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A Survey of Ultraproduct Constructions in General Topology

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Dedication: To my early mentors and colleagues: B. Banaschewski, G. Bruns, H. J. Keisler, K. Kunen, E. Nelson and M. E. Rudin.

1. Introduction.

This survey is intended primarily for two readerships: general topologists who know a little model theory, and model theorists who know a little general topology. Both camps can boast a healthy constituency, and most of this paper should be readily accessible to anyone in either.

The ultraproduct construction has a long and distinguished history. While its beginnings go back to the 1930s with K. Gödel (who was proving his completeness theorem) and T. Skolem (who was building nonstandard models of arithmetic), it was not until 1955, with the publication of the Fundamental Theorem of Ultraproducts, due to J. Łoś, that the construction was described explicitly, and its importance to first-order logic became apparent. The understanding of the structure and use of ultraproducts developed rapidly during the next fifteen years or so, culminating in the Ultrapower Theorem of H. J. Keisler and S. Shelah (UT). (The gist of the theorem is that two relational structures are elementarily equivalent if and only if an ultrapower of one is isomorphic to an ultrapower of the other. Keisler established a much stronger statement in the early 1960s using the Generalized Continuum Hypothesis (GCH); and toward the end of the decade, Shelah provided a GCH-free proof of a second stronger statement that is somewhat weaker than Keisler's.) By the late 1960s, the theory of ultraproducts had matured into a major area of investigation in its own right (see [20, 23, 29, 41] for a vastly more detailed account than is possible here), and was ready for export beyond the confines of classical model theory.

Actually the exportation process had already begun by the early 1960s, when I. Fleischer [30] observed that classic ultrapowers are directed limits of powers (and, by implication, that classic ultraproducts are directed limits of products). This observation, illustrating a major strength of category theory (see [50]), provides an abstract reformulation of a concrete construction. One may now start with a category **C** endowed with products (which construction being itself an abstract reformulation of the cartesian product) and directed limits, and define ultraproducts within that category. Going further, any *bridging theorem*, i.e., one that translates a concrete notion into abstract terms involving the ultraproduct, becomes available as a definitional vehicle to reformulate that notion in a suitably rich category. T. Ohkuma [56] (and A. Day and D. Higgs [26] a bit later) made good use of this idea, introducing a notion of *finiteness in a category* by means of the important elementary result that says a relational structure is finite if and only if all diagonal maps from that structure into its ultrapowers are isomorphisms. (In the setting of concrete categories; i.e., those endowed with a suitable "forgetful" functor to the category of sets and functions, this notion of finiteness and that of having finite underlying set can easily fail to coincide. Two examples: If **C** is the category **CH** of compacta (compact Hausdorff spaces) and continuous maps, then "**C**-finite" means "having at most one point." If **C** is the category **BAN** of Banach spaces and nonexpansive homomorphisms, then "**C**-finite" means "being finite dimensional.")

I became aware of Fleischer's limit approach to ultraproducts in 1974, while visiting McMaster University, late in my career as a graduate student. It was there that I had the idea of using the UT as a bridging theorem, mimicking Ohkuma's use of the ultrapower characterization of finiteness. My aim was not the abstract reformulation of set-theoretic notions, however, but model-theoretic ones; namely elementary equivalence and elementary embedding (as well as their various derivative notions). I can attribute much of my own development as a mathematician to enlightening talks I had with the universal algebra group at McMaster at that time (namely B. Banaschewski, G. Bruns and E. Nelson), and the papers [7, 8, 19] extend and develop the ideas introduced in [56, 26]. Moreover, my coinage of the term "ultracoproduct," along with my own investigations of how ultraproducts behave in the *opposite* of the concrete category **CH** (to be discussed in §5) can also be traced to Fleischer's approach.

What Fleischer started in 1963 might be regarded as the beginning of the idea of a "model-theoretic study of a class (or category) **C**." This should be immediately contrasted with what might be called "**C**-based model theory." While the two subject areas may overlap a great deal, there is a difference in emphasis. In the former, one perhaps fixes an autonomous notion of ultraproduct in **C** (hence a mechanism for generating conjectures that stem from known classical results), then tries to establish (functorial) links between **C** and particular classes of models of first-order theories (hence a mechanism for settling some of those conjectures). In the latter, one enriches objects of **C** with extra "functions" and "relations," possibly nonclassical in nature but recognizable nonetheless, views these enriched objects as models of logical languages, and proceeds to develop new model theories, using more established model theories for guidance. Our study of compacta in [11] and elsewhere exemplifies the former emphasis, while the Banach model theory initiated by C. W. Henson (see [37, 38, 40]), as well as the approaches to topological model theory found in [31, 32, 53, 62], exemplify the latter.

In this paper, our primary focus is on how classical ultraproducts can be exported to purely topological contexts, with or without category-theoretic considerations as motivation. (So the Banach ultraproduct [27], for example, the Fleischer ultraproduct in **BAN**, is not directly a subject of our survey.) We begin in the next section with a quick introduction to ultraproducts in model theory; then on, in §3, to consider the topological ultraproduct, the most straightforward and naïve attempt at exporting the ultraproduct to the topological context. The motivation in §3 is purely model-theoretic, with no overt use of category-theoretic concepts. This is also true in §4, where we look at a variation of this construction in the special case of ultrapowers. It is not until §5, where ultracoproducts are introduced, that the Fleischer approach to defining ultraproducts plays a significant role. Although the ultracoproduct may be described in purely concrete (i.e., set-theoretic) terms, and is of independent interest as a topological construction, the important point is that category-theoretic language allows one to see this construction as a natural gateway out of the classical model-theoretic context.

The ultraproduct construction in model theory is a quotient of the direct product, where an ultrafilter on the index set dictates how to specify the identification. When we carry out the analogous process in general topology, at least from the viewpoint of §3, the "product" in question is not the usual Tychonov product, but the less sophisticated (and much worse-behaved) *box* product. (While one could use the Tychonov product instead of the box product, the result would be an indiscrete (i.e., trivial) topological space, unless the ultrafilter were countably complete.)

The identification process just mentioned does not require the maximality of the designated ultrafilter in order to be well defined, and may still be carried out using any filter on the index set. The resulting construction, called the *reduced product*, serves as a generalization of both the direct (box) product and the ultraproduct constructions. In §6 we survey some of the recent work on furthering this generalization to include the Tychonov product and some of its relatives. Finally, in §7, we list some of the more resistant and intriguing open problems in the topological study of ultraproducts.

2. Preliminaries from Model Theory.

First we recall some familiar notions from model theory, establishing our basic notation and terminology in the process.

Given a set I, the power set of I is denoted $\wp(I)$, and is viewed as a bounded lattice under unions and intersections. (The alphabet of bounded lattices consists of two binary operation symbols, $\sqcup$ (join) and $\sqcap$ (meet), plus two constant symbols, $\top$ (top) and $\bot$ (bottom).) A **filter on** I is a filter *in* the lattice $\wp(I)$; i.e., a collection $\mathcal{F}$ of subsets of I satisfying: (*i*) $I \in \mathcal{F}$, (*ii*) any superset of an element of $\mathcal{F}$ is also an element of $\mathcal{F}$, and (iii) the intersection of any two elements of $\mathcal{F}$ is also an element of $\mathcal{F}$. A filter $\mathcal{F}$ is called **proper** if $\emptyset \notin \mathcal{F}$; an **ultrafilter on** I is a proper filter on I that is not contained in any other (distinct) proper filter on I; i.e., a maximal proper filter in the lattice $\wp(I)$. In power set lattices, the maximal proper filters are precisely the **prime** ones; that is, any proper filter $\mathcal{F}$ on I is an ultrafilter, if for each $J, K \subseteq I$, if $J \cup K \in \mathcal{F}$, then either $J \in \mathcal{F}$ or $K \in \mathcal{F}$. If $\mathcal{S}$ is any family of subsets of I, $\mathcal{S}$ is said to satisfy the **Finite Intersection Property (FIP)** if no finite intersection of elements of $\mathcal{S}$ is empty. Our underlying set theory of choice is Zermelo-Fraenkel Set Theory *with* Choice (ZFC); consequently, any family of subsets of I that satisfies the FIP must be contained in an ultrafilter on I. (More generally, if a subset of a bounded distributive lattice satisfies the Finite *Meet* Property, then that subset is contained in a maximal proper filter in the lattice.)

We start with an alphabet L of finitary relation and function symbols (with the equality symbol $\approx$ tacitly assumed to be included). An L-**structure** consists of an underlying set A and an interpretation of each symbol of L, in the usual way. Like many authors (and unlike many others), we use the same font to indicate both a relational structure and its underlying set; being careful to make the distinction clear whenever there is a threat of ambiguity.

If $\langle A_i : i \in I \rangle$ is an indexed family of L-structures, and $\mathcal{F}$ is a filter on I, the ordinary direct product of the family is denoted $\prod_{i \in I} A_i$, with the ith coördinate of an element a being denoted $a(i)$. (Each symbol of L is interpreted in the standard way.) The binary relation $\sim_{\mathcal{F}}$ on the product, given by $a \sim_{\mathcal{F}} b$ just in case $\{i \in I : a(i) = b(i)\} \in \mathcal{F}$, is easily seen to be an equivalence relation; and we define $a/\mathcal{F} := \{b : a \sim_{\mathcal{F}} b\}$. We denote by $\prod_{\mathcal{F}} A_i$ the corresponding **reduced product**; i.e., the set of $\sim_{\mathcal{F}}$-equivalence classes, with the standard interpretation of each symbol of L. When $A_i = A$ for each $i \in I$, we have the **reduced power**, denoted $A^I/\mathcal{F}$. The **canonical diagonal map** $d : A \to A^I/\mathcal{F}$, given by $a \mapsto$ (constantly a)$/\mathcal{F}$, is clearly an embedding of L-structures.

From here on, unless we specify otherwise, we concentrate on reduced products (powers) in which the filter is an ultrafilter. The corresponding constructions are called **ultraproducts (ultrapowers)**, and the Fundamental Theorem of Ultraproducts is the following. (We follow the standard notation regarding satisfaction of substitution instances of first-order formulas. That is, if $\varphi(x_0, \ldots, x_{n-1})$ is a first-order L-formula with free variables from the set $\{x_0, \ldots, x_{n-1}\}$, and if A is an L-structure with n-tuple $\langle a_0, \ldots, a_{n-1} \rangle \in A^n$, then $A \models \varphi[a_0, \ldots, a_{n-1}]$ means that the sentence got from φ by substituting each free occurrence of x_i with a new constant symbol denoting a_i, $i < n$, is true in A. (See also [20, 23, 41].))

2.1. **Theorem.** (Łoś' Fundamental Theorem of Ultraproducts [23]) Let $\langle A_i : i \in I \rangle$ be a family of L-structures, with $\mathcal{D}$ an ultrafilter on I and $\varphi(x_0, \ldots, x_{n-1})$ a first-order L-formula. Given an n-tuple $\langle a_0/\mathcal{D}, \ldots, a_{n-1}/\mathcal{D} \rangle$ from the ultraproduct, then $\prod_{\mathcal{D}} A_i \models \varphi[a_0/\mathcal{D}, \ldots, a_{n-1}/\mathcal{D}]$ if and only if $\{i \in I : A_i \models \varphi[a_0(i), \ldots, a_{n-1}(i)]\} \in \mathcal{D}$.

