Intuitive Explanations

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  Extremal axioms: logical, mathematical and cognitive aspects.
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- Two modest scholarships will be offered in the years 2017–2018 for PhD students willing to participate in the project.
- For applications, check the announcements of the National Scientific Center by the end of 2016.

Plan for today

- Claims about intuitions in the context of transmission
- Examples of intuitive explanations
- Views of mathematicians and philosophers
- Reflections on efficiency of intuitive explanations

- The term *mathematical intuition* covers many topics, notably the beliefs of professional mathematicians (context of discovery).
- Mathematical results are accepted on the basis of proof (context of justification).
- Below we carefully limit ourselves to intuitive explanations of already known mathematical ideas contained in the textbooks or presented in the classroom (context of transmission).

Poincaré: *Science and Method*

We are in a class of the fourth grade. The teacher is dictating: 'A circle is the position of the points in a plane which are at the same distance from an interior point called the centre.' The good pupil writes this phrase in his copy-book and the bad pupil draws faces, but neither of them understands. Then the teacher takes the chalk and draws a circle on the board. 'Ah', think the pupils, 'why didn't he say at once, a circle is a round, and we should have understood.'

- Quotation after Sierpińska 1994 (*Understanding in Mathematics*, p. 1). Each chapter of the book starts with a quotation from Poincaré.
- Sierpińska's treatment of understanding in mathematics is based on Ajdukiewicz's ideas form his *Pragmatic logic*.

Sierpińska: *Understanding in Mathematics*

- The quest for an explanation in mathematics cannot be a quest for proof, but it may be an attempt to find a rationale of a choice of axioms, definitions, methods of constructing of a theory. A rationale does not reduce to logical premisses. An explanation in mathematics can reach for historical, philosophical, pragmatic arguments. In explaining something in mathematics, we speak *about* mathematics: our discourse becomes more metamathematical than mathematical (76).
- Explanation of an abstract mathematical theory may consist in a construction of its model, in which the variables, rules and axioms of the theory are interpreted and acquire meaning. The model becomes a certain 'reality', ruled by its own 'laws'. In explaining a theory, we deduce its rules, axioms, definitions, and theorems from the 'laws' of the model (77).

Ideas under consideration

- The meanings of mathematical concepts are determined in the underlying theory. They are like chimeras. The *intuitive meanings* (in didactic explanations) of mathematical ideas are not determined once and for ever. We do our best in order to domesticate the chimeras.
- *Intuitive explanation* is a relational concept. It is understood here in a pragmatic sense (clarification of ideas, heuristic methods, hints facilitating understanding), without reference to the methodology of science.

- Context of transmission: the entirety of the didactic process.
- Purpose of intuitive explanations: evocation of understanding.
- Tools used in intuitive explanations: paraphrase, translation, analogy, metaphor, model building, etc.

Something to warm up

- *Euclid*: the surface of the sphere is obtained by *rotating* the semicircle around its diameter.
- *Archimedes*: sphere, cone and cylinder. Mechanical argumentation versus mathematical proof based on the method of exhaustion.

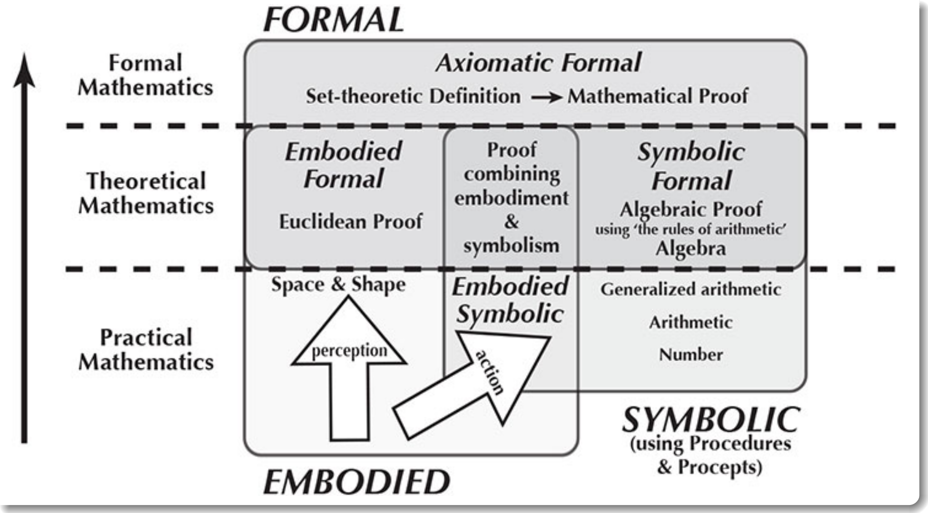
- *Riemann Hypothesis*: truth with probability one. Random variables in number theory.
- *Hilbert and Gödel*: axiom of completeness. Pragmatic justification of new axioms for set theory.
- *Cohen*: generic sets and forcing. Model theoretic investigations in set theory explained by analogy to transcendental field extensions.

Never ending story of innovations

- Piaget: stages in development.
- Vygotsky: higher cognitