence of $\psi$ was proved in [11], pp. 50-60. Take as $\Phi$ the formula $\Phi(x_1, \ldots, x_k, y)$ and $(y)'(y < y) \supset \sim \Phi(x_1, \ldots, x_k, y)'$. Using axiom $\Omega_0$ we easily see that $\Phi(x_1, \ldots, x_k, y)$ is provable in $R^*$.

**Lemma 5.** The set $P = \{ z \}$, the relation $J(m, n) \leq J(p, q)$ and the formulas $z = x$, $x < x, x \leq x$ satisfy conditions III, IV for the theory $R^*$ and its arbitrary consistent extensions.

Proof. III is obvious. It is also obvious that the formulas $z = x$ and $x < x, x < x, x \leq x, x < x, x < x, x \leq x, x < x, x < x, x \leq x, x < x, x < x, x \leq x$ strongly represent the set $P$ and the relation $J(m, n) \leq J(p, q)$ (cf. Lemma 3). Formula IV (i) results from axioms $\Omega_0$ and $\Omega_0$; formula IV (ii) results from axiom $\Omega_0$. Let us finally assume that $\Phi(\lambda, \lambda, \lambda)$ is provable in $R^*$ (or its extension $T$) for arbitrary $p, q$ such that $J(p, q) \leq J(p_0, q_0)$. Since

$$\vdash \psi(x, y) \equiv \lambda(\lambda, \lambda) = \lambda(\lambda, \lambda) = \lambda(\lambda, \lambda)$$

and since every $x \in J(p_0, q_0)$ is representable as $J(p, q)$, we conclude that the formula $(y)'(y < y) \supset \sim \Phi(x, y)$ is provable in $R^*$ (or in $T$).
To obtain a theory to which theorem 2 is applicable we shall add to \( R^* \) several new operation symbols and axioms. The new symbols are: a unary symbol \( x \) and binary symbols \( \pi, \gamma, \sigma \).\( \ast \) We shall write \( \pi^* \) instead of \( \pi(x, y) \). We also introduce the following abbreviations:

\[
\begin{align*}
\pi_0(x) &= x, \\
\pi_{i+1}(x_1, \ldots, x_{i+1}) &= \pi(\pi_i(x_1, \ldots, x_i), x_{i+1}) \\
\gamma_0(x) &= x, \\
\gamma_{i+1}(x) &= \gamma_\lambda(x) \quad \text{for} \quad i = 1, \ldots, k, \\
\sigma_{i+1}(x) &= \sigma_{i+1}(x) \quad \text{for} \quad i = 1, \ldots, k, \\
\lambda(x) &= \lambda(x).
\end{align*}
\]

The new axioms are

\[
\begin{align*}
(\Omega_3) \quad & x \cdot y = 0 = ([x = 0] \lor [y = 0]), \\
(\Omega_4) \quad & \phi([x = z]) \quad \text{for} \quad \phi \in \{\pi, \gamma, \sigma\}, \\
(\Omega_5) \quad & \phi(\lambda, y) = y + \lambda, \\
(\Omega_6) \quad & \phi(\lambda, y) = \phi(y), \\
(\Omega_7) \quad & \phi(\lambda, y) = \phi(\lambda, y) + \phi(\lambda, \lambda), \\
(\Omega_8) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
(\Omega_9) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
(\Omega_{10}) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
(\Omega_{11}) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
(\Omega_{12}) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
(\Omega_{13}) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
(\Omega_{14}) \quad & \phi(\lambda, y) = \phi(\lambda, y) = \phi(\lambda, \lambda), \\
\end{align*}
\]

Let \( R^* \) be a theory with the primitive symbols enumerated above and based on axioms (\(\Omega_3\))-(\(\Omega_{14}\)).

In order to make the content of the axioms (\(\Omega_3\))-(\(\Omega_{14}\)) more accessible we shall sketch informally the proof of the following

**Lemma 6.** Theory \( R^* \) is interpretable in \( P \).

We interpret \( x \cdot y, x \cdot y, \pi^* \) in the usual way, \( \pi(x, y), \pi(x, \lambda) \) as the pairing functions \( J, K, L \) in \( P \) as the excess of \( x \) over the nearest square not greater than \( x \). \( \pi(x, y) \) is interpreted as the function \( U(n, m) = U_0(m) \) defined by induction. The values of \( U_0(m) \), \( U_0(m) \) are determined according to \( (\Omega_{15}) \), \( (\Omega_{16}) \), the values of \( U_0(m), U_1(m), U_2(m), U_3(m) \) are arbitrary, the values of \( U_0(n+1), U_0(m) \) for \( f = 0, 1, 2 \) are respectively \( U_0(0), U_0(1), U_0(2) \). The value of \( U_1(n+1) \) is either the least \( p \) such that \( U_0(p) = m \) or \( \beta \) if such a \( p \) does not exist. The value of \( U_2(n+1) \) is equal to \( \sum_{j=0}^{\beta} U_0(L(n, f)) \) and the value of \( U_3(n+1) \) is equal to \( \prod_{j=0}^{\beta} U_0(L(n, f)) \), where the product is extended over pairs \( u, v \) such that \( J(u, v) < J(L(n, f)) \), \( L(n, f) \). Finally, the value of \( U_4(n+1) \) is \( U_0(f(n, p)) \) and the values of \( U_4(n+1) \) for \( f = 1, 2, 3, 4 \) are arbitrary.

**Lemma 7.** For every general recursive function \( f(n_1, \ldots, n_k) \) there is an integer \( e \) such that

\[
\neg \phi(A_1, A_2, \ldots, A_k) = A_{n=0}(n_0).
\]

(\( \neg \phi \) means here "provably in \( R^* \)).