By a **level zero formula**, we mean a Boolean combination of atomic formulas. If k is any natural number, define a **level $k+1$ formula** to be a level k formula φ preceded by a string Q of quantifiers of like parity (i.e., either all universal or all existential) such that, if φ begins with a quantifier, then the parity of that quantifier is not the parity of the quantifiers of Q. Formulas with a well-defined level are said to be in **prenex form**, and elementary first-order logic provides an effective procedure for converting any L-formula to a logically equivalent formula (with the same free variables) in prenex form. A function $f : A \to B$ between L-structures is a **level $\geq k$ embedding** if for each L-formula $\varphi(x_0, \ldots, x_{n-1})$ of level k, and n-tuple $\langle a_0, \ldots, a_{n-1}\rangle \in A^n$, it is the case that $A \models \varphi[a_0, \ldots, a_{n-1}]$ if and only if $B \models \varphi[f(a_0), \ldots, f(a_{n-1})]$. It is easy to see that the level ≥ 0 embeddings are precisely the algebraic embeddings; the level ≥ 1 embeddings are also called **existential embeddings**. (Existential embeddings have been of considerable interest to algebraists and model theorists alike.) If a function f is of level $\geq k$ for all $k < \omega$, we call it a **level $\geq \omega$ embedding**. Now an **elementary embedding** is one that preserves the truth of all first-order formulas, even those without an obvious level; so elementary embeddings are clearly of level $\geq \omega$. The effective procedure mentioned above, then, assures us of the converse. We are taking pains to make this point because, as we shall see, the notion of level $\geq k$ embedding can be given a precise abstract meaning, devoid of reference to first-order formulas. Moreover, it can be extended into the transfinite, giving rise to an abstract notion of level $\geq \alpha$ *morphism*. There is no *a priori* reason that this hierarchy should terminate at level ω. (See, e.g., §5.)

2.2. **Corollary.** (Diagonal Theorem) The canonical diagonal embedding from a relational structure into an ultrapower of that structure is an elementary embedding.

A first-order formula containing no free variables is called a **sentence**, and two L-structures A and B are called **elementarily equivalent** (denoted $A \equiv B$) if they satisfy the same L-sentences. Clearly if there is an elementary embedding from one L-structure into another, then the two structures are elementary equivalent; in particular, because of 2.2, if some ultrapower of A is isomorphic to some ultrapower of B, then $A \equiv B$. By the same token, if $f : A \to B$ is a map between L-structures, then f is an elementary embedding as long as there are ultrafilters $\mathcal{D}$ and $\mathcal{E}$ (on sets I and J respectively) and an isomorphism $h : A^I/\mathcal{D} \to B^J/\mathcal{E}$ such that the compositions $e \circ f$ and $h \circ d$ with the corresponding diagonal embeddings are equal. The converses of these two statements are also true. (Indeed, the converse of the second follows from the converse of the first via the method of expanding the alphabet L by adding constants denoting all the elements of A.) This fact, called the (Keisler-Shelah) Ultrapower Theorem (UT), is a milestone in model theory, with a very interesting history (see, e.g., [23]). Its importance, in part, is that it allows many basic notions of first-order model theory to be formulated in abstract terms, i.e., in terms of mapping diagrams; it is what we called a bridging theorem in the Introduction. The obvious central notions are elementary equivalence and elementary embedding, but there are also derivative notions (e.g., prime model) readily definable in terms of these. Other derivative notions are less obvious. The following result is stated and used extensively in [67], and is an application of Keisler's Model Extension Theorem (see [64]).

2.3. **Theorem.** A function $f : A \to B$ between L-structures is a level $\geq k+1$ embedding if and only if there is an elementary embedding $e : A \to C$ and a level $\geq k$ embedding $g : B \to C$ such that $e = g \circ f$.

Theorem 2.3, in conjunction with the UT, is another bridging theorem; as the elementary embedding e may be taken to be an ultrapower diagonal embedding. Thus the notion of level $\geq k$ embedding has an abstract reformulation. Indeed, because of the inductive flavor of 2.3, that notion may be formally carried into the realm of transfinite levels. What is more, the notion of level ≥ 1 (existential) embedding is now available in abstract form. That means we can export model completeness to the category-theoretic setting.

We begin to see how these ideas may be exploited when we survey the topological ultracoproduct in §5. (We use the infix *co* because we are dealing with the *opposite* of the concrete category **CH**.) Using the UT as a bridge, abstract model-theoretic notions are imported, only in dual form, and made concrete once again. In order for this to be a productive enterprise, however, it is necessary to use more of the theorem than simply the "gist" form stated above. We therefore end this section with statements of both Keisler's GCH version and Shelah's subsequent GCH-free version. (We employ standard notation as regards cardinals and ordinals; see, e.g., [23]. In particular, if κ and λ are cardinals, then κ^+ is the cardinal successor of κ; and κ^λ is the cardinal exponential, the cardinality of the set of all functions from λ into κ. If S is any set, its cardinality is denoted $|S|$.)

2.4. **Theorem.** (Keisler's Ultrapower Theorem, [23, 63]) Let λ be an infinite cardinal where the GCH holds (i.e., $2^\lambda = \lambda^+$), and let I be a set whose cardinality is λ. Then there is an ultrafilter $\mathcal{D}$ on I such that if L is an alphabet with at most λ symbols, and if A and B are elementarily equivalent L-structures of cardinality at most λ^+, then $A^I/\mathcal{D} \cong B^I/\mathcal{D}$.

2.5. **Theorem.** (Shelah's Ultrapower Theorem. [63]) Let λ be an infinite cardinal, with $\mu := \min\{\alpha : \lambda^\alpha > \lambda\}$, and let I be a set whose cardinality is λ. Then there is an ultrafilter $\mathcal{D}$ on I such that if L is an alphabet, and if A and B are elementarily equivalent L-structures of cardinality less than μ, then $A^I/\mathcal{D} \cong B^I/\mathcal{D}$.

3. Topological Ultraproducts.

Following established usage, a **topological space** consists of an underlying set X and a family $\mathcal{T}$ of subsets of X, called a **topology**; members of $\mathcal{T}$ being called **open sets**. All a family of subsets has to do to be called a topology is to be closed under arbitrary unions and finite intersections. As with the case of relational structures, we use the same symbol to indicate both a topological space and its underlying set (using disambiguating notation, such as $\langle X, \mathcal{T}\rangle$, only when necessary). If $\mathcal{B}$ is an open base for a topology $\mathcal{T}$ on X (so arbitrary unions of members of $\mathcal{B}$ form a topology), then we write $\mathcal{T} = \tau(\mathcal{B})$, the topology **generated by** $\mathcal{B}$.

Let $\langle\langle X_i, \mathcal{T}_i\rangle : i \in I\rangle$ be an indexed family of topological spaces, with $\mathcal{D}$ an ultrafilter on I. Then the ultraproduct $\prod_{\mathcal{D}} \mathcal{T}_i$ may easily be identified with a family of subsets of the ultraproduct $\prod_{\mathcal{D}} X_i$, and this family qualifies as an open base for a topology $\tau(\prod_{\mathcal{D}} \mathcal{T}_i)$ on $\prod_{\mathcal{D}} X_i$, which we call the **ultraproduct topology**. The resulting **topological ultraproduct** is denoted (when we can get away with it) $\prod_{\mathcal{D}} X_i$; and the canonical basic open sets $\prod_{\mathcal{D}} U_i \in \prod_{\mathcal{D}} \mathcal{T}_i$ are called **open ultraboxes**. Clearly the quotient map $x \mapsto x/\mathcal{D}$ from $\prod_{i\in I} X_i$ to $\prod_{\mathcal{D}} X_i$ is a continuous open map from the box product to the ultraproduct. Also there is a certain amount of flexibility built into the definition of topological ultraproduct; in that one may obtain an open base for the ultraproduct topology by taking open ultraboxes $\prod_{\mathcal{D}} U_i$, where, for each $i \in I$, the sets U_i range over an open base for the topology $\mathcal{T}_i$. This flexibility extends to *closed* bases as well. Recall that a family $\mathcal{C}$ is a **closed base** for $\mathcal{T}$ if $\mathcal{T}$-closed sets (i.e., complements in X of members of $\mathcal{T}$) are intersections of subfamilies taken from $\mathcal{C}$. One may obtain a closed base for the ultraproduct topology by taking *closed* ultraboxes $\prod_{\mathcal{D}} C_i$, where, for each $i \in I$, the sets C_i range over a closed base for the topology $\mathcal{T}_i$. (The reader interested in nonstandard topology may want to compare the topological ultrapower topology with A. Robinson's *Q-topology* [59].)

The connection between topological ultraproducts and usual ultraproducts should be rather apparent, but we will find it convenient to spell things out. By the **basoid alphabet** we mean the alphabet $L_{BAS} := \{P, B, \varepsilon\}$, where the first two symbols are unary relation symbols standing for "points" and "basic open sets," respectively, and the third, a binary relation symbol, stands for "membership." If X is any set and $\mathcal{S} \subseteq \wp(X)$, then $\langle X, \mathcal{S}\rangle$ may be naturally viewed as the L_{BAS}-structure $\langle X \cup \mathcal{S}, X, \mathcal{S}, \in\rangle$, where set-theoretic membership is restricted to $X \times \mathcal{S}$. An L_{BAS}-structure is called a **basoid** if it is (isomorphic to) such a structure, where $\mathcal{S}$ is an open base for a topology on X. The basoid is called **topological** if $\mathcal{S}$ is itself a topology. Every basoid has a uniquely associated topological basoid; the second is said to be **generated** from the first. It is a routine exercise to show that there is a first-order L_{BAS}-sentence whose models are precisely the basoids. Thus ultraproducts of basoids are basoids by Theorem 2.1, and we obtain $\prod_{\mathcal{D}}\langle X_i, \mathcal{T}_i\rangle$ as the topological basoid generated from the usual ultraproduct of the basoids $\langle X_i, \mathcal{T}_i\rangle$.

The alphabet L_{BAS} is a natural springboard for topological model theory: Allow extra relation and function symbols to range over points, and build various languages from there. This is a one-sorted approach, which is quite sensible, but which turns out to be somewhat cumbersome in practice for the purposes of exposition. Other approaches in the literature start with a first-order alphabet L, and expand the first-order language over L in various ways. For example, there is the extra-quantifiers approach, exemplified by J. Sgro's L_Q [62] (patterned after Keisler's L_Q [44]); also the two-sorted approach, exemplified by the "invariant" languages L_t of T.A. McKee [53] and S. Garavaglia [32]. (The two worked independently, with McKee confining himself to the case $L = \{\approx\}$. See also [31].) There is an extensive model theory for L_t which we cannot possibly survey adequately. (The interested reader is urged to consult the Flum-Ziegler monograph [31].) However, since this model theory includes a nice ultrapower theorem, we take a few lines to describe these languages and state the theorem.

One starts with an ordinary first-order alphabet L, adds new variables to stand for sets, and then adds the intersorted binary relation symbol ε for membership. Atomic formulas consist of the first-order atomic formulas from L, plus the intersorted formulas of the form $t\varepsilon U$, where t is a first-order term (from L) and U is a set variable. The language L_2 consists of the closure of the atomic formulas under the logical connectives $\neg$ (not), $\vee$ (or) and $\wedge$ (and), and the quantifiers $\exists$ (there exists) and $\forall$ (for all), applied to variables of either sort. A formula φ of L_2 is **positive** (resp., **negative**) in the set variable U if each free occurrence of U in φ lies within the scope of an even (resp., odd) number of negation symbols. We then define L_t to be the smallest subset K of L_2 satisfying: (i) the atomic formulas are in K; (ii) K is closed under the logical connectives, as well as quantification over point variables; and (iii) if t is a first-order term and $\varphi \in K$ is positive (resp., negative) in U, then $(\forall U(\neg(t\varepsilon U) \vee \varphi)) \in K$ (resp., $(\exists U((t\varepsilon U) \wedge \varphi)) \in K$).

