SET-THEORETICAL APPROACH TO GENERAL SYSTEMS THEORY[1]

JERZY POGONOWSKI

Department of Applied Logic,
Adam Mickiewicz University
www.logic.amu.edu.pl

The purpose of this paper is to give a set-theoretical definition of the concept of a general system. After presenting this definition we discuss its efficacy in empirical applications, viz. in the study of the hierarchical structure of natural languages.

0. Introductory remarks

There is a widespread opinion that any formal definition of the concept of a general system should follow the principle of the minimality of mathematical structure (cf. for instance [2]). This means that any such definition should be as general as possible in order to embrace all the existing general systems. To specify several sorts of general systems, further conditions are added to the main definition. We think that the formalism od set theory is an adequate machinery to obtain this goal. The idea of the set-theoretical definition of the concept of a general system presented below was introduced in [3].

1. Notation

For any set X, $\wp(X)$ is the powerset of X, X^n is the n-th Cartesian power of X and the hierarchy V_X is defined by transfinite induction as follows:

- $V_X^0 = X$
- $V_X^{\alpha+1} = \wp(X \cup V_X^\alpha)$
- $V_X^\lambda = \bigcup\limits_{\alpha<\lambda} V_X^\alpha$ for limit ordinals λ

[1]Published in: R. Trappl (ed.) *Cybernetics and Systems Research*, North Holland Publishing Company, 1982, 15–18.

- $V_X = \bigcup\limits_{\alpha} V_X^{\alpha}$.

If $\mathfrak{A}$ is a relational structure, then $dom(\mathfrak{A})$ is the domain of $\mathfrak{A}$ and $rel(\mathfrak{A})$ is the set of all relations of $\mathfrak{A}$. If Ω is a finite set of predicate symbols, then $Str(\Omega)$ denotes the class of all finite non-empty relational structures of type Ω, i.e. structures whose relations are realizations of predicate symbols from Ω. If $\mathcal{A}$ is a family of sets, then $\mathcal{A}^*$ denotes the family of all finite Cartesian products of sets from $\mathcal{A}$ and $\mathcal{A}^+$ is the family of all finite Cartesian products from $\mathcal{A}$ except Cartesian powers of sets from $\mathcal{A}$, i.e.:

$$\mathcal{A}^+ = \mathcal{A}^* - \bigcup_{A\in\mathcal{A}} \bigcup_{n} A^n.$$

The remaining notation is standard.

2. Main definition

Let $\mathcal{U} = \{U_i : i \in I\}$ be an arbitrary family of sets and put $U = \bigcup \mathcal{U}$. We say that Σ is a ***general system based on*** $\mathcal{U}$ if $\Sigma = (\mathcal{U}, C)$, where $C \subseteq V_U$. Elements of $\mathcal{U}$ are called ***levels*** of Σ, U is the set of ***objects*** of Σ and C is the ***signature*** of Σ.

The intuitive idea behind this definition is that the signature C of Σ is a network of relationships between objects of Σ grouped into sorts (levels of Σ) as well as between set-theoretical constructs over those objects. Indeed, as C consists of set-theoretical constructs over U, it may contain any kind of relation between objects of Σ, relations between sets of those objects, relations between relations, etc. Thus, for instance the set

$$C \cap V_U^1$$

is exactly the set of all 1-argument relations defined on U and belonging to the signature C of Σ, i.e. it is the set of all properties (features) of objects of Σ. Similarly, binary relations over objects of Σ are included in $C \cap V_U^3$. Notice that the above definition of a general system is purely extensional.

Most likely, the best way to give a comprehensive intuitive explanation of the concept of general system defined above is to present examples of such systems.

3. Examples

a) Let $\mathfrak{A} = (A, R_1, \ldots, R_k)$ be any relational structure. Then

$$\Sigma = (\{A\}, \{R_1, \ldots, R_k\})$$

is a general system based on $\{A\}$. Observe that the signature of this general system consists entirely of relations between its objects (this systems has exactly one level, viz. A, i.e. the domain of the corresponding relational structure).

This example can be easily extended to many sorted structures (such as for instance vector spaces, abstract automata, etc.) as well as to structures with an infinite number of relations (e.g. topological spaces). For instance, if (I, O, Q, ψ, φ) is an abstract automaton with the set I of inputs, O of outputs, Q of states, transition function

$$\psi : I \times Q \to Q$$

and output function

$$\varphi : I \times Q \to O$$

then it can be identified with a general system $(\{I, O, Q\}, \{\psi, \varphi\})$, because obviously $\{\psi, \varphi\} \subseteq V_{I \cup O \cup Q}$.

b) Let L be any (formal) language over an alphabet A. Of course, the grammar of L (rules of formation of well formed formulas of L), its semantics (the concept of validity), as well as logic supplying L (rules of inference) can be constructed within set-theoretical framework. Hence any set of formulas of L can be considered as a general system (in the sense of main definition) based on a suitable family of levels (determined by the alphabet A of L, i.e. logical constants, variables, specific non-logical symbols, etc.).

The above examples clearly show that our definition of a general system covers already known definitions — for instance such common approaches to general systems theory as an algebraic, a black-box or linguistic approach can be easily presented in the proposed set-theoretical framework.

4. Relational characteristics of general systems

If $\Sigma = (\mathcal{U}, C)$ is an arbitrary general system based on $\mathcal{U} = \{U_i : i \in I\}$ and $U = \bigcup \mathcal{U}$, then define the function

$$c_\Sigma : \mathcal{U}^* \to \wp(C)$$

by $c_\Sigma(K) = C \cap \wp(K)$ for any $K \in \mathcal{U}^*$. The function c_Σ is called the ***relational characteristic*** of Σ. If, for instance

$$K = U_1 \times U_2 \times \ldots \times U_n,$$

then $\wp(K)$ equals the family of all n-argument relations between elements of the sets $U_1, U_2, \ldots, U_n$. Consequently, in this case $c_\Sigma(K)$ contains those n-argument relations between elements of $U_1, U_2, \ldots, U_n$ which belong to the signature of Σ. For example:

- if $K = U_1$, then $c_\Sigma(K) = C \cap \wp(U_1)$ equals the set of all properties of objects from the level U_1;
- if $K = U_1 \times U_2$, then $c_\Sigma(K)$ equals the family of all binary relations (belonging to C) between objects from levels U_1 and U_2.

Finally, for any level U_i, the relational structure

$$(U_i, \bigcup_n c_\Sigma(U_i^n))$$

is a formal description of the internal structure of this level. Notice that

$$\{c_\Sigma(K) : K \in \mathcal{U}^+\}$$

is the family of all inter-level relationships between objects from Σ.

5. Representation theorem

Let $\Sigma = (\mathcal{U}, C)$ be a general system based on $\mathcal{U}$. Then there exists a general system $\Phi = (\mathcal{W}, C)$ based on some family of sets $\mathcal{W}$ such that

$$C = \bigcup_{K \in \mathcal{W}^*} c_\Phi(K).$$

A trivial example of a general system satisfying the thesis of the above theorem is the general system of the form $\Phi = (C, C)$, where $c_\Phi(A) = \{A\}$ for $A \in C$; cf. also part 8 of this paper. This theorem shows that any general system can be represented by a many sorted structure, being also a general system, whose signature consists entirely of relations between its objects.

6. Sorts of general systems

Several further conditions added to the definition of a general system give us different sorts of more specific general systems. Without going into details, let us only point out a few possibilities.

If $\Sigma = (\mathcal{U}, C)$ is an arbitrary general system based on $\mathcal{U} = \{U_i : i \in I\}$ and $U = \bigcup \mathcal{U}$, then we say that Σ is:

- a) ***non-cumulative***, if for every $i, j \in I$

  $$U_i \cap V_{U_j} = \begin{cases} U_i, & \text{if} \quad i = j \\ \emptyset, & \text{if} \quad i \neq j; \end{cases}$$

- b) ***bounded***, if there is α such that $C \subseteq V_U^\alpha$;
- c) ***separable***, if

  $$U_i \cap U_j = \begin{cases} U_i, & \text{if} \quad i = j \\ \emptyset, & \text{if} \quad i \neq j; \end{cases}$$

- d) ***atomic***, if no element of U is a set.[2]

The reader can easily find the intuitive interpretation of the above conditions.

[2] It is assumed that we work in set theory with urelements.

7. Hierarchical analyses of language

We will present an application of the above defined concept of a general system to the study of language in the next section. The construction of a hierarchical analysis, presented below, is a formal counterpart of „an image" of a natural language obtained by a linguistic theory. From the purely formal point of view, however, hierarchical analyses may also serve as models of hierarchical systems.

W focus our attention on linguistic analyses, i.e. theories which:

- have concrete utterances as observational data,
- take into account the hierarchical structure of language,
- have the reconstruction of the internal structure of language as their ultimate goal.

Such theories can be informally characterized by the following postulates:

- 1. Concrete utterances (of arbitrary length) are the only data for any linguistic analysis.
- 2. The decomposition of utterances into constituent segments is the principle in any linguistic analysis.
- 3. Relations between linguistic units (i.e. between segments of utterances) form a basis for decomposition of utterances.
- 4. Any linguistic analysis distinguishes levels in language. Two segments belong to the same level if and only if the relations between their constituent parts are of the same kind.
- 5. For any two adjacent levels of language,individual segments which belong to one of these levels are looked upon as combinations of segments in the other level.
- 6. Each utterance is an individual concrete object.

A very good example of a linguistic analysis is Hjelmslev's glossematics (cf. [1]).

The construction of hierarchical analysis presented below satisfies the above informal postulates.

Let $(\Omega_1, \ldots, \Omega_k)$, $k > 1$, be a sequence of finite sets of predicate symbols. We say that a sequence of sets of relational structures $(S_1, \ldots, S_k)$ is a ***hierarchical analysis with respect to*** $(\Omega_1, \ldots, \Omega_k)$ if the following conditions hold:

- 1. if $1 \leqslant i \leqslant k$, then $S_i \subseteq Str(\Omega_i)$;
- 2. S_k is non-empty and at most denumerable;
- 3. if $1 \leqslant i < k$ and $\mathfrak{A} \in S_i$, then there is $\mathfrak{B} \in S_{i+1}$ such that $\mathfrak{A} \in dom(\mathfrak{B})$;

- 4. if $1 \leqslant i < k$ and $\mathfrak{A} \in S_{i+1}$, then $dom(\mathfrak{A}) \subseteq S_i$;
- 5. if $1 \leqslant i \leqslant k$, $\mathfrak{A}, \mathfrak{B} \in S_i$ and $\mathfrak{A} \neq \mathfrak{B}$, then $dom(\mathfrak{A}) \cap dom(\mathfrak{B}) = \emptyset$.

In linguistic terms, the sets S_i correspond to ***language levels***, structures from each S_i to ***analyzed tokens*** and relations from $rel(\mathfrak{A})$, $\mathfrak{A} \in S_i$, to ***syntagmatic relations***. More details concerning hierarchical analyses can be found in chapter 7 of [4].

Besides syntagmatic relations, there are two other kinds of linguistic relations: ***paradigmatic*** and ***inter-level*** relations. If $S = (S_1, \ldots, S_k)$ is a hierarchical analysis with respect to $(\Omega_1, \ldots, \Omega_k)$, then by an ***expanded*** hierarchical analysis with respect to $(\Omega_1, \ldots, \Omega_k)$ we understand any system (S, Prd, Ilv), where:

- 1. $Prd = (Prd_1, \ldots, Prd_k)$ and each Prd_i is a set of paradigmatic relations on S_i, i.e. for any $R \in Prd_i$ there is $Q \in S_i^*$ such that $R \subseteq Q$.
- 2. Ilv is a set of inter-level relations between elements of the sets $S_1, \ldots, S_k$, i.e. for any $R \in Ilv$ there is $Q \in S^+$ such that $R \subseteq Q$.[3]

8. Representations of hierarchical analyses by general systems

Given an arbitrary hierarchical analysis $(S_1, \ldots, S_k)$ there are at least two linguistically relevant ways of representing it as a general system. Let us discuss both of them.

a) Define:

- $\Sigma_1 = (\mathcal{U}_1, C_1)$
- $\mathcal{U}_1 = \{dom(\mathfrak{A}) : \mathfrak{A} \in S_1\}$
- $C_1 = \bigcup_i \bigcup_{\mathfrak{A} \in S_i} rel(\mathfrak{A})$.

Here each domain of an analyzed token from the lowest language level S_1 is a level of a general system Σ_1. Hence objects of Σ_1 are non-analyzable in terms of the underlying linguistic theory. Observe that relations from $rel(\mathfrak{A})$ where $\mathfrak{A} \in S_i$, $i \geqslant 2$, ***are not*** relations between objects of Σ_1. Hence the equality

$$C_1 = \bigcup_{K \in \mathcal{U}_1^*} c_{\Sigma_1}(K)$$

does not hold. The system Σ_1 is bounded, separable and non-cumulative (in the assumed interpretation of $(S_1, \ldots, S_k)$).

b) Let:

[3] Added in 2008. This is not the most general form of inter-level relations. Actually, Q here should be taken from S^* with the proviso that Q is not a subset of any S_i^n, for any n, i.e. that R is not a paradigmatic relation on any S_i. In such a case we can count as an inter-level relation e.g. a relation $R \subseteq S_i^2 \times S_j^3$, i.e. a 5-ary relation, as well as, say, $R \subseteq S_i \times S_j^2$, a ternary relation, etc.

- $\Sigma_2 = (\mathcal{U}_2, C_2)$
- $\mathcal{U}_2 = \{ \bigcup_{\mathfrak{A} \in S_i} dom(\mathfrak{A}) : 1 \leqslant i \leqslant k\}$
- $C_2 = C_1$.

Each level of Σ_2 consists of all elements of domains of analyzed tokens from one language level S_i. For Σ_2 the equality

$$C_2 = \bigcup_{K \in \mathcal{U}_2^*} c_{\Sigma_2}(K)$$

certainly holds. The system Σ_2 is bounded and separable but it is not non-cumulative.

Observe that the relationship between Σ_1 and Σ_2 is that described in representation theorem.

The main difference between the two representations of hierarchical analyses discussed above may be summarized as follows. Σ_2 has its signature nicely stratified — it consists entirely of relations between its objects. The set of levels of Σ_2 is, however, more redundant than the corresponding set of levels of Σ_1. At the same time, Σ_1 has a much more complicated signature with respect to its set of levels. From the purely extensional point of view both Σ_1 and Σ_2 have exactly the same set as its signature, viz. $C_1 = C_2$. Thus the difference between Σ_1 and Σ_2 sets apart two distinct points of view on the structure of language: we are left to decide whether we want to have relatively simple structure of levels or of signatures of the considered general systems. All these remarks remain true if we take into account expanded hierarchical analyses, i.e. hierarchical analyses with paradigmatic and inter-level relations.

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