**Proof.** First assume that \( k = 1 \) and let \( \mathcal{F} \) be the family of functions \( f(n) \) such that there is an \( e \) satisfying

\[
\neg \phi(A_1, A_2) = A_{n=0}(n_0)
\]

for each \( n \). Functions \( S(n) = n + 1 \) and \( E(n) = \text{excess of } n \) over the nearest lower square belong to \( \mathcal{F} \) because we can take \( e = 0 \) or \( e = 1 \). If \( f, g \in \mathcal{F} \), then there are integers \( e, f \) such that (16) and the following formula (17) hold for all \( n \):

\[
\neg \phi(A_1, A_2) = A_{n=0}(n_0),
\]

whence by axioms (\(\Omega_{15}\)), (\(\Omega_{16}\)), and (\(\Omega_{17}\)),

\[
\neg \phi(A_1, A_2, A_3) = A_{n=0}(n_0),
\]

and hence the functions \( f(n) + g(n) \) and \( f(g(n)) \) belong to \( \mathcal{F} \).

If \( f \) is in \( \mathcal{F} \) and \( f \) assumes all natural numbers as values, then using axioms (\(\Omega_{18}\)), (\(\Omega_{19}\)), and (\(\Omega_{20}\)) we easily obtain

\[
\neg \phi(A_1, A_2, A_3) = A_{n=0}(n_0),
\]

and hence \( f^{-1} \in \mathcal{F} \). Thus \( \mathcal{F} \) contains all general recursive functions of one argument.

If \( k > 1 \) and \( f(n_1, \ldots, n_k) \) is general recursive, then so is \( g(m) = f(K_1(m), \ldots, K_k(m)) \) and hence there is a \( d \) satisfying (17). Substituting \( n = J_1(n_1, \ldots, n_k) \) we obtain the desired result.

Let \( H^n(a_1, a_2, \ldots, a_k) \) be the formula \( \phi(x, u(a_1, \ldots, a_k)) = 0 \).
LEMMA 8. Formulas $H^{(n)}$ satisfy condition VI.

Proof. VI (i) is satisfied with $\sigma(x, y, z) = 12y[1, 1] + 7$; in the proof we use axioms $(\Omega_2)$ and $(\Omega_3)$.

Formula VI (ii), which becomes in this case

$$\vdash H^{(n)}(d_{j+2}, x, t_1, t_2, t_3, t_4) = \text{Eval}(x, y) \{ \{ \{ x < y \} \} \}
\& H^{(2)}(d_{j+1}, x, y, t_1, t_2),$$

is but a different formulation of $(\Omega_4)$ for $x = d_{j+1}$ we get $r(j) = 12j + 10$.

From $(\Omega_4)$, $(\Omega_2)$, and $(\Omega_3)$ we obtain by substitution $\sigma = d_{j+1}$, $y = t_{j+1}$, $12j + 10$. Then

$$\vdash \varphi(d_{j+1}, x, t_1, t_2, t_3, t_4) = 0 \Rightarrow \varphi(d_{j+1}, t_{j+1}, t_2, t_3, t_4, d_{j+1}) = 0$$

and repeating the same argument

$$\vdash \varphi(d_{j+1}, x, t_1, t_2, t_3, t_4) = 0 \Rightarrow \varphi(d_{j+1}, t_{j+1}, t_2, t_3, t_4, d_{j+1}) = 0$$

where $\psi(j, m, n) = 6 \cdot 2^{m+1}(2j + 1) + 1$. Thus formula VI (iii) is provable.

THEOREM 4. For every recursively enumerable family $(A_j)$ of consistent sets containing axioms of $R^{**}$ and closed with respect to the rules of proof there is a formula free for that family.

Proof. Relations $\delta$, $\gamma$, defined in the proof of lemma 1 are in the present case recursive in the four arguments $p, j, i, n$. By lemma 7 there are integers $r_1, r_2$ such that the formulas $H^{(2)}(d_{j+1}, x, y, z, d_{k+1})$ and $H^{(2)}(d_{j+1}, x, y, z, d_{k+1})$ strongly represent the relations $\delta(p, j, i, n)$ and $\gamma(p, j, i, n)$. These formulas satisfy condition (g) (because with our choice of formulas $M$ and $M$) the antecedent of both formulas in (g) becomes (by $(\Omega_4)$) $i(x, y) = i(x', y')$, and thus (by $(\Omega_2)$ and $(\Omega_3)$) is equivalent to $(x = x')$). This proves that theorem 4 follows from theorem 4.

COROLLARY 1. There is a formula free for the theory $P$ and each of its recursively enumerable extensions.

The corollary follows from theorem 4 and lemma 6.

COROLLARY 2. There is a formula free for every sub-theory of $P$.

Indeed, a formula free for a theory is free for an arbitrary sub-theory. It is an open question whether for every recursively enumerable extension of $R$ there is a formula free for that extension.

5. In this section we shall briefly discuss a theory $F$ obtained from $R^{**}$ by enlarging the set of axioms by all formulas $(\sigma)_A(x)$ such that $A(x)$ is provable in $R^{**}$. For $n = 0, 1, 2, ..., (cf. [3]). Let \(On_\sigma(X)\) denote the smallest set containing $X$ and the axioms of $F$ and closed with respect to the rules of proofs of $F$.

THEOREM 5. If $\{B_i\}$ is a recursively enumerable family of sets consistent in $F$ each of which consists of closed formulas of $F$, then there is a formula free for the family $A_1 = On_\sigma(B_1)$.

Proof. Choose $P, \leq, H$, and $M$ as in lemma 5. It is obvious that conditions I-V are satisfied. It is also obvious that the family $(A_j)$ satisfies conditions (a)-(d) and (g).

If $R$ is a recursive subset of $\mathbb{N}^{n+1}$, then the set

$$\{(a_1, ..., a_n) : (E)(\sigma)(\sigma)((x, y, z, t_1, ..., t_n) \in K)$$

is weakly representable in $F$ and the set

$$\{(a_1, ..., a_n) : \forall(x, y, z, t_1, ..., t_n \in K)$$

is strongly representable in $F$. Actually sets of the form (a) are most general sets weakly representable in $F$ but we shall not need this fact in our discussion.

Let $\langle W \rangle$ be the set of Gödel numbers of formulas with one free variable $\sigma$, and $\Gamma$—the provability relation for $F^{**}$. Both sets are obviously primitive recursive.

$$\Phi \ast A_1 = (E)(\sigma)(\sigma)[(x, y, z, t_1, ..., t_n) \in W] \& (E)(\sigma)(\sigma)(x, y, z, t_1, ..., t_n) \in K)$$

it follows that $(A_j)$ satisfies conditions (c) and (t).