By a **basoid L-structure**, we mean a pair $\langle A, \mathcal{B}\rangle$, where A is an L-structure and $\mathcal{B}$ is an open base for some topology on A. It should then be clear what it means for a basoid structure to be a *model* of a sentence φ of L_2, as well as what it means for two basoid L-structures to be *isomorphic*. If $\langle A_1, \mathcal{B}_1\rangle$ and $\langle A_2, \mathcal{B}_2\rangle$ are two basoid L-structures, then these structures are **homeomorphic** just in case $\langle A_1, \tau(\mathcal{B}_1)\rangle$ and $\langle A_2, \tau(\mathcal{B}_2)\rangle$ are isomorphic. We may now state the topological version of the ultrapower theorem, due to Garavaglia, as follows.

3.1. **Theorem.** (Garavaglia's Ultrapower Theorem [31, 32]) Let A and B be two basoid L-structures. Then A and B satisfy the same L_t-sentences if and only if some ultrapower of A is homeomorphic to some ultrapower of B.

In [4], two spaces X and Y are said to be **power equivalent** if some ultrapower of X is homeomorphic to some ultrapower of Y. It is not hard to show directly (Theorem A2.3 in [4]) that power equivalence is really an equivalence relation, and it is of some interest to see just how strong an equivalence relation it is. Recall that a space is said to be **self-dense** if it has no isolated points. We use the well-known T_n-numbering of the separation axioms (à la [70]); but note that, for the purposes of this paper, we assume the T_1 axiom (i.e., singletons are closed) whenever we talk about separation axioms involving arbitrary closed sets. Thus *regularity* (resp., *normality*), the property of being able to separate a point and a non-containing closed set (resp., two disjoint closed sets) with disjoint open sets, presupposes the T_1 axiom, and is synonymous with the T_3 (resp., T_4) axiom. Similarly, we assume T_1 when we define *complete regularity* (or, the *Tychonov property*, sometimes referred to as the $T_{3.5}$ axiom) as the property of being able to separate a point and a non-containing closed set with a continuous real-valued function. The following tells us that power equivalence is not very discriminating.

3.2. **Theorem.** (Theorem A2.6 in [4]) Any two self-dense T_3-spaces are power equivalent.

3.3. **Remark.** The proof of 3.2 uses a combination of model theory and topology. In particular, it makes use of the Löwenheim-Skolem theorem and a result of W. Sierpiński [65], to the effect that any two countable, second countable, self-dense T_3-spaces are homeomorphic. One could claim that 3.2 is a corollary of 3.1, but that would be a stretch. One would still need to employ the theorems of Löwenheim-Skolem and Sierpiński to show that any two topological basoids that are self-dense and T_3 must satisfy the same L_t-sentences (where $L = \{\approx\}$).

With any apparatus that produces new objects from old, an important issue concerns the idea of preservation. In the context of the topological ultraproduct construction, a preservation problem takes the following general form.

3.4. **Problem.** (General Preservation) Given topological properties P and Q, and a property R of ultrafilters, decide the following: For any I-indexed family $\langle X_i : i \in I\rangle$ of topological spaces and any ultrafilter $\mathcal{D}$ on I, if $\{i \in I : X_i \text{ has property } P\} \in \mathcal{D}$ (i.e., "$\mathcal{D}$-almost every X_i has property P") and $\mathcal{D}$ has property R, then $\prod_{\mathcal{D}} X_i$ has property Q.

3.5. **Remark.** The general problem, as stated in 3.4, is not quite as general as it could be. The property P could actually be a family $\mathcal{P}$ of properties, and the clause "$\mathcal{D}$-almost every X_i has property P" could read "$\mathcal{D}$-almost every X_i has property P for all $P \in \mathcal{P}$." The vast majority of instances of this problem do not require the added generality, however. (One obvious exception: Consider, for $n < \omega$, the property P_n that says that there are at least n points, and set $\mathcal{P} := \{P_n : n < \omega\}$. If R is the property of being countably incomplete and Q is the property of being infinite, then this instance of the more general version of 3.4 has an affirmative answer.)

The question of the preservation of the separation axioms T_0–T_4 under ultraproducts turns out to be a very rich topic. In [4] we define a topological property P to be **closed** if 3.4 has an affirmative answer for $Q = P$ and R nonrestrictive. P is **open** if its negation is closed. It is a straightforward exercise in definition manipulation to show [4] that the axioms T_0 through T_3 are both closed and open; a little less straightforward to show is the fact that $T_{3.5}$ is closed. It should come as no surprise to general topologists that neither $T_{3.5}$ nor T_4 is open, and that T_4 is not closed. The proofs of these negative facts are fairly involved, but they are valuable more for how they themselves involve an assortment of new ideas and positive (general) results. For this reason we take a few paragraphs to expatiate on some of their key points.

Consider first why $T_{3.5}$ is a closed property. Recall the well-known characterization of O. Frink [68] that a T_1-space X is completely regular if and only if it has a **normal disjunctive lattice base**; that is, if there is a bounded sublattice $\mathcal{C}$ of the bounded lattice of closed subsets of X satisfying: (i) $\mathcal{C}$ is a closed base for the topology on X (i.e., $\mathcal{C}$ is meet-dense in the closed set lattice); (ii) (normality) for each disjoint pair $C, D \in \mathcal{C}$ there exist $C', D' \in \mathcal{C}$ with $C \cap C' = D \cap D' = \emptyset$ and $C' \cup D' = X$; and (iii) (disjunctivity) for each two distinct elements of $\mathcal{C}$, there is a nonempty element of $\mathcal{C}$ that is contained in one of the first two elements and is disjoint from the other. (A good source on basic distributive lattice theory is [1].) If $\langle X_i : i \in I\rangle$ is a family of spaces such that $\mathcal{D}$-almost every X_i is completely regular, then for $\mathcal{D}$-almost every $i \in I$, there is a normal disjunctive lattice base $\mathcal{C}_i$ for X_i. It follows quickly that $\prod_{\mathcal{D}} \mathcal{C}_i$ is a normal disjunctive lattice base for $\prod_{\mathcal{D}} X_i$.

Of the twelve preservation results above concerning T_0–T_4, only the first nine are apparently positive. Nevertheless, it so happens that the last three are corollaries of positive results. Indeed, one can show that both $T_{3.5}$ and T_4 are not open properties in one go, with the help of 3.2. A space X is **linearly orderable** (a **LOTS**) if X has a linear ordering whose open intervals constitute an open base for X. X is **linearly uniformizable** (a **LUTS**) if the topology on X is induced by a uniformity that has a linearly ordered base under inclusion. (See, e.g., [70]. For example, if ρ is a metric inducing the topology on X, then $\{\{\langle x, y\rangle : \rho(x, y) < \epsilon\} : \epsilon > 0\}$ is a linearly ordered uniform base that witnesses the fact that X is a LUTS.) Suppose $\mathcal{D}$-almost every X_i is a LOTS with inducing linear order $\leq_i$ (resp., a LUTS with inducing linearly ordered uniform base $\mathcal{U}_i$). Then $\prod_{\mathcal{D}} \leq_i$ (resp., $\prod_{\mathcal{D}} \mathcal{U}_i$) is a linear ordering (resp., a linearly ordered uniform base) that induces the ultraproduct topology on $\prod_{\mathcal{D}} X_i$. Now every LOTS is hereditarily normal; indeed every LUTS is hereditarily paracompact Hausdorff. So let X be any regular space. Then $X \times \mathbb{R}$, the topological product of X with the real line, is self-dense and regular. By 3.2, there is an ultrapower $(X \times \mathbb{R})^{\mathbb{I}}/\mathcal{D}$ that is homeomorphic to an ultrapower of $\mathbb{R}$, and is hence both a LOTS and a LUTS. It is easy to show that ultrapowers commute with finite products. Thus the ultrapower $X^I/\mathcal{D}$ embeds in an ultrapower of the reals, and is hence hereditarily normal (indeed, hereditarily paracompact Hausdorff). The following theorem, whose proof we have just outlined, immediately implies the failure of $T_{3.5}$ and T_4 to be open properties.

3.6. **Theorem.** (Corollary A2.7 in [4]) Every regular space has a hereditarily paracompact Hausdorff ultrapower.

We now turn to the problem of showing that normality is not a closed property. First some notation: If κ and λ are cardinals, we write κ^λ to indicate the λ-fold topological (Tychonov) power of the ordinal space κ (as well as the cardinal exponentiation). The following positive result clearly implies that normality fails to be closed.

3.7. **Theorem.** (Corollary of Theorem 8.2 in [4]) Let X be any space that contains an embedded copy of 2^{ω_2}, and let $\mathcal{D}$ be any nonprincipal ultrafilter on a countable set I. Then $X^I/\mathcal{D}$ is not normal.

The proof of 3.7, being far more interesting than the statement, deserves a bit of discussion.

Of course, if X fails to be regular, so does any ultrapower. Thus it suffices to confine our attention to regular X (or even Hausdorff; it does not matter). In that case, any embedded copy Y of 2^{ω_2} is closed in X; hence $Y^I/\mathcal{D}$ is closed in $X^I/\mathcal{D}$. It is therefore enough to show that $Y^I/\mathcal{D}$ is nonnormal.

This brings us to the important class of P-spaces. Following the terminology of [33, 68], we call a space X a **P-space** if every countable intersection of open sets is an open set. More generally, following the Comfort-Negrepontis text [24], let κ be an infinite cardinal. A point x in a space X is called a **P_κ-point** if for every family $\mathcal{U}$ of fewer than κ open neighborhoods of x, there is an open neighborhood of x that is contained in each member of $\mathcal{U}$. X is a **P_κ-space** if each point of X is a P_κ-point. In P_κ-spaces, intersections of fewer than κ open sets are open; the P-spaces are just the P_{ω_1}-spaces. (In [3, 4], the P_κ-spaces are called *κ-open*. While it is convenient to have a concise adjectival form of "being a P_κ-space," there was already one in the literature, *κ-additive*, due to R. Sikorski [66], which we adopt here.)

It is very hard for a topological ultraproduct not to be a P-space. To be specific, define an ultrafilter $\mathcal{D}$ on I to be **κ-regular** if there is a family $\mathcal{E} \subseteq \mathcal{D}$, of cardinality κ, such that each member of I is contained in only finitely many members of $\mathcal{E}$. It is well known [24] that $|I|^+$-regular ultrafilters cannot exist, that $|I|$-regular ultrafilters exist in abundance, that ω-regularity is the same as countable incompleteness, and that nonprincipal ultrafilters on countable sets are countably incomplete. The following not only says that κ-regularity in ultrafilters produces κ^+-additivity in topological ultraproducts (deciding affirmatively an instance of Problem 3.4); it actually *characterizes* this property of ultrafilters.

3.8. **Theorem.** (Additivity Lemma, Theorem 4.1 in [4]) An ultrafilter is κ-regular if and only if all topological ultraproducts via that ultrafilter are κ^+-additive (P_{κ^+}-spaces).

3.9. **Remark.** There is a model-theoretic analogue to 3.8: Just replace "additive" with "universal." (See Theorem 4.3.12 and Exercise 4.3.32 in [23].)

Given any space X and cardinal κ, we denote by $(X)_\kappa$ the space whose underlying set is X, and whose topology is the smallest κ-additive topology containing the original topology of X. If κ is a regular cardinal (so κ is not the supremum of fewer than κ smaller cardinals; for example κ could be a successor cardinal), then one may obtain an open base for $(X)_\kappa$ by taking intersections of fewer than κ open subsets of X. (See, e.g., [24] for an extensive treatment of this kind of topological operation.)

3.10. **Terminological Remark.** The adjective *regular*, as used in technical mathematics, is probably the most overloaded word in the English language. Already in this paper it has three senses; modifying the nouns *space*, *ultrafilter* and *cardinal* in completely unrelated ways. In other areas of mathematics as well, the word is used with abandon. In algebra, functions, rings, semigroups, permutations and representations can all be regular; in homotopy theory, fibrations can be regular; and in analysis, Banach spaces, measures and points can be regular too. (*Regular* modifies *ring* in the same way that it modifies *semigroup*, but otherwise there are no apparent similarities in the senses to which it is used.) The list, I am sure, goes on.