Let $H^F(\sigma, t_1, ..., t_n)$ be the formula $(\sigma)(E)(\sigma)(\sigma)((x, y, z, t_1, ..., t_n) \in W) = 0$.

It is obvious that every set of the form (a) with a general recursive $\sigma$ is strongly representable in $F$ by a formula $H^F(\sigma, t_1, ..., t_n)$. It follows that there are integers $r_1, r_2$ such that the formulas $H^F(\sigma, t_1, ..., t_n)$ and $H^F(\sigma, t_1, ..., t_n)$ strongly represent the relations $\delta$ and $\gamma$, respectively, of $A_j$.

Finally it is not difficult to show that conditions VI (i)-(iii) are satisfied with our choice of the formulas $H^F$. Thus by theorem 2 the assertion of theorem 5 is proved.

6. In this section we shall deal with the system $A_n$ of analysis defined in [2] and with its extensions. It will be convenient to eliminate from $A_n$ function variables with more than one argument. Since the pairing functions are definable in $A_n$, it is clear that this simplification of $A_n$ is not essential. A further (essential) change is that we shall add to the axioms of $A_n$ the following weak form of the axiom of choice (9):

$$(A): (x)(\sigma)(\xi)(\sigma)(\sigma)(\sigma)(\sigma)(\sigma)(\sigma)(\sigma)(\xi)\beta(x, \beta) = y(A_1, y), \beta) \ast \Phi(x, \beta).$$

Footnote 9: We use in $A_n$ the notation of [2] with the only change that the $n$-th numeral is denoted by $\kappa_n$ and multiplication by juxtaposition of terms.

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It is known that for every arithmetically definable relation (function) there is an elementary propositional (or numerical) formula which strongly represents the relation (function) in $A_\omega$. We shall use the notation $\bar{p}(x, y)$, $i(x, y, z), \bar{A}(x)$ for elementary (numerical) formulas which represent in $A_\omega$ the following functions: $U_\omega(m)$ (cf. the proof of lemma 6), $F(m, n)$, $E(m, l)(m)$.

Let $W$ be the set of integers $c$ such that the relation $\leq c$ defined as $U_\omega(d(m, n)) = 0$ is a well-ordering of $N_\omega$. The order type of $\leq c$ will be denoted by $|c|$. Since the relation $\leq c$ is recursive, it follows that $|c| = c_1$ (cf. [5], p. 412). On the other hand, every recursive relation is representable as $U_\omega(d(m, n)) = 0$ for a suitable $c$ (cf. lemma 7), and hence, by a theorem of Markwald ([7], p. 142) every infinite $x < c_1$ is representable as $|c|$ for a suitable $c$ in $W$. Thus we have proved

**Lemma 9.** $(|c|: c \in W) = (x: x < c_1)$.

**Lemma 10.** $W$ is weakly representable in $A_\omega$ by a formula $\Pi(x)$ of the form $(\alpha, \beta)C_\omega(\alpha, \beta, x)$, where $C_\omega$ is an elementary formula.

**Proof.** Let $ord(a)$ be the formula

$$[(\alpha, \beta)(x, y, z) = 0 \land \alpha(t(x, z)) = 0] \lor [\alpha(t(y, z)) = 0 \lor \alpha(t(x, z)) = 0] \lor \exists \alpha[\alpha(t(x, y)) = 0 \land \alpha(t(y, z)) = 0 \lor \alpha(t(x, z)) = 0] \lor \exists \alpha[\alpha(t(x, z)) = 0 \land \alpha(t(y, z)) = 0 \lor \alpha(t(x, z)) = 0]$$

and $F(\alpha, \beta)$ the formula

$$[(\alpha, \beta)(x, y, z) = 0 \lor \exists \alpha[\alpha(t(x, z)) = 0 \lor \alpha(t(y, z)) = 0 \lor \alpha(t(x, z)) = 0] \lor \exists \alpha[\alpha(t(x, z)) = 0 \lor \alpha(t(y, z)) = 0 \lor \alpha(t(x, z)) = 0]$$

Take as $C_\omega(\alpha, \beta, x)$ the formula

$$(x, y) [\alpha(t(x, y)) = 0 \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0] \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0]$$

We immediately see that the formula $(\alpha, \beta)C_\omega(\alpha, \beta, A_\omega)$ is true in the principal model $R_\omega$ (cf. [2], p. 190) if and only if $c \in W$. Hence, by theorem 3.13 of [2]

$c \in W$ if and only if $\exists \alpha(\beta)(x, y, z)$.

This proves the lemma.

Let $Im(a, x, y)$, $Im(a, y, x, z)$ be the following formulas

$$(a, x, y) [\bar{y}(x, y, z) = 0 \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0] \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0]$$

We assert that the formula $\alpha C_\omega(\alpha, \beta, A_\omega)$ is true in the principal model $R_\omega$ (cf. [2], p. 190) if and only if $c \in W$. Hence, by theorem 3.13 of [2]

$c \in W$ if and only if $\exists \alpha(\beta)(x, y, z)$.

This proves the lemma.

Let $Im(a, x, y)$, $Im(a, y, x, z)$ be the following formulas

$$(a, x, y) [\bar{y}(x, y, z) = 0 \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0] \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0]$$

We assert that the formula $\alpha C_\omega(\alpha, \beta, A_\omega)$ is true in the principal model $R_\omega$ (cf. [2], p. 190) if and only if $c \in W$. Hence, by theorem 3.13 of [2]

$c \in W$ if and only if $\exists \alpha(\beta)(x, y, z)$.

This proves the lemma.

**Lemma 11.** If $\phi$ is a function and $e_1, e_2, n$ are integers, then $\forall x, y, z$ satisfy $Im(a, x, y)$ in $R_\omega$ if and only if $\phi$ maps $N_\omega$ into itself and $\phi(\alpha, \beta) = \phi(\alpha(\beta, x) = 0 \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0] \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0]$.

We assert that the formula $\alpha C_\omega(\alpha, \beta, A_\omega)$ is true in the principal model $R_\omega$ (cf. [2], p. 190) if and only if $c \in W$. Hence, by theorem 3.13 of [2]

$c \in W$ if and only if $\exists \alpha(\beta)(x, y, z)$.

This proves the lemma.

Let $Im(a, x, y)$, $Im(a, y, x, z)$ be the following formulas

$$(a, x, y) [\bar{y}(x, y, z) = 0 \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0] \lor \exists \alpha[\alpha(t(x, y)) = 0 \lor \alpha(t(x, y)) = 0]$$

We assert that the formula $\alpha C_\omega(\alpha, \beta, A_\omega)$ is true in the principal model $R_\omega$ (cf. [2], p. 190) if and only if $c \in W$. Hence, by theorem 3.13 of [2]

$c \in W$ if and only if $\exists \alpha(\beta)(x, y, z)$.

This proves the lemma.
(ξ − η is here an elementary numerical formula which represents the function as η in ∆).

Proof. The relation (m = n) ∨ (m < n < p) ∨ (p ≤ m) & (n < p) is recursive and hence equivalent to $U_n[f_d(n, m, p)] = 0$ for a suitable $\gamma$. The proof of this equivalence being formalized in $A_w$, we obtain

$$\vdash \varphi(\delta, f, y, v) = 0 = (x = y) \forall (x < y) & (y < v) \forall (v < z) & (y < \alpha).$$

Since axiom (Ω) of $R^\star$ is provable in $A_w$, we obtain

$$\vdash \varphi(\delta, f, y, v) = 0 = (x = y) \forall (x < y) & (y < \alpha) \forall (\delta < x) & (y < \alpha),$$

where $f_d(n) = 6 \cdot 2^n + 1$. Arguing similarly we obtain primitive recursive functions $f_3(n), f_4(n), f_5(n), f_6(n)$ such that

$$\vdash \varphi(\delta, f_3(n), i(x, y)) = 0 = (x = y) \forall (x < y) & (y < \alpha),$$

$$\vdash \varphi(\delta, f_4(n), i(x, y)) = 0 = \varphi_2(\gamma(x, y)) \forall (x < y) & (y < \alpha),$$

(17)

$$\vdash \varphi(\delta, f_5(n), i(x, y)) = 0 = \varphi_2(\gamma(x, y)) \forall (x < y) & (y < \alpha).$$

From the last formula and from (Ω), which is valid in $A_w$ we obtain

$$\vdash \varphi(\delta, f_6(n), i(x, y)) = 0 = \varphi_2(\gamma(x, y)) \forall (x < y) & (y < \alpha).$$

where $f_d(n) = 12J_f, f_3(n) + 8$. Using this formula and (17) and observing that $\vdash \varphi(x + y = 0) = (x = 0) \forall (y = 0)$ and that the axiom (Ω) is valid in $A_w$, we obtain

$$\vdash \varphi(\delta, f_6(n), i(x, y)) = 0 = \varphi_2(\gamma(x, y)) \forall (x < y) & (y < \alpha).$$

Continuing in this way finally we obtain a primitive recursive function $f(e, n) = f_d(e, n)$ such that if $\Gamma$ represents $f$ in $\Delta$, then (18) is valid for $w = \Delta$, $v = \Delta$, $e = 0, 1, 2, \ldots$.

Using rule (ω) we see that (18) is valid for this choice of $f$.

LEMMA 15. $\vdash \Pi(w) \supset \Pi(I(w, v))$.

To prove this lemma we formalize in $A_w$ the set-theoretical theorem saying that the formula on the right-hand side of (18) defines a well-ordering of $\mathfrak{N}_w$ whenever $\varphi(m, i(x, y)) = 0$ does so.

LEMMA 16. If $v \in \mathfrak{W}$, then $f(e, n) \in \mathfrak{W}$ and $f(e, n) = \omega(N) + n$.

Proof. If $\omega_c$ has the order type $\xi$, then the formula on the right-hand side of (18) with $v$ replaced by $\Delta$ and $v$ replaced by $\Delta$ defines a relation of type $\omega: \xi + n$.

LEMMA 17. $\vdash \Pi(w) \supset \Pi(w') \supset \Pi(w, v) \supset \Pi(w', v) \supset \Pi(w, v) \supset \Pi(w', v)$.

We prove by formalizing in $A_w$ the proof of a set-theoretical theorem stating that $\omega: \xi + n = \omega: \xi' + n$ implies $\omega = \omega'$ and $\xi = \xi'$.

We now take $P \in \mathfrak{W}$ and define $\omega = P \supset \Pi(w, q)$. Let $\Gamma$ be the formula weakly representing $P$ constructed in lemma 10 and let $\Delta$ be the formula $\Pi(x, y) \supset \Gamma(x, t)$.

LEMMA 18. Conditions III and IV are satisfied for the above choice of $P$, $\Pi$, $\Gamma$, and $\Delta$.

Proof. III is obvious. In lemmas 15, 13, and 16 we proved that $\Pi$ weakly represents $P$ and $\Gamma$ strongly represents $\Pi$ relatively to $P \times \mathfrak{N}_w \times P \times \mathfrak{N}_w$. Formula IV (i) follows from lemma 17 and IV (ii) from lemmas 15 and 12 (ii). Thus it remains to prove IV (iii).

Assume that $E \in \mathfrak{W}$ and that $\vdash \Pi(A, \Delta) \supset \Pi(w, v)$ for every pair $(p, q)$ in $W \times \mathfrak{N}_w$ such that $f(p, q) \leq f(p, q)$.
It is then easy to prove that
\[ \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

Let \( \tau_i \) (\( i = 1, 2 \)) be integers such that \( \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha) \) represents \( f_i \) in \( \lambda \alpha \) and let \( g(k) = 6 \cdot 2^k \cdot (2k + 1) \). From axiom (\( \Omega_{\alpha} \)) it follows that \( \varphi(\alpha, \lambda \beta, \lambda \alpha, \lambda \beta) \) represents \( f_i(n, n, k + 1) \) considered as the function of \( n, m \) along with using rule \( \omega \) so we infer that
\[ \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

Obviously in both formulas we can replace \( t \) by \( \lambda \alpha \). From these formulas we infer that (19) (implies in \( \lambda \alpha \)) the following formula
\[ (E\beta)(t, t) D(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