Returning to the proof outline of 3.7, recall the diagonal map d from a set X into an ultrapower $X^I/\mathcal{D}$ of that set. If the ultrapower is a topological one, d is not necessarily continuous; consider, for example the case where X is the real line and $\mathcal{D}$ is a countably incomplete ultrafilter. The image $d[X]$ of X under d then carries the discrete topology. The following uses 3.8.

3.11. **Theorem.** (Theorem 7.2 in [4]) Let $\mathcal{D}$ be a regular ultrafilter on a set of cardinality κ, with X a topological space. Then the diagonal map, as a map from $(X)_{\kappa^+}$ to $X^I/\mathcal{D}$, is a topological embedding.

Suppose Y is a compactum (i.e., a compact Hausdorff space), and that $\mathcal{D}$ is an ultrafilter on I. Then for each $a/\mathcal{D} \in X^I/\mathcal{D}$, there is a unique point $x \in X$ such that for each open set U containing x, the open ultra*cube* $U^I/\mathcal{D}$ contains $a/\mathcal{D}$. Let $\lim_{\mathcal{D}}(a/\mathcal{D})$ denote this unique point. Then the function $\lim_{\mathcal{D}}$ is continuous (Theorem 7.1 in [4]), and is related to the standard part map in nonstandard analysis [59]. But more is true, thanks to 3.11.

3.12. **Theorem.** (A consequence of Corollary 7.3 of [4]) Let $\mathcal{D}$ be a regular ultrafilter on a set of cardinality κ, with Y a compactum. Then the limit map $\lim_{\mathcal{D}}$, as a map from $Y^I/\mathcal{D}$ to $(Y)_{\kappa^+}$, is a continuous left inverse for the diagonal map d. As a result, the diagonal $d[Y]$, a homeomorphic copy of $(Y)_{\kappa^+}$, is a closed subset of $Y^I/\mathcal{D}$.

We are just about done with 3.7. In a preliminary version of [46], K. Kunen shows that $(2^{\mathfrak{c}^+})_{\omega_1}$ is nonnormal, where $\mathfrak{c} := 2^{\aleph_0}$ is the power of the continuum; and in [28], E. K. van Douwen uses an earlier result of C. Borges [22] to replace $\mathfrak{c}$ with ω_1. So let Y now be the compactum 2^{ω_2}, with $\mathcal{D}$ any nonprincipal ultrafilter on a countable set I. In order to show $Y^I/\mathcal{D}$ is nonnormal, it suffices to show some closed subset is nonnormal. This is true, though, since $(Y)_{\omega_1}$ is nonnormal and, by 3.12, sits as a closed subset of $Y^I/\mathcal{D}$. This completes our discussion of 3.7.

What Borges' result cited above actually says is that the space $(\kappa^{\kappa^+})_\kappa$ is nonnormal whenever κ is a regular cardinal. It is quite easy to show from this that, for any infinite cardinal κ, $(2^{\kappa^{++}})_{\kappa^+}$ is not normal either. This, together with the additivity lemma 3.8 and some arguments to show how easy it is for paracompactness to be present in P-spaces, gives rise to a characterization of the GCH in terms of topological ultraproducts.

Recall that the **weight** of a space X is the greater of $\aleph_0$ and the least cardinality of an open base for the topology on X. For each infinite cardinal κ, let UP_κ be the following assertion.

UP$_\kappa$: If I is a set of cardinality κ, $\mathcal{D}$ is a regular ultrafilter on I, and $\langle X_i : i \in I\rangle$ is an I-indexed family of spaces, $\mathcal{D}$-almost each of which is regular and of weight at most 2^κ, then $\prod_{\mathcal{D}} X_i$ is paracompact Hausdorff.

The main result of [3] (see also W. Comfort's survey article [25]) is the following.

3.13. **Theorem.** (Theorem 1.1 in [3]) UP$_\kappa$ holds if and only if the GCH holds at level κ (i.e., $2^\kappa = \kappa^+$).

3.14. **Remarks.** (i) The proof of 3.13 allows several alternatives to UP$_\kappa$. In particular, *regular* (as the word applies to spaces) may be replaced by *normal*; even by *compact Hausdorff*. Also *paracompact Hausdorff* may be replaced by *normal*.

(ii) Topological ultraproducts are continuous open images of box products, and there are many inevitable comparisons to be made between the two constructions. In particular, let BP$_\kappa$ be the statement that the box product of a κ-indexed family of compact Hausdorff spaces, each of weight at most 2^κ, is paracompact Hausdorff. In [46] it is proved that the CH (i.e., the GCH at level ω) implies BP$_\omega$. Since $(2^{\omega_2})_{\omega_1}$ is nonnormal, the compactum 2^{ω_2} stands as a counterexample to BP$_\omega$ if the CH fails, and as an *absolute* counterexample to BP$_\kappa$ for $\kappa > \omega$.

We now turn to the exhibition of Baire-like properties in topological ultraproducts. If κ is an infinite cardinal, define a space X to be **κ-Baire** (or, a **B_κ-space**) if intersections of fewer than κ dense open subsets of X are dense. Of course, every space is a B_ω-space, and various forms of the Baire category theorem say that completely metrizable spaces and compact Hausdorff spaces are ω_1-Baire. Finally, one topological form of Martin's Axiom (MA, see, e.g., [21]) says that if X is compact Hausdorff and satisfies the *countable chain condition* (i.e., there is no uncountable family of pairwise disjoint nonempty open subsets of X), then X is $\mathfrak{c}$-Baire.

What we are working toward is an analogue of 3.8, with P replaced with B. What has been achieved in this connection is interesting, if imperfect, and begs for improvement.

For any set S and cardinal λ, let $\wp_\lambda(S)$ be the set of all subsets of S of cardinality less than λ. If $\mathcal{D}$ is an ultrafilter on a set I, a map $F : \wp_\omega(S) \to \mathcal{D}$ is **monotone** (resp., **multiplicative**) if $F(s) \supseteq F(t)$ whenever $s \subseteq t$ (resp., $F(s \cup t) = F(s) \cap F(t)$). The ultrafilter $\mathcal{D}$ is called **λ-good** if: (i) $\mathcal{D}$ is countably incomplete, and (ii) for every $\mu < \lambda$ and every monotone $F : \wp_\omega(\mu) \to \mathcal{D}$, there exists a multiplicative $G : \wp_\omega(\mu) \to \mathcal{D}$ such that $G(s) \subseteq F(s)$ for all $s \in \wp_\omega(\mu)$. (This notion is due to Keisler.)

Every countably incomplete ultrafilter is ω_1-good, and every λ-good ultrafilter is μ-regular for all $\mu < \lambda$. Consequently, if $|I| = \kappa$, the maximal degree of goodness an ultrafilter on I could hope to have is κ^+. The existence of good ultrafilters (i.e., κ^+-good ultrafilters on sets of cardinality κ) was first proved by Keisler under the hypothesis $2^\kappa = \kappa^+$, and later by Kunen without this hypothesis. (See [24]. There it is shown that there are as many good ultrafilters on a set as there are ultrafilters.) Good ultrafilters produce saturated models (see Theorem 6.1.8 in [23]), and the production of saturated models necessitates goodness (see Exercise 6.1.17 in [23]). Finally, and most importantly, good ultrafilters play a crucial role in the proofs of both ultrapower theorems 2.4 and 2.5. Our analogue of 3.8 is the following affirmative answer to the general preservation problem (3.4).

3.15. **Theorem.** (Theorem 2.2 in [5]) If an ultrafilter is κ-good, then all topological ultraproducts via that ultrafilter are κ-Baire (B_κ-spaces) (as well as being λ^+-additive for all $\lambda < \kappa$).

3.16. **Remark.** Theorem 3.8 is actually key to the proof of 3.15. We do not know whether producing topological ultraproducts that are κ-Baire as well as λ^+-additive for all $\lambda < \kappa$ is sufficient to show an ultrafilter to be κ-good.

Topological ultraproduct methods have proven useful in the study of the η_α-sets of F. Hausdorff [36]. Recall that, for any infinite cardinal α, a linear ordering $\langle A, < \rangle$ is an η_α**-set** if whenever $B, C \subseteq A$ each have cardinality less than α, and every element of B lies to the left of every element of C, then there is some element of A lying to the right of every element of B and to the left of every element of C. The η_ω-sets are just the dense linear orderings without endpoints, and Hausdorff [36] invented the famous "back and forth" method to show that any two η_α-sets of cardinality α are order isomorphic. He was also able to establish the existence of η_{α^+}-sets of cardinality 2^α (and L. Gillman showed how to exhibit two distinct such orderings whenever $\alpha^+ < 2^\alpha$). Gillman and B. Jónsson proved that η_α-sets of cardinality α exist precisely under the condition that $\alpha = \sup\{\alpha^\lambda : \lambda < \alpha\}$. (The interested reader should consult [33, 24].) Denote by $\mathbb{Q}_\alpha$ the (unique, when it exists) η_α-set of cardinality α. ($\mathbb{Q}_\omega$ is, of course, the rational line $\mathbb{Q}$.) In [10], we use topological ultraproduct methods to establish properties of $\mathbb{Q}_\alpha$, viewed as a LOTS. In particular, $\mathbb{Q}_\alpha$ is both α-additive and α-Baire, and the following is true.

3.17. **Theorem.** (Theorem 3.14 of [10]) If X is a nonempty space that embeds in $\mathbb{Q}_\alpha$, then $\mathbb{Q}_\alpha$ can be partitioned into homeomorphic copies of X, each of which is closed and nowhere dense in $\mathbb{Q}_\alpha$.

We end this section with one more preservation result about topological ultraproducts. Its main interest is that its proof apparently needs to involve two cases, depending upon whether the ultrafilter is countably complete or countably incomplete. Also it involves a topological property that illustrates a general machinery for producing new properties from old.

By 3.8, every topological ultraproduct via a countably incomplete ultrafilter is a P-space. Now if a P-space is also T_1, then it has the peculiar property of being **pseudofinite** (or, a cf**-space**, see [42]); i.e., one having no infinite compact subsets.

Another way of saying this is that the only compact subsets of X are the ones that have to be, based on cardinality considerations alone.

There is a general phenomenon afoot here. Namely, if P is any topological property, let spec(P) be the set of cardinals κ such that every space of cardinality κ has property P; and denote by anti-P the class of spaces X such that if Y is a subspace of X and Y has property P, then $|Y| \in \text{spec}(P)$. For example, if P is the property *compact* (resp., *connected*, *self-dense*), then anti-P is the property *pseudofinite* (resp., *totally disconnected*, *scattered*). The modifier *anti-* was introduced in [6], and it has been studied in its own right by a number of workers. (See, e.g., [58, 51, 52].) Concerning topological ultraproducts, what we showed in [6] is the following affirmative answer to 3.4.

3.18. **Theorem.** (Corollary 3.6 of [6]) Topological ultraproducts of pseudofinite Hausdorff spaces are pseudofinite Hausdorff.

3.19. **Remark.** Of course, topological ultraproducts of Hausdorff spaces, via countably incomplete ultrafilters, are pseudofinite Hausdorff (by 3.8 plus basic facts). One must argue quite differently when the ultrafilters are countably complete. In this case cardinal measurability is involved, and pseudofiniteness on the part of the factor spaces is essential; moreover the argument does not work if the Hausdorff condition is eliminated (or even weakened to T_1). One needs to know that if a set has a certain cardinality, then the cardinality of its closure cannot be too much greater. The T_2 axiom assures us of this, but the T_1 axiom does not. (Consider any set with the cofinite topology.) So, for example, we do not know whether topological ultraproducts of pseudofinite T_1-spaces are pseudofinite in general.