Conversely this formula implies (19), as we easily see using the theorem
\[ \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

We can simplify the formula obtained above by observing that
\[ \vdash \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha), \quad \vdash \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha), \quad \vdash \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha). \]

In view of lemma 19 it will be sufficient to prove only the second part of the lemma. The formula in square brackets on the right-hand side is equivalent to
\[ (E\beta)(t', t'', t', t') D(\alpha, \beta, t', t') \supset \varphi(\alpha, \beta, t', t'), \quad \vdash \varphi(\alpha, \beta, t', t'), \quad \vdash \varphi(\alpha, \beta, t', t'). \]

Consider the primitive recursive functions
\[ b(m, q) = \prod_{i\in \mathbb{N}} \rho_i^{m_i}, \quad f_i(m, q, k) = \prod_{i\in \mathbb{N}} \rho_i^{m_i q + k}. \]

Let \( \tau_i \) (\( i = 1, 2 \)) be integers such that \( \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha) \) represents \( f_i \) in \( \lambda \alpha \) and let \( g(k) = 6 \cdot 2^k \cdot (2k + 1) \). From axiom (\( \Omega_{\alpha} \)) it follows that \( \varphi(\alpha, \lambda \beta, \lambda \alpha, \lambda \beta) \) represents \( f_i(n, n, k + 1) \) considered as the function of \( n, m \) along with using rule \( \omega \) so we infer that
\[ \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

Obviously in both formulas we can replace \( t \) by \( \lambda \alpha \). From these formulas we infer that (19) (implies in \( \lambda \alpha \)) the following formula
\[ (E\beta)(t, t) D(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta), \quad \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

Conversely this formula implies (19), as we easily see using the theorem
\[ \vdash \varphi(\alpha, \beta, \lambda \alpha, \lambda \beta) \supset D(\alpha, \beta, \lambda \alpha, \lambda \beta). \]

We can simplify the formula obtained above by observing that
\[ \vdash \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha), \quad \vdash \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha), \quad \vdash \varphi(\alpha, \lambda \alpha, \lambda \beta, \lambda \alpha). \]

In view of lemma 19 it will be sufficient to prove only the second part of the lemma. The formula in square brackets on the right-hand side is equivalent to
\[ (E\beta)(t', t'', t', t') D(\alpha, \beta, t', t') \supset \varphi(\alpha, \beta, t', t'), \quad \vdash \varphi(\alpha, \beta, t', t'), \quad \vdash \varphi(\alpha, \beta, t', t'). \]

Consider the primitive recursive functions
\[ b(m, q) = \prod_{i\in \mathbb{N}} \rho_i^{m_i}, \quad f_i(m, q, k) = \prod_{i\in \mathbb{N}} \rho_i^{m_i q + k}. \]
Proof. Again we shall prove only the second equivalence. Let \( S'(y, \beta_1, \beta_2) \) be the formula \( \{ y(z) = i(\beta_1(z), \beta_2(z)) \} \) and \( S'(s, s', s'') \) the formula \( \{ k(s) = \lambda(s') \wedge (s) \} \). It is obvious that
\[
\vdash \alpha(\beta_1, \beta_2) \langle E \rangle S(y, \beta_1, \beta_2), \quad \vdash \alpha(\gamma) \langle E \rangle \beta_1, \beta_2) \langle E \rangle S(y, \beta_1, \beta_2) \]
and
\[
\vdash D(y, s, s', t) \wedge D(\beta_1, s', t) \wedge D(\beta_2, s', t) \wedge S'(s, s', s'') \]
\[
\vdash S'(s, s', s'').
\]
If \( m, m', m'' \) satisfy \( S' \) in \( R_n \), then \( m' = E_{(m_0, \ldots, m'_0, \ldots)} \cdot E_{(m_0, \ldots, m'_0, \ldots)} \cdot E_{(m_0, \ldots, m'_0, \ldots)} \cdot E_{(m_0, \ldots, m'_0, \ldots)} = f''(m) \), \( m'' = E_{(m_0, \ldots, m'_0, \ldots)} \cdot E_{(m_0, \ldots, m'_0, \ldots)} \cdot E_{(m_0, \ldots, m'_0, \ldots)} = f''(m) \). The function \( f', f'' \) are primitive recursive; let \( r_1, r_2 \) be integers such that \( \varphi(\lambda, s) \) represents \( f' \) in \( A_m \) and \( \varphi(\lambda, s) \) represents \( f'' \) in \( A_m \). We then have
\[
\vdash S'(s, \varphi(\lambda, s), \varphi(\lambda, s)) \quad \text{and} \quad \vdash \alpha(\gamma) \langle E \rangle S'(s, s', s'').
\]
It is now easily seen that the right-hand side of the second formula in the lemma is equivalent to
\[
(D) \{ x_1, \ldots, \}
\]
and
\[
\vdash D(y, s, s', t) \wedge S'(s, s', s'').
\]
We reduce this formula to the desired form as follows: since the function \( \delta \) is primitive recursive, there is an integer \( r' \) such that
\[
\vdash \varphi(\lambda, s) = \mathcal{T}(\varphi(\lambda, s), \ldots, \varphi(\lambda, s)) = 0
\]
This gives
\[
\vdash \varphi(y, \varphi(\lambda, s), \ldots, \varphi(\lambda, s)) = 0
\]
This is equivalent to \( D(\alpha, \beta, \gamma, s, t) \)
\[
\vdash D(y, s, s', t) \wedge S'(s, s', s'').
\]
Hence \( \alpha(\gamma) \langle E \rangle S(y, \beta_1, \beta_2, \gamma) \)
\[
\vdash \varphi(y, s, s', t) = 0
\]
\[
\vdash \varphi(y, s, s', t) = 0.
\]
Lemma 29. There are primitive recursive functions \( \varphi, \varphi_1, \varphi_2 \) such that if \( l \geq 1 \) and \( \alpha, \beta \) are numerical formulas representing \( \varphi_1 \) and \( \beta_1 \), then
\[
\vdash \alpha(\beta_1, \beta_2) \langle E \rangle S(y, \beta_1, \beta_2), \quad \vdash \alpha(\gamma) \langle E \rangle \beta_1, \beta_2) \langle E \rangle S(y, \beta_1, \beta_2)
\]
and
\[
\vdash D(y, s, s', t) \wedge D(\beta_1, s', t) \wedge D(\beta_2, s', t) \wedge S'(s, s', s'') \]
\[
\vdash S'(s, s', s'').
\]
Proof. By axiom \((\Omega_0)\) we obtain
\[
\vdash \varphi(s, s_0, s_1, \ldots, s_{n-1}) = \varphi_1(s_0, t) \wedge \varphi_2(s_1, \ldots, s_{n-1})
\]
\[
\vdash \varphi(\alpha, \beta, \gamma, s, t) \wedge S'(s, s', s'') \Rightarrow \varphi(s, s_0, s_1, \ldots, s_{n-1}) = 0.
\]
Using axiom \((\Omega_0)\) of \( A_m \) (cf. p. 217) we transform this formula to
\[
\vdash \varphi(\alpha, \beta, \gamma, s, t) \wedge S'(s, s', s'') \Rightarrow \varphi(s, s_0, s_1, \ldots, s_{n-1}) = 0.
\]
Put \( \varphi(m, n) = \{ \lambda(m) \cdot s^{n-1} \} \), \( h(m, n) = \sum_{n < \xi < m} p_{(m, n)}a_{m+n} \), and \( r \) be an integer such that \( \varphi(\lambda, \xi, y, y) \) represents \( A_m \) in the function
\[
\sum_{n < \xi < m} p_{(m, n)}a_{m+n} \cdot g(\lambda, \xi, y, y).
\]
We shall show (informally) that (21) is equivalent to
\[
\varphi(\alpha, \beta, \gamma, s, t) \wedge S'(s, s', s'') \Rightarrow \varphi(s, s_0, s_1, \ldots, s_{n-1}) = 0.
\]
Assume (21) and choose \( s, s_1, s_2 \) so that \( D(\alpha, \beta, \gamma, s, t) \) and \( s_0 < t \).
If \( \beta(\gamma) = \varphi(\alpha, \beta, \gamma, s, t) \) satisfies the condition
\[
\varphi(\alpha, \beta, \gamma, s, t) \wedge S'(s, s', s'') \wedge \varphi(s, s_0, s_1, \ldots, s_{n-1}) = 0
\]
and hence, by (21), we obtain
\[
\varphi(s, s_0, s_1, \ldots, s_{n-1}) = 0.
\]
The converse implication is proved similarly. It remains to reduce (22) to the form indicated in the lemma. By the technique already used we find a primitive recursive function \( \varphi_2 \) such that if \( \xi \) is an elementary formula representing \( \varphi_4 \), then
\[
\varphi(s, s_0, s_1, \ldots, s_{n-1}) = 0
\]
It follows by axiom \((\Omega_\alpha)\) that
\[
\vdash \exists \bar{z} \left[ \bar{z} \in \text{D}(y, \bar{d}) \right] = 0
\]
\[
= \left[ \left( \exists \bar{z} \right) \left[ \bar{z} \in \text{D}(y, \bar{d}) \right] = 0 \right].
\]