4. Coarse Topological Ultrapowers.

There is a natural variation on the definition of the ultraproduct topology in cases where all the factor spaces are the same. In this section, we consider ultrapowers only, and restrict the ultrapower topology to the one generated by just the open ultracubes. This is what we call the **coarse topological ultrapower**. That is, if $\langle X, \mathcal{T} \rangle$ is a topological space and $\mathcal{D}$ is an ultrafilter on a set I, then the family of open ultracubes $\{U^I/\mathcal{D} : U \in \mathcal{T}\}$ forms an open base for the coarse ultrapower topology. Note that, with regard to this topology, the natural diagonal map $d : X \to X^I/\mathcal{D}$ is a topological embedding. We denote the coarse topological ultrapower by $[X^I/\mathcal{D}]$. (For those interested in nonstandard topology, there is a connection between coarse topological ultrapowers and Robinson's S-*topology* [59].)

Quite straightforwardly, one may obtain a closed base for the coarse ultrapower topology by taking all closed ultracubes. However, it is generally *not* true that an open (resp., closed) base for the coarse ultrapower topology may be obtained by taking ultracubes from an open (resp., closed) base for the original space. (Indeed, let X be infinite discrete, with $\mathcal{B}$ the open base of singleton subsets of X.)

Our main interest in this section is the question of when coarse topological ultrapowers satisfy any of the usual separation axioms. If the ultrafilter is countably complete, then the diagonal map is a homeomorphism unless the base space has cardinality exceeding the first measurable. While this may be an interesting avenue of research, there are no results at this time that we know of; and we therefore confine attention to countably incomplete ultrafilters. For each $r \in \{0, 1, 2, 3, 3.5, 4\}$, define an ultrafilter $\mathcal{D}$ to be a **T_r-ultrafilter** if it is countably incomplete, and for some infinite space X, the coarse ultrapower $[X^I/\mathcal{D}]$ is a T_r-space. The reader should have no difficulty in constructing coarse topological ultrapowers that are not T_0-spaces, so the question of the mere existence of T_0-ultrafilters will doubtless come to mind. The good news is that T_0-ultrafilters are closely related to ones whose combinatorial properties are fairly well understood; so their existence follows from MA. We currently do not know whether T_0-ultrafilters exist absolutely.

First, we may reduce the existence question to the case of ultrafilters on a countable set; ω, say. The reason is that if $\mathcal{D}$ is a T_0-ultrafilter on an infinite set I and $X^I/\mathcal{D}$ is T_0, then we may partition I into countably many subsets, none of which is in $\mathcal{D}$, and build a function f from I onto ω such that the images of the members of the partition of I partition ω into infinite sets. Then $\mathcal{E} := \{S \subseteq \omega : f^{-1}[S] \in \mathcal{D}\}$ is clearly a countably incomplete ultrafilter. Moreover f induces an embedding of $X^\omega/\mathcal{E}$ into $X^I/\mathcal{D}$; hence $\mathcal{E}$ is a T_0-ultrafilter.

In [61], B. Scott defines an ultrafilter $\mathcal{D}$ on ω to be **separative** if whenever $f, g : \omega \to \omega$ are two functions that are **$\mathcal{D}$-distinct** (i.e., $\{n < \omega : f(n) \neq g(n)\} \in \mathcal{D}$), then their Stone-Čech lifts f^β and g^β disagree at the point $\mathcal{D} \in \beta(\omega)$ (i.e., there is some $J \in \mathcal{D}$ such that $f[J] \cap g[J] = \emptyset$). Scott's main results in [61] include the facts that selective ultrafilters are separative, and the properties of selectivity and being a P-point ultrafilter (i.e., a P-point in $\beta(\omega)\setminus\omega$) are not implicationally related. From MA, one may infer the existence of selective ultrafilters; hence the consistency of separative ultrafilters is assured. By the famous Shelah P-point independence theorem [71], P-point ultrafilters cannot be shown to exist in ZFC. We do not know whether the same can be said for separative ultrafilters, but strongly suspect so. The following is an amalgam of several results in [9].

4.1. **Theorem.** An ultrafilter on ω is T_r, for $r \in \{0, 1, 2, 3, 3.5\}$, if and only if it is separative.

4.2. **Remark.** That $\mathcal{C}$ is separative if it is T_0 is straightforward (Proposition 2.1 in [9]). Assuming $\mathcal{D}$ is separative, it is shown in [9] that a coarse $\mathcal{D}$-ultrapower of X is: (*i*) T_1 if X is a *weak P-space* (i.e., no point is in the closure of any countable subset of the complement of the point) ; (*ii*) T_2 if X is T_2 and a P-space; (*iii*) $T_{3.5}$ if X is T_4 and a weak P-space; and (*iv*) *strongly zero dimensional* (i.e., disjoint zero sets are separable via disjoint closed open sets) if X is T_4 and a P-space. We do not know whether coarse topological ultrapowers (of infinite spaces, via countably incomplete ultrafilters) can *ever* be normal.

5. Topological Ultracoproducts.

Most algebraists at all familiar with the classical reduced product construction know how to define it in terms of direct limits of products (à la Fleischer [30]). Indeed, in his introductory article in the "Handbook of Mathematical Logic," Paul Eklof [29] goes this route, but then says:

> "Although the shortest approach to the definition of reduced products is via the notion of direct limit, this approach is perhaps misleading since it is the concrete construction of the direct limit rather than its universal mapping properties which will be of importance in the sequel."

Eklof quite sensibly proceeds immediately to the concrete construction (i.e., in terms of elements), because classical model theory has no use for the abstract approach. However, there is more to ultraproducts than just first-order logic. Consider, for example, the problem of giving an explicit concrete description of the Stone space of an ultraproduct of Boolean lattices, in terms of the Stone spaces of those lattices. (Note: We speak of Boolean *lattices*, rather than Boolean *algebras*, because we do not include complementation as a distinguished unary operation. Of course, in the context of bounded distributive lattices, complements are unique when they exist. Thus a bounded sublattice of a Boolean algebra is a subalgebra just in case the sublattice is Boolean itself.) Because of the duality theorem of M. H. Stone (see [43]), this "ultracoproduct" must be an inverse limit of coproducts. To be more definite, suppose $\langle X_i : i \in I\rangle$ is an I-indexed family of Boolean (i.e., totally disconnected compact Hausdorff) spaces, with $\mathcal{D}$ an ultrafilter on I. Letting $B(X)$ denote the Boolean lattice of "clopen" (i.e., closed open) subsets of X, the operator $B(\ )$ is contravariantly functorial, with "inverse" given by the maximal spectrum functor $S(\ )$. Given any Boolean lattice A, the points of $S(A)$ are the maximal proper filters in A. If $a \in A$ and $a^\sharp := \{M \in S(A) : a \in M\}$, then the set $A^\sharp := \{a^\sharp : a \in A\}$ forms a (closed) lattice base for a totally disconnected compact Hausdorff topology on $S(X)$.

So Stone Duality tells us that $S(\prod_\mathcal{D} B(X_i))$ is an inverse limit of coproducts; hence a subspace of $\beta(\bigsqcup_{i\in I} X_i)$, the Stone-Čech compactification of the disjoint union of the spaces X_i. Here is one way (out of many) to describe this space in purely topological terms. Let Y be $\bigsqcup_{i\in I} X_i$, and let $q : Y \to I$ take an element to its index. Then there is the natural Stone-Čech lift $q^\beta : \beta(Y) \to \beta(I)$ (I having the discrete topology), and it is not hard to show that $S(\prod_\mathcal{D} B(X_i))$ is naturally homeomorphic to $(q^\beta)^{-1}[\mathcal{D}]$, the inverse image of $\mathcal{D} \in \beta(I)$ under q^β. Let us denote this space $\sum_\mathcal{D} X_i$. It is rightfully called an "ultracoproduct" because it is category-theoretically dual to the usual ultraproduct in a very explicit way. What makes this whole exercise interesting is that our explicit description of $\sum_\mathcal{D} X_i$ requires nothing special about the spaces X_i beyond the Tychonov separation axiom. Indeed, the construction just described, what we call the **topological ultracoproduct**, is the Fleischer-style ultraproduct for the *opposite* of the category **CH** of compacta and continuous maps. And while the topological ultracoproduct makes sense in the Tychonov context ($\sum_\mathcal{D} X_i$ is actually a compactification of the topological ultraproduct $\prod_\mathcal{D} X_i$), one does not get anything new in the more general setting. That is, $\sum_{\mathcal{D}} X_i$ is naturally homeomorphic to $\sum_{\mathcal{D}} \beta(X_i)$ (see [11]). For this reason we confine our attention to ultracoproducts of compacta.

If each X_i is the same compactum X, then we have the **topological ultracopower** $XI\backslash\mathcal{D}$, a subspace of $\beta(X \times I)$. In this case there is the Stone-Čech lifting p^β of the natural first-coördinate map $p : X \times I \to X$. Its restriction to the ultracopower is a continuous surjection, called the **codiagonal map**, and is officially denoted $p_{X,\mathcal{D}}$ (with the occasional notation-shortening alias possible). This map is dual to the natural diagonal map from a relational structure to an ultrapower of that structure, and is not unlike the standard part map from nonstandard analysis. (It is closely related to, indeed an extension of, $\lim_{\mathcal{D}}$, introduced after 3.11.) Recalling from the Introduction that being abstractly finite in **CH** means having at most one point (because abstract ultraproducts via countably incomplete ultrafilters must have trivial topologies), it is natural to ask what being abstractly finite means in the opposite of **CH**. The answer is simple, satisfying and easy to prove: it means "having a finite number of points."

Stone Duality is a contravariant equivalence between the categories **BS** of Boolean spaces and continuous maps and **BL** of Boolean lattices and homomorphisms. From our perspective, **BL** is an interesting participant in the duality because it has abstract products, all cartesian, and its class of objects is one that is first-order definable. This tells us its Fleischer-style ultraproduct construction is the usual one. For the purposes of this paper, let us call a concrete category **C** **Stone-like** if there is a contravariant equivalence between **C** and some concrete category **A**, with usual (cartesian) products; where the objects of **A** are the models of a first-order theory, and the morphisms of **A** are the functions that preserve atomic formulas. Then clearly any Stone-like category has an ultracoproduct construction, in the Fleischer sense of forming inverse limits of coproducts. Thus **BS** is Stone-like, as is the category **CAG** of compact Hausdorff abelian groups and continuous group homomorphisms. The reason **CAG** is Stone-like is that there is a celebrated duality theorem, due to L. Pontryagin (see, e.g., [70]), that matches this category with the category **AG** of abelian groups and homomorphisms. But while the ultraproduct constructions in **BL** and **AG** are exactly the same, the ultracoproduct constructions in **BS** and **CAG** are quite different [13].

Any time a concrete category **C** has an abstract ultra(co)product construction, there are two clear lines of investigation that present themselves. First one may study the construction *per se* in set-theoretic terms, via the underlying set functor; second one may view the construction as a vehicle for establishing abstract formulations of various model-theoretic notions (thanks to the Ultrapower Theorem). The second line is more "global" in flavor; it is part of a study of the category **C** as a whole. For example, one may wish to know whether **C** is Stone-like. (As explained in [7], the full subcategory **TDCAG** of totally disconnected compact Hausdorff abelian groups, a category with an abstract ultracoproduct construction, is not Stone-like because it has "cofinite" objects with infinite endomorphism sets.) As one might expect, it is a combination of these two lines that gives the best results.

Now we have seen that there is an abstract ultraproduct construction, as well as an abstract ultracoproduct construction, in the category **CH**. As we saw earlier, the first construction is uninteresting because it almost always has the trivial topology. The story is quite different for the second, however. For one thing, it extends the corresponding construction in the full subcategory **BS**, so there is an immediate connection with model-theoretic ultraproducts. (In fact there is generally a natural isomorphism between $B(\sum_{\mathcal{D}} X_i)$ and $\prod_{\mathcal{D}} B(X_i)$. This implies, of course, that ultracoproducts of connected compacta (=continua) are connected [7, 11].) For another thing, there is the fact that a compactum X is finite if and only if all codiagonal maps $p_{X,\mathcal{D}}$ are homeomorphisms ("cofinite" = finite).

In light of the above, a natural conjecture to make is that **CH** is Stone-like; and after over twenty years, everything known so far about the topological ultracoproduct points to an affirmative answer (in contrast to the situation with **TDCAG**). I first posed the question in the McMaster algebra seminar in 1974, and expressed then my belief that the conjecture is false. At the time I had little more to go on than the empirical observation that there were already quite a few duality theorems involving **CH**, e.g., those of Banaschewski, Morita, Gel'fand-Kolmogorov and Gel'fand-Naĭmark, and none of them were of the right kind. Almost ten years (and several partial answers, see [7]) later, there came confirmation of my belief from two independent quarters.

5.1. **Theorem.** (B. Banaschewski [2] and J. Rosický [60]) **CH** is not a Stone-like category.

Of course, what Banaschewski and Rosický independently prove are two somewhat different-sounding statements that each imply 5.1. The importance of their result is that it underscores the point that dualized model-theoretic analogues of classical results, automatically theorems in Stone-like categories, are merely conjectures in **CH**. (Shining example: R. L. Vaught's Elementary Chains Theorem.)

Because of the failure of **CH** to be Stone-like (perhaps this "failure" is a virtue in disguise), one is forced to look elsewhere for model-theoretic aids for a reasonable study of topological ultracoproducts. Fortunately there is a finitely axiomatizable universal-existential Horn class of bounded distributive lattices, the so-called **normal disjunctive** lattices (also called Wallman lattices), comprising precisely the (isomorphic copies of) **lattice bases**, those lattices that serve as bases for the closed sets of compacta. (To be more specific: The normal disjunctive lattices are precisely those bounded lattices A such that there exists a compactum X and a meet-dense sublattice $\mathcal{A}$ of the closed set lattice $F(X)$ of X such that A is isomorphic to $\mathcal{A}$.) We go from bounded distributive lattices to spaces, as in the case of Stone duality, via the **maximal spectrum** $S(\ )$, pioneered by H. Wallman [69]. $S(A)$ is the space of maximal proper filters of A; a typical basic closed set in $S(A)$ is the set $a^\sharp$ of elements of $S(A)$ containing a given element $a \in A$. $S(A)$ is generally compact with this topology. Normality, the condition that if a and b are disjoint ($a \sqcap b = \bot$), then there are a', b' such that $a \sqcap a' = b \sqcap b' = \bot$ and $a' \sqcup b' = \top$, ensures that the maximal spectrum topology is Hausdorff. Disjunctivity, which says that for any two distinct lattice elements there is a nonbottom element that is below one and disjoint from the other, ensures that the map $a \mapsto a^\sharp$ takes A isomorphically onto the canonical closed set base for $S(A)$. $S(\ )$ is contravariantly functorial: If $f : A \to B$ is a homomorphism of normal disjunctive lattices and $M \in S(B)$, then $f^S(M)$ is the unique maximal filter extending the prime filter $f^{-1}[M]$. (For normal lattices, each prime filter is contained in a unique maximal one.)

It is a relatively easy task to show, then, that $S(\ )$ converts ultraproducts to ultracoproducts. Furthermore, if $f : A \to B$ is a **separative** embedding; i.e., an embedding such that if $b \sqcap c = \bot$ in B, then there exists $a \in A$ such that $f(a) \geq b$ and $f(a) \sqcap c = \bot$, then f^S is a homeomorphism. Because of this, there is much flexibility in how we may obtain $\sum_{\mathcal{D}} X_i$: Simply choose a lattice base $\mathcal{A}_i$ for each X_i and apply $S(\ )$ to the ultraproduct $\prod_{\mathcal{D}} \mathcal{A}_i$. So, taking each $\mathcal{A}_i$ to be $F(X_i)$, we infer very quickly that $\sum_{\mathcal{D}} X_i$ contains the topological ultraproduct $\prod_{\mathcal{D}} X_i$ (à la §3) as a densely-embedded subspace. (Also we get an easy concrete description of the codiagonal map $p : XI\backslash\mathcal{D} \to X$: If $\mathcal{A}$ is a lattice base for X and $y \in XI\backslash\mathcal{D} = S(\mathcal{A}^I/\mathcal{D})$, then $p(y)$ is that unique $x \in X$ such that if $A \in \mathcal{A}$ contains x in its interior, then $A^I/\mathcal{D} \in y$. So p does indeed extend $\lim_{\mathcal{D}}$. Note that we may view the $\mathcal{D}$-equivalence class of the "constantly A" function in $\mathcal{A}^I$ as an ultrapower itself.)

So we officially define two compacta X and Y to be **co-elementarily equivalent** if there are ultracopowers $p : XI\backslash\mathcal{D} \to X$, $q : YJ\backslash\mathcal{E} \to Y$, and a homeomorphism $h : XI\backslash\mathcal{D} \to YJ\backslash\mathcal{E}$. (Recall the definition of *power equivalence* in §3.) A function $f : X \to Y$ is a **co-elementary map** if there are p, q, and h as above such that the compositions $f \circ p$ and $q \circ h$ are equal. These definitions come directly from the UT. Furthermore, because of Theorem 2.3, we may define the **level** of a map $f : X \to Y$ as follows: f is a map **of level** ≥ 0 if f is a continuous surjection. If α is any ordinal, f is a map **of level** $\geq \alpha + 1$ if there are maps $g : Z \to Y$ and $h : Z \to X$ such that g is co-elementary, h is of level $\geq \alpha$, and $f \circ h = g$. If α is a positive limit ordinal, f is a map **of level** $\geq \alpha$ if f is a map of level $\geq \beta$ for all $\beta < \alpha$. (Because of the definition of *co-elementary map*, $g : Z \to Y$ may be taken to be an ultracopower co-diagonal map.) A map of level ≥ 1 is also called **co-existential**.

The reader may be wondering whether we are justified in the terminology "co-elementary *equivalence*," as there is nothing in the definition above that ensures the transitivity of this relation. The answer is that we are so justified; but we need the maximal spectrum functor $S(\ )$, plus the full power of the UT (i.e., Theorem 2.5) to show it (Theorem 3.2.1 in [11]). By the same token, one also shows that compositions of co-elementary maps are co-elementary (Theorem 3.3.2 in [11]), and that compositions of maps of level $\geq \alpha$ are of level $\geq \alpha$ (Proposition 2.5 in [18]).

Because of how it translates ultraproducts of lattices to ultracoproducts of compacta, the maximal spectrum functor also translates elementary equivalence between lattices to co-elementary equivalence between compacta. Furthermore, if $f : A \to B$ is an elementary (resp. level $\geq \alpha$) embedding, then $f^S : S(B) \to S(A)$ is a co-elementary (resp. level $\geq \alpha$) map. Nevertheless, the spectrum functor falls far short of being a duality, except when restricted to the Boolean lattices. For this reason, one must take care not to jump to too many optimistic conclusions; such as assuming, e.g., that if $f : X \to Y$ is a co-existential map, then there must be lattice bases $\mathcal{A}$ for X and $\mathcal{B}$ for Y and an existential embedding $g : \mathcal{B} \to \mathcal{A}$ such that $f = g^S$. (Of course, for level ≥ 0, this is obvious: Pick $\mathcal{A} := F(X)$, $\mathcal{B} := F(Y)$, and $g := f^F$. However, f^F is not an existential embedding, unless it is already an isomorphism (a slight adjustment of the proof of Proposition 2.8 in [16]).) This "representation problem" has yet to be solved.

The infrastructure for carrying out a dualized model-theoretic study of compacta is now in place. Because of Stone Duality, dualized model theory for Boolean spaces is perfectly reflected in the ordinary model theory of Boolean lattices, but 5.1 tells us there is no hope for a similar phenomenon in the wider context. For example, one may use the Tarski Invariants Theorem [23], plus Stone Duality, to show that there are exactly $\aleph_0$ co-elementary equivalence classes in **BS**; however, one must work directly to get the number of co-elementary equivalence classes in **CH**.

5.2. **Theorem.** (Diversity, Theorem 3.2.5 in [11]) There are exactly $\mathfrak{c}$ co-elementary equivalence classes in **CH**. Moreover (Theorem 1.5 in [13]), for each $0 < \alpha \leq \omega$, there is a family of $\mathfrak{c}$ metrizable compacta, each of dimension α, no two of which are co-elementarily equivalent. Finally (Theorem 2.11 in [14]), there is a family of $\mathfrak{c}$ locally connected metrizable (i.e., *Peano*) continua, no two of which are co-elementarily equivalent.

Another example concerns various statements of the Löwenheim-Skolem Theorem. The weakest form, for Boolean lattices, says that every Boolean lattice is elementarily equivalent to a countable one ("countable" = "countably infinite or finite"). Now Stone Duality equates the cardinality of a Boolean lattice with the weight of its maximal spectrum space (in symbols, $|A| = w(S(A))$); hence we infer immediately that every Boolean space is co-elementarily equivalent to a metrizable one (since, for compacta, metrizability = weight $\aleph_0$). The same is true for compacta in general, by use of the Löwenheim-Skolem Theorem for normal disjunctive lattices. This was first proved by R. Gurevič [35], in response to a question raised in [11].

5.3. **Theorem.** (Löwenheim-Skolem Theorem, Proposition 16 in [35]) For every compactum X, there is a metrizable compactum Y and a co-elementary map $f : X \to Y$. In particular, every compactum is co-elementarily equivalent to a metrizable one.

Theorem 5.3 has several sharper versions; one is Theorem 1.7 in [13], which sees the Löwenheim Theorem as a factorization of maps. The strongest version appears in [17].

5.4. **Theorem.** (Löwenheim-Skolem Factorization Theorem, Theorem 3.1 in [17]) Let $f : X \to Y$ be a continuous surjection between compacta, with κ an infinite cardinal such that $w(Y) \leq \kappa \leq w(X)$. Then there is a compactum Z and continuous surjections $g : X \to Z$ and $h : Z \to Y$ such that $w(Z) = \kappa$, g is a co-elementary map, and $f = h \circ g$.

5.5. **Remark.** When restricted to spaces in **BS**, 5.4 is an immediate corollary of classical model theory. In the absence of a Stone-like duality, though, one must resort to other techniques. The proof of 5.4 actually makes use of some Banach space theory.

Another line of inquiry regarding topological ultracoproducts concerns the General Preservation Problem 3.4, with $\sum_{\mathcal{D}} X_i$ in place of $\prod_{\mathcal{D}} X_i$. In this new setting, we define a property P of compacta to be **closed** if for any indexed family $\langle X_i : i \in I\rangle$ of compacta, and any ultrafilter $\mathcal{D}$ on I, $\sum_{\mathcal{D}} X_i$ has property P whenever $\{i \in I : X_i \text{ has property } P\} \in \mathcal{D}$. P is **open** if the complement of P in **CH** is closed. (Frequently we speak of a subclass **K** of **CH** as being closed or open.)

5.6. **Theorem.** The following properties of compacta are both closed and open:

> "Being a continuum" (Proposition 1.5 in [11]); "being Boolean" (Proposition 1.7 in [11]); "having Lebesgue covering dimension n, $n < \omega$," (essentially Theorem 2.2.2 in [11]); and "being a decomposable/indecomposable continuum" (Proposition 2.4.4 in [11], and Proposition 11 in [35]).