The left-hand side of this equivalence can obviously be represented in the form \(\exists \bar{z} \left[ \bar{z} \in \text{D}(y, \bar{d}) \right] = 0\) where \(\bar{z}\) is an elementary term representing a primitive recursive function. Since \(\vdash \exists \bar{z} \left[ \bar{z} \in \text{D}(y, \bar{d}) \right] = 0\), we see that (22) is equivalent to
\[
H^{(\text{D}(y, \bar{d}))}(y, \bar{d}), a, a_0, a_1, a_2, \ldots, a_{n-1}, a_{n-2}, \ldots, a_{n-3}).
\]

**Lemma 24.** There are primitive recursive functions \(g_1, g_2, \ldots, g_k\) such that if \(\eta, \eta_1, \xi, \eta_2, \bar{z}\) are elementary terms representing them in \(\text{A}_n\), then for \(j = 0, 1, \ldots, \lambda - 1\)
\[
\vdash H^{(\text{D}(y, \bar{d}))}(y, \bar{d}), a, a_0, a_1, a_2, \ldots, a_{n-1}, a_2, a_3, \ldots, a_{n-1}).
\]

The proof is similar to that of previous lemmas.

**Lemma 25.** For every \(l\) there are primitive recursive functions \(g_1, g_2, \ldots, g_k\) such that if \(\xi, \lambda, \xi_1, \xi_2, \bar{z}\) are elementary terms representing them in \(\text{A}_n\), then for \(j = 0, 1, \ldots, \lambda - 1\)
\[
\vdash H^{(\text{D}(y, \bar{d}))}(y, \bar{d}), a, a_0, a_1, a_2, \ldots, a_{n-1}, a_{n-2}, \ldots, a_{n-3}).
\]

**Proof.** Let us consider only the second formula. The right-hand side is equivalent to
\[
(\text{D}(z, \bar{d}))_0 \left[ \text{D}(z, \bar{d}), \bar{d}, a, a_0, a_1, a_2, \ldots, a_{n-1}, a_{n-2}, \ldots, a_{n-3} \right] = 0\]

We can replace \(\beta(y, \bar{d})\) by \((\text{D}(y, \bar{d}))_0\) and then use the technique of the preceding lemmas to reduce the right-hand side to the desired form.

**Lemma 26.** Let \(\Gamma\) be an elementary numerical formula and \(\mathcal{C}\) an elementary propositional formula and let the variables (free and bound) occurring in them be some of the variables \(a_0, \ldots, a_{n-1}, a_n, \ldots, a_{n-1}\). Then there exist \(j_1, j_1, j_2, \bar{e}, \bar{e}\) such that
\[
\vdash \Gamma \Rightarrow H^{(\text{D}(y, \bar{d}))}(y, \bar{d}), a, a_0, a_1, a_2, \ldots, a_{n-1}).
\]

**Proof.** From lemmas 23 and 26 it is obvious that if lemma 24 holds for the formula \(\mathcal{C}\), then it does so for the formulas \((\xi_1)\mathcal{C}\) and \((\text{D}(y, \bar{d}))_0\).
It is obvious that we can determine integers $f', f$ such that
\[ \vdash \{ \bar{h}(s) \geq x \} \land \{ [\bar{v} = (s)_{a_0}] \land \bar{v}(A_{f', f} (s, x, \bar{x})) = 0 \} \]
\[ = \bar{v}(A_{f', f} (s, x, \bar{x}, v)) = 0, \]
\[ \vdash \{ \bar{h}(s) < x \} \land \{ [\bar{v} = (s)_{a_0}] \land \bar{v}(A_{f', f} (s, x, \bar{x})) = 0 \} \]
\[ = \bar{v}(A_{f', f} (s, x, \bar{x}, v)) = 0. \]

We thus obtain
\[ \vdash \alpha(\bar{I}_t) = (a) (E_3^1) R^{\alpha(i)} \bar{A}_{f', f} (s, x, \bar{x}, v), \]
\[ \vdash \alpha(\bar{I}_t) = (a) (E_3^1) R^{\alpha(i)} \bar{A}_{f', f} (s, x, \bar{x}, v), \]
whence we obtain the desired result using lemmas 23 and 25.

**Lemma 27.** For every $k, l$ there are primitive recursive functions $g_t, h_t$ such that if $\bar{v}, \bar{v}$ are elementary numerical formulas representing them in $A_{s}$ and $a' = (a_1, \ldots, a_{k-1})$, then
\[ \vdash \bar{h}^{\alpha(k)} (y, A_{s}, a', s) = (a_2) \bar{h}^{\alpha(k)} (y, a, s, \bar{x}), \]
\[ \vdash \bar{h}^{\alpha(k)} (y, A_{s}, a', s) = (a_2) \bar{h}^{\alpha(k)} (y, a, s, \bar{x}), \]

whence we obtain the desired result using lemmas 23 and 25.

**Proof.** We shall prove the first formula. The right-hand side of this formula is equivalent to
\[ (\gamma)(E_3^1) \bar{A}_{f', f} (s, x, y) \bar{S}(\bar{y}, a, \bar{v} \& D(\bar{a}, \bar{v} (x, t) \& \bar{v}(y, t_0 + (s, x))) = 0), \]
where $S$ is the formula used in the proof of lemma 21. Let $h(m, k)$ be the primitive recursive function
\[ h(m, k) = \bar{h}^{\alpha(i)} (y, A_{s}, a', s), \]
where $f'$ and $f''$ have the same meaning as in the proof of lemma 21 and let $r$ be an integer such that $\bar{h}(A_{s}, t_0 (s, y))$ represents $h(m, k)$ in $A_{s}$.

Then
\[ \vdash \bar{S}(\gamma, a, \bar{v} \& D(\bar{a}, \bar{v} (x, t) \& \bar{v}(y, t_0 + (s, x))) = 0), \]
and we infer that (23) is equivalent to
\[ \vdash (\gamma)(E_3^1) \bar{A}_{f', f} (s, x, y) \bar{S}(\bar{y}, a, \bar{v} \& D(\bar{a}, \bar{v} (x, t) \& \bar{v}(y, t_0 + (s, x))) = 0). \]

This formula is reducible to the form required in the lemma in the way used several times in the preceding proofs.

**Lemma 28.** Formulas $H^{\alpha(i)} (y, \bar{x})$ satisfy condition VI.

**Proof.** (i) follows from lemma 21. To prove (iii) we notice that
\[ \vdash \bar{v}(A_{f', f} (s, x, \bar{x}) \bar{v}(\bar{A}_{f', f} (s, x, \bar{x}, v)) = 0, \]
where $\xi$ is primitive recursive (cf. axiom (v)). Hence
\[ \vdash \alpha(\bar{I}_t) = (a) (E_3^1) \bar{h}^{\alpha(i)} (y, A_{s}, \bar{x}, v), \]
\[ \vdash \alpha(\bar{I}_t) = (a) (E_3^1) \bar{h}^{\alpha(i)} (y, A_{s}, \bar{x}, v), \]

The proof is completed by transfinite induction on $\xi$ as follows:

\[ S_{a, \xi} = B_{t}, \quad S_{a, \xi} = \bigcup_{\xi < \lambda} S_{a, \xi} \text{ for limit ordinals } \lambda, \]
\[ S_{a, 0} = \{ \Phi: (\Phi \text{ is a closed formula}) \land (E_3^1)[(\exists \bar{v}(y, A_{s} (x, \bar{x}))) \land \bar{v}(\bar{A}_{f', f} (s, x, \bar{x}, v)) = 0). \]

Thus $S_{a, \xi}$ is the set of (closed) formulas which can be derived from $B_{t}$ by $\xi$ applications of the rule $o$.

Spector [10] proved that $A_{s} = G_{a, \xi} (B_{t}) = S_{a, \xi}$.

Let $F_n (a)$, $T_n (x)$ be elementary formulas which (strongly) represent in $A_{s}$ the set of formulas with one free variable $\bar{a}_i$ and the set of Gödel numbers of closed formulas $\Phi$ such that $\vdash \bar{v}$. 

Further, let \( ab(x, y), gen(x), ximp y \) be elementary numerical formulas which represent in \( A_0 \), the primitive recursive functions \( Sb(1, 1, \Gamma \Phi \bar{P}) \), \( \text{Neg} Br[1, 1, \text{Neg} \Gamma \Phi \bar{P}] \), \( \Gamma \Phi \bar{P} \text{Imp} \Gamma \Phi \bar{P} \) (cf. II, p. 206).