It follows from the above that the property of having infinite covering dimension is a closed property. It is not an open property because $\sum_{\mathcal{D}} X_i$ will be infinite dimensional as long as $\{i \in I : X_i \text{ has dimension } n\} \in \mathcal{D}$ for arbitrarily large $n < \omega$. The reader may be wondering whether other dimension functions behave as well as covering dimension *vis à vis* ultracoproducts, and the short answer is no: There is a compactum X, due to A. L. Lunc [55, 57] such that $\dim(X) = 1$ and $\mathrm{ind}(X) = \mathrm{Ind}(X) = 2$ (where $\dim(\ )$, $\mathrm{ind}(\ )$ and $\mathrm{Ind}(\ )$ are covering dimension, small inductive dimension and large inductive dimension, respectively). Using 5.3, find a metrizable $Y \cong X$. Then $\dim(Y) = 1$ by 5.6. Since all three dimension functions agree for separable metrizable spaces, we see that the two inductive dimension functions are not preserved by co-elementary equivalence.

Recall that decomposability in a continuum X means that X is the union of two proper subcontinua; equivalently, it means that X has a proper subcontinuum with nonempty interior. It is relatively easy to show that the class of decomposable continua is closed; much less trivial [35] to show the same for the class of indecomposable continua. In [17] the class of **κ-wide** compacta is defined, for each cardinal κ. Membership in this class amounts to having a family of λ pairwise disjoint proper subcontinua with nonempty interiors, for each cardinal $\lambda < \kappa$; so decomposability for a continuum is equivalent to being 1-wide, and all infinite locally connected compacta are $\aleph_1$-wide. Using a technique similar to the one Gurevič used to prove Proposition 11 in [35], one can show that the class of n-wide compacta is both open and closed for each $n < \omega$; consequently that any compactum co-elementarily equivalent to a locally connected compactum is $\aleph_0$-wide. The class of $\aleph_0$-wide compacta is closed under co-elementary equivalence, but this is hardly the case for the locally connected compacta.

5.7. **Theorem.** (Corollary 14 in [35]) Let $\mathcal{D}$ be a nonprincipal ultrafilter on a countable set, with X an infinite compactum. Then $XI\backslash\mathcal{D}$ is not locally connected.

This result was used in [14] (along with regular ultrafilters and the Löwenheim-Skolem Theorem) to obtain the following.

5.8. **Theorem.** (Theorem 2.10 in [14]) Let κ be an infinite cardinal, and X an infinite compactum. Then there is a compactum Y, of weight κ, that is co-elementary equivalent to X, but not locally connected.

The central role of local connectedness in the study of topological ultracoproducts was discovered by R. Gurevič in solving a problem I raised in [11]. In an exact analogy with the concept of $\aleph_0$-categoricity in model theory, define a metrizable compactum X to be **$\aleph_0$-categorical** if there is no homeomorphically distinct metrizable compactum that is co-elementarily equivalent to X. For example, the Cantor discontinuum 2^ω is $\aleph_0$-categorical because its Boolean lattice of clopen sets is the unique (up to isomorphism) countable atomless Boolean lattice, and the class of Boolean spaces is both closed and open. The problem I raised was whether the closed unit interval $[0,1]$ (or *any* nontrivial metrizable continuum, for that matter) is $\aleph_0$-categorical, and Theorem 5.8 provides a negative answer. (The same negative answer was given in [35], but the proof of Proposition 15, a key step, was significantly incomplete.) The question of the existence of $\aleph_0$-categorical continua remains open, but we know from 5.8 that any $\aleph_0$-categorical compactum must fail to be locally connected. (There is even more: Using a Banach version of the classic Ryll-Nardzewski Theorem from model theory, C. W. Henson [39] has informed me that $\aleph_0$-categorical compacta must fail to be $\aleph_0$-wide.)

The concept of categoricity may be relativized to a subclass **K** of **CH** in the obvious way. Thus we could ask about the existence of metrizable compacta in **K** that are $\aleph_0$-categorical *relative to* **K**. When **K** is the locally connected compacta, there is a satisfying answer. Recall that an **arc** (resp **simple closed curve**) is a homeomorphic copy of the closed unit interval (resp. the standard unit circle).

5.9. **Theorem.** (Theorem 0.6 in [12]) Arcs and simple closed curves are $\aleph_0$-categorical relative to the class of locally connected compacta.

Getting back to the General Preservation Problem 3.4, there is not much known about properties of a topological ultraproduct that are conferred solely by the ultrafilter involved (in analogy with 3.8 and 3.15). One such is due to K. Kunen [47], and uses a Banach space argument.

5.10. **Theorem.** (Kunen [47]) Let $\mathcal{D}$ be a regular ultrafilter on I, with X an infinite compactum. Then $w(XI\backslash\mathcal{D}) = w(X)^{|I|}$.

Recall that a P_κ-space is one for which intersections of fewer than κ open sets are open. When κ is uncountable, such spaces are pseudofinite; hence infinite compacta can never be counted among them. There is a weakening of this property, however, that compacta can subscribe to. Call a space an **almost-P_κ-space** if nonempty intersections of fewer than κ open sets have nonempty interior. The following is an easy consequence of the Additivity Lemma 3.8, plus the fact that the topological ultracoproduct contains the corresponding topological ultraproduct as a dense subspace.

5.11. **Theorem.** (Theorem 2.3.7 in [11]) If an ultrafilter is κ-regular, then all topological ultracoproducts via that ultrafilter are almost-P_{κ^+}-spaces.

A little more significant is the following result about ultraco*powers*. (Compare with 3.15.)

5.12. **Theorem.** (Theorem 2.3.17 in [11]) If an ultrafilter is κ-good, then all topological ultracopowers via that ultrafilter are B_{κ^+}-spaces.

The rest of this section concerns what we have informally referred to as the "dualized model theory" of compacta, in exact parallel (only with the arrows reversed) with model-theoretic investigations of well-known classes of relational structures (e.g., linear orders, graphs, groups, fields, etc.). As we saw above, the topological ultracoproduct allows for the definition of co-elementary maps between compacta, as well as for the creation of the hierarchy of classes of maps of level $\geq \alpha$ for any ordinal α. When we restrict our attention to Boolean spaces, co-elementary maps and maps of level $\geq \alpha$ are the Stone duals of elementary embeddings and embeddings of level $\geq \alpha$, respectively, between Boolean lattices. This basic correspondence provides us with an abundance of facts about the Boolean setting that we would like to extend to the compact Hausdorff setting. Any failure of extendability would give a new proof of the Banaschewski-Rosický Theorem 5.1; so far, however, there has been nothing but success (or indecision).

The first obvious question that needs clearing up is whether the levels really go beyond ω, and the answer is no.

5.13. **Theorem.** (Hierarchy Theorem, Theorem 2.10 in [18]) Let α be any infinite ordinal. Then the maps between compacta that are of level $\geq \alpha$ are precisely the co-elementary maps.

This leads us to the second question, whether the composition of two maps of level $\geq \alpha$ is also of level $\geq \alpha$. As mentioned above, the answer is yes, but a much stronger result is true. The following is a dualized version of (an easy generalization of) the Elementary Chains Theorem of R. L. Vaught. (I first conjectured the result in the mid 1970s, and it took twenty years to find a proof.)

5.14. **Theorem.** (α-Chains Theorem, Theorem 3.4 in [18]) Let $\langle X_n \stackrel{f_n}{\leftarrow} X_{n+1} : n < \omega \rangle$ be a sequence of maps of level $\geq \alpha$ between compacta, with inverse limit X and limit maps $g_n : X \to X_n$, $n < \omega$. Then each g_n is a map of level $\geq \alpha$.

5.15. **Remark.** Without much ado, 5.14 may be extended to arbitrary inverse systems of compacta.

In any model-theoretic study of algebraic systems, the most commonly investigated homomorphisms are the level ≥ 1, or existential, embeddings. These are the ones arising from the classical study of algebraically closed fields, for example. When we look at the dual notion of level ≥ 1, or co-existential, maps between compacta, a very rich theory emerges. First of all, let us recall some properties of compacta that are preserved by level ≥ 0 maps (alias continuous surjections). These include: "having cardinality (or weight) $\leq \kappa$ (κ any cardinal); "being connected;" and "being locally connected." When we consider preservation by co-existential maps, we obtain preservation for several important properties that are not generally preserved by continuous surjections.

5.16. **Theorem.** (various results of [17]) The following properties are preserved by co-existential maps:

> "Being infinite;" "being disconnected;" "having covering dimension $\leq n$" ($n < \omega$);" "being an indecomposable continuum;" and "being a hereditarily indecomposable continuum."

5.17. **Remark.** Co-existential maps cannot raise covering dimension, but they can lower it (Example 2.12 in [17]). It is not hard to show that level ≥ 2 maps between compacta must preserve covering dimension.

An important tool in the proof of results such as 5.16 is the following result, of interest in its own right.

5.18. **Theorem.** (Covering Lemma, Theorem 2.4 in [17]) Let $f : X \to Y$ be a co-existential map between compacta. Then there exists a $\bigcup$-semilattice homomorphism f^* from the subcompacta of Y to the subcompacta of X such that for each subcompactum K of Y: (i) $f[f^*(K)] = K$; (ii) $f^{-1}[U] \subseteq f^*(K)$ whenever U is a Y-open set contained in K; (iii) the restriction of f to $f^*(K)$ is a co-existential map from $f^*(K)$ to K; and (iv) $f^*(K) \in \mathbf{K}$ whenever $K \in \mathbf{K}$ and $\mathbf{K} \subseteq \mathbf{CH}$ is closed under ultracopowers and continuous surjections.

An easy corollary of 5.18 is the fact that co-existential maps between compacta are **weakly confluent**; i.e., possessed of the feature that subcontinua of the range are themselves images of subcontinua of the domain. If a subcontinuum of the range is the image of each component of its pre-image, then the map is called **confluent**. Stronger still, a continuous surjection is **monotone** if pre-images of subcontinua of the range are subcontinua of the domain.

5.19. **Theorem.** (Theorem 2.7 in [17]) Let $f : X \to Y$ be a co-existential map between compacta, where Y is locally connected. Then f is monotone.

Theorem 5.19 is a main ingredient for the following result; another is Proposition 2.7 in [15].

5.20. **Theorem.** Let $f : X \to Y$ be a function from an arc to a compactum. The following are equivalent:

(i) f is a co-existential map.
(ii) f is a co-elementary map.
(iii) Y is an arc and f is a monotone continuous surjection.

Recall that a class of relational structures is called **elementary** if it is the class of models of a first-order theory. (This is the usage in [23]. In [20], elementary classes are the classes of models of a single sentence; what in [23] are called *basic* elementary classes.) From early work (1962) of T. E. Frayne, A. C. Morel and D. S. Scott (see Theorem 4.1.12 in [23]), a class is elementary if and only if it is closed under the taking of ultraproducts and ultra*roots* (i.e., a structure is in the class if some ultrapower of the structure is in the class). This characterization is another bridging theorem, allowing us to define a class $\mathbf{K} \subseteq \mathbf{CH}$ to be **co-elementary** if it is closed under the taking of ultracoproducts and ultracoroots. For example, all the classes (properties) mentioned in Theorem 5.6 are co-elementary, since they are both closed and open. The class of compacta of infinite covering dimension, while not being open, is still co-elementary. The same may be said for the class of $\aleph_0$-wide compacta (but certainly not for the class of locally connected compacta, by 5.7).