We consider the following formulas:

\[
Z_0(u, x, z) : (y) [\bar{p}(u, i(x, y)) = 0] \quad (x \text{ is the minimal element of } \leq u) ,
\]

\[
Z_0^+(u, x, z') : (y) [\bar{p}(u, i(x, y)) = 0] \equiv [\bar{p}(u, i(y, x')) = 0] \quad (x \text{ is the successor of } x' \text{ in the ordering } \leq u) ,
\]

\[
Z_0^+(u, x, z) : \sim Z_0(u, x, z) \land (y) [\bar{p}(u, i(x, y)) = 0] \land (y \neq x)
\]

\[
\vee (\forall y') [y' \neq y' \land x \neq x' \land [\bar{p}(u, i(x, y')) = 0] \land [\bar{p}(u, i(y', x)) = 0] \]  
\[
(x \text{ is a limit element of the ordering } \leq u) .
\]

Let \( Z(u, u, v) \) be the formula

\[
(z) [Z_0(u, x) \supset (y) [a[i(x, y)] = 0 = \bar{p}(u, i(v, y)) = 0]]
\]

\[
& (x, x') [Z_0(u, x, x') \supset (y) [a[i(x, y)] = 0]] (\forall t) [f(t, x)]
\]

\[
& \land \quad (x, x') [Z_0(u, x, x') \supset (y) [a[i(x, y)] = 0]] (\forall t) [f(t, x)]
\]

\[
(z) [Z_0(u, x) \supset (y) [a[i(x, y)] = 0]] (\forall t) [f(t, x)]
\]

\[
& (x, x') [Z_0(u, x, x') \supset (y) [a[i(x, y)] = 0]] (\forall t) [f(t, x)]
\]

The following lemma explains the meaning of this formula:

**Lemma 29.** Let \( p \) be in \( W \) and let \( \tau, \psi, p \) satisfy \( Z \) in \( R_0 \). Then

\[
[p(U(x, z)) = 0] = [x \text{ is the Gödel number of a formula } \Phi \text{ in } R_{2,3} \text{ where } z_0 \text{ is the order type of a segment of } X_0 \text{ determined by } n \text{ in the well-ordering } \leq u].
\]

Proof by induction on \( z_0 \) presents no difficulties.

**Lemma 30.** \( \sim \sim U(u) \supset (\forall t) [f(t, x)] Z(u, u, v) \).

Proof by the formalization in \( A_0 \) of the usual existence and uniqueness proofs of formulas defined by transfinite induction.

**Lemma 31.** \( \sim \sim U(u) \supset (\forall t) [f(t, x)] Z(u, u, v) \).

Proof by the formalization in \( A_0 \) of the usual existence and uniqueness proofs of formulas defined by transfinite induction.

**Lemma 32.** The formulas

\[
\Gamma(u, v, w) : \{ (a) [Z_0(u, v, w) \supset (\forall z) [a[i(x, w)] = 0]]
\]

\[
\Gamma^*(u, v, w) : \{ (a) [Z_0^+(u, v, w) \supset (\forall z) [a[i[\bar{p}(u, i(x, y)) = 0]]
\]

(where \( \gamma(w) \) is an elementary numerical formula which represents the function \( \text{Neg} Br[1, 1, \text{Neg} \Gamma \Phi \bar{P}] \) strongly represent in \( A_0 \) relatively to \( W \times X^2 \) the relations

\[
C(p, j, n) : (\forall U) [(u) \equiv (\forall t) [f(t, x)] Z_0(u, x) \supset (\forall t) [f(t, x)] Z_0(u, x, v)]
\]

\[
C^*(p, j, n) : (\forall U) [(u) \equiv (\forall t) [f(t, x)] Z_0^+(u, x) \supset (\forall t) [f(t, x)] Z_0^+(u, x, v)]
\]

Proof. Let \( p \in W \), whence \( \sim \sim (A_0) \) (cf. lemma 10). Assume that

\[
\sim \sim (A_0) \sim \sim (A_0)
\]

\( C(p, j, n) \) that \( n = \sim \sim \Gamma \Phi \bar{P} \) and \( \Phi \in \mathcal{R} \) where \( \xi < [p] \). We shall show that \( \Gamma^*(A_0, A_1, A_2) \) is true in \( R_0 \). Indeed, if \( \Phi \) together with \( p, j \) satisfies \( Z \) in \( R_0 \) then there is an integer \( g \) such that (with the notation of lemma 29) \( \xi = \xi_0 \) and hence

\[
\psi[U(q, n)] = 0, \text{ i.e. } \psi \text{ and } n \text{ satisfy in } R_0 \text{ the formula } (\forall t)[a[i(x, w)] = 0].
\]

It follows that \( \sim \sim \Gamma^*(A_0, A_1, A_2) \).

Now assume that \( \sim \sim \Gamma^*(A_0, A_1, A_2) \). According to lemma 30

\[
\sim \sim \Gamma^*(A_0, A_1, A_2) \Rightarrow \sim \sim \Gamma^*(A_0, A_1, A_2)
\]

Thus the implication \( \sim \sim \Gamma^*(A_0, A_1, A_2) \Rightarrow \sim \sim \Gamma^*(A_0, A_1, A_2) \) will be proved if we succeed in showing that \( \Gamma^*(A_0, A_1, A_2) \) is false in \( R_0 \). However, this is obvious because by lemma 30 there is exactly one function \( \psi \) which satisfies in \( R_0 \) the formula \( Z(u, v, r) \) and this function (by lemma 29) we have \( \psi[U(q, n)] = 0 \) for every \( q \).

Proof of the second part of the lemma is similar.

**Theorem 6.** If \( (B_1) \) is a recursive family of closed formulas and if the sets \( A_0 = \text{Cn}(B_1) \) are consistent, then there is a formula free for the family \( A_0 \).

Proof. Starting with formulas \( \Gamma, \Gamma^* \) of lemma 32 we construct formulas \( \Gamma_0, \Gamma^*_0 \) as in lemma 1. It has been proved in lemma 1 that these formulas strongly represent relations \( \mathcal{C}_0, \mathcal{C}^*_0 \) relatively to \( W \times X^2 \) and satisfy condition (g), p. 207. From lemmas 25, 27, 23, 24, and 20 it follows that there are integers \( r_1, r_2 \) such that

\[
\sim \sim \Gamma_0(U_0, u, v, w) = \Gamma_0(u, x, y, z),
\]

\[
\sim \sim \Gamma_0(U_0, u, v, w) = \Gamma_0(u, x, y, z)
\]

and hence that all assumptions of theorem 2 are satisfied.

It is rather remarkable that theorem 1 fails for the system \( A_n \) of analysis discussed in [8]. Indeed, it has been proved in [8] that there is a finite complete extension of \( A_n \). It is extremely unlikely that there be a formula free for \( A_n \); this question, however, is open and seems to be rather difficult.
References