An elementary class of relational structures is called **model complete** (see [49]) if every embedding between members of that class is elementary. Thus we may define, in parallel fashion, the notion of **model cocomplete co-elementary class**. (I apologize for so many *co*s.) Because of Stone Duality, plus the fact that the class of atomless Boolean lattices is model complete, the class of self-dense Boolean spaces is a model cocomplete class of compacta. The following is an exact analogue of Robinson's Test for model completeness, and uses the ω-Chains Theorem 5.14.

5.21. **Theorem.** (Robinson's Test, Theorem 5.1 in [17]) A co-elementary class of compacta is model cocomplete if and only if every continuous surjection between members of the class is a co-existential map.

In model theory, the Chang-Łoś-Suszko Theorem (see [23, 64]) tells that an elementary class is the class of models of a set of universal-existential sentences if and only if the class is **inductive**; i.e., closed under arbitrary chain unions. In the compact Hausdorff setting, we then define a co-elementary class to be **co-inductive** if that class is closed under inverse limits of chains of continuous surjections. Examples of co-inductive co-elementary classes are **CH**, **BS** and **CON** (the class of continua).

The co-elementary class of decomposable continua is not co-inductive; indeed a favorite method of constructing indecomposable continua is to take inverse limits of decomposable ones (see [54]). Define a class **K** of compacta to be **κ-categorical**, where κ is an infinite cardinal, if: (i) **K** contains compacta of weight κ; and (ii) any two members of **K** of weight κ are homeomorphic. The class of self-dense Boolean spaces, for example, is $\aleph_0$-categorical. The following is an exact analogue of Lindström's Test for model completeness, and uses Theorem 5.21 above, as well as a fair amount of topology.

5.22. **Theorem.** (Lindström's Test, Theorem 6.4 in [17]) Any co-inductive co-elementary class of compacta is model cocomplete, provided it contains no finite members and is κ-categorical for some infinite cardinal κ.

5.23. **Remark.** Theorems 5.21 and 5.22 are interesting and fairly hard to prove. Unfortunately, they have proven useless in finding interesting model cocomplete classes. In particular, we know of no model cocomplete classes of continua.

Model cocomplete co-elementary classes are interesting because, in some sense, it is difficult to distinguish their members from one another. This is especially true if they are also **cocomplete**; i.e., consisting of exactly one co-elementary equivalence class. (It is not especially hard to prove that every co-elementary equivalence class is closed, so there is no problem finding cocomplete co-elementary classes.) One way to try to look for examples is via the study of *co-existential closure*. Recall that in model theory, an L-structure A is **existentially closed** relative to a class **K** of L-structures, of which A is a member, if every embedding from A into a member of **K** is existential. Let $\mathbf{K}^e$ donote the members of **K** that are existentially closed relative to **K**. It is well known (see [23]) that if **K** is an inductive elementary class, then each infinite $A \in \mathbf{K}$ embeds in some $A' \in \mathbf{K}^e$, of cardinality $|A|$. In certain special cases, $\mathbf{K}^e$ has a very elegant characterization. For example, if **K** is the class of fields, then $\mathbf{K}^e$ is the class of algebraically closed fields (Hilbert's *Nullstellensatz*). Other examples include: (i) $\mathbf{K} =$ the linear orderings without endpoints, $\mathbf{K}^e =$ the dense linear orderings without endpoints; (ii) $\mathbf{K} =$ the abelian groups, $\mathbf{K}^e =$ the divisible abelian groups with infinitely many elements of each prime order.

We thus define a compactum $X \in \mathbf{K} \subseteq \mathbf{CH}$ to be **co-existentially closed** relative to **K** if every continuous surjection from a member of **K** onto X is co-existential. Let $\mathbf{K}^c$ denote the members of **K** that are co-existentially closed relative to **K**. An exact analogue to the existence result just cited is the following.

5.24. **Theorem.** (Level ≥ 1 Existence, Theorem 6.1 in [17]) Let **K** be a co-inductive co-elementary class, with $X \in \mathbf{K}$ infinite. Then X is a continuous image of some $X' \in \mathbf{K}$, of weight $w(X)$.

Theorem 5.24 applies, then, to the three co-inductive co-elementary classes, **CH**, **BS** and **CON** mentioned above. The following is not difficult to prove.

5.25. **Theorem.** (Proposition 6.2 in [17]) $\mathbf{CH}^c = \mathbf{BS}^c =$ the self-dense Boolean spaces.

The nature of $\mathbf{CON}^c$ is apparently much more difficult to discern. If we can show it to be a co-elementary class, then, by Robinson's Test 5.21, it is model-cocomplete. (It is not hard to show that $\mathbf{K}^c$ is closed under co-existential images when $\mathbf{K}$ is a co-elementary class. Thus to show $\mathbf{K}^c$ to be co-elementary, it suffices to show it is closed under ultracoproducts.) With a slight abuse of language, call a member of $\mathbf{CON}^c$ a *co-existentially closed continuum*. We know from 5.24 that co-existentially closed continua abound, but the process used to construct them involves direct limits of lattices, and is not very informative. We have very few criteria to decide whether a given continuum is co-existentially closed; what we know so far is the following.

5.26. **Theorem.** (Theorem 4.5 in [18]) Every co-existentially closed continuum is an indecomposable continuum of covering dimension one.

6. Related Constructions.

Starting with an I-indexed family $\langle X_i : i \in I\rangle$ of topological spaces, the box product topology on the set $\prod_{i\in I} X_i$ is defined by declaring the *open boxes* $\prod_{i\in I} U_i$ as basic open sets, where the sets U_i are open subsets of X_i, $i \in I$. Alternatively, one forms the Tychonov product topology by restricting attention to those open boxes having the property that $\{i \in I : U_i \neq X_i\}$ is finite. In [45], C. J. Knight combines these two formations under a common generalization, the **$\mathcal{I}$-product topologies** for any ideal $\mathcal{I}$ of subsets of I (so $\emptyset \in \mathcal{I}$, and $\mathcal{I}$ is closed under subsets and finite unions) as follows: Take as open base all open boxes $\prod_{i\in I} U_i$ such that $\{i \in I : U_i \neq X_i\} \in \mathcal{I}$. Then the box (resp., Tychonov) product topology is the $\mathcal{I}$-product topology for $\mathcal{I} := \wp(I)$ (resp., $\mathcal{I} := \{J \subseteq I : J \text{ finite }\}$). (For the trivial ideal $\mathcal{I} := \{\emptyset\}$, one trivially obtains the trivial topology.) The collective name for these $\mathcal{I}$-product formations, for various ideals $\mathcal{I}$, is known as the **ideal product topology**.

In [34], M. Z. Grulović and M. S. Kurilić add a new ingredient to the pot, creating a further generalization that now takes in all ideal product topologies, as well as all reduced product topologies. Known as the **reduced ideal product topology**, it comprises the **$\mathcal{FI}$-product topologies** for any pair $\langle\mathcal{F}, \mathcal{I}\rangle$, where $\mathcal{F}$ (resp., $\mathcal{I}$) is a filter (resp., an ideal) on I: First one takes the $\mathcal{I}$-product topology on $\prod_{i\in I} X_i$; then forms the obvious quotient topology on the reduced product $\prod_{\mathcal{F}} X_i$ of underlying sets. Denote this new space by $\prod_{\mathcal{F}}^{\mathcal{I}} X_i$. Then the topological reduced product $\prod_{\mathcal{F}} X_i$ of §3 is $\prod_{\mathcal{F}}^{\wp(I)} X_i$ in this notation. Also, when $\mathcal{F}$ includes all the complements of members of $\mathcal{I}$, it follows that $\prod_{\mathcal{F}}^{\mathcal{I}} X_i$ has the trivial topology.

Define a filter-ideal pair $\langle\mathcal{F}, \mathcal{I}\rangle$ on I to satisfy the **density condition** if for every $A \in \mathcal{F}$ and every $B \notin \mathcal{F}$, there exists a $C \in \mathcal{I}$ such that $C \subseteq A\backslash B$ and $I\backslash C \notin \mathcal{F}$. (The use of the word *density* in this definition is justified by the following observation. Consider the quotient partially ordered set $\wp(I)/\mathcal{F}$, where $A, B \subseteq I$ are identified if $A\cap F = B\cap F$ for some $F \in \mathcal{F}$. Then the density condition amounts to the condition that every nonbottom element of $\wp(I)/\mathcal{F}$ dominates a nonbottom element of $\mathcal{I}/\mathcal{F}$.) Note that $\langle \mathcal{F}, \wp(I)\rangle$ satisfies the density condition when $\mathcal{F}$ is a proper filter (given A and B, just let C be $A\backslash B$), and that $\langle \{I\}, \mathcal{I}\rangle$ satisfies the density condition when every nonempty subset of I contains a nonempty member of $\mathcal{I}$. Also note that if $\mathcal{F}$ includes all the complements of members of $\mathcal{I}$, then $\langle \mathcal{F}, \mathcal{I}\rangle$ does not satisfy the density condition.

The main contribution of [34] is to connect the density condition with the preservation of the separation axioms by reduced ideal products (in a manner not entirely unlike the style of Theorem 4.1). For a topological property P, say that a filter-ideal pair $\langle \mathcal{F}, \mathcal{I}\rangle$ **preserves** P if for any I-indexed family $\langle X_i : i \in I\rangle$, $\prod_{\mathcal{F}}^{\mathcal{I}} X_i$ has property P whenever $\{i \in I : X_i \text{ has property } P\} \in \mathcal{F}$.

6.1. **Theorem.** (Grulović-Kurilić [34]) Let property P be any of the separation axioms T_r, $r \in \{0, 1, 2, 3, 3.5\}$. Then a filter-ideal pair $\langle \mathcal{F}, \mathcal{I}\rangle$ preserves P if and only if it satisfies the density condition.

7. Open Problems.

Here are some of the problems that have remained tantalizingly open.

7.1. (See 3.6.) Can a topological ultraproduct be normal without being paracompact?

7.2. (See 3.15, 3.16.) If all topological ultraproducts via $\mathcal{D}$ are κ-Baire, as well as λ^+-additive for all $\lambda < \kappa$, is $\mathcal{D}$ necessarily κ-good?

7.3. (See 3.17.) Is there a nice topological characterization of $\mathbb{Q}_\alpha$ for uncountable α? (Candidate: being regular self-dense, of cardinality = weight = α, α-additive and α-Baire. It is definitely not enough to exclude the "α-Baire" part, as Example 3.11 in [10] shows.)

7.4. (See 3.18, 3.19.) Is pseudofiniteness (i.e., anticompactness) preserved by topological ultraproducts?

7.5. (See 4.1, 4.2.) Are there (consistently) any T_4-ultrafilters?

7.6. (A representation problem.) If X and Y are co-elementarily equivalent compacta, can one always find lattice bases $\mathcal{A}$ for X and $\mathcal{B}$ for Y such that $\mathcal{A} \equiv \mathcal{B}$?

7.7. (See 5.8 and subsequent discussion.) Do nontrivial $\aleph_0$-categorical continua exist?

7.8. (See 5.9.) What Peano continua are there (besides arcs and simple closed curves) that are $\aleph_0$-categorical relative to the class of locally connected compacta?

7.9. (See 3.15, 5.12.) Do κ-good ultrafilters create ultracoproduct compacta that are B_{κ^+}-spaces?

7.10. (See 5.18, 5.19.) Are co-existential maps always confluent?

7.11. (See 5.18.) Is there a true version of the Covering Lemma where *co-existential* is replaced by *of level* $\geq \alpha$ $(\alpha \geq 1)$?

7.12. (See 5.23.) Are there any model cocomplete co-elementary classes that contain nontrivial continua?

7.13. (See 5.26.) Is the class of co-existentially closed continua a co-elementary class?

7.14. (See 5.26.) Are any of the familiar one-dimensional indecomposable continua (e.g., the solenoids, the bucket handle continua, the lakes of Wada) co-existentially closed?

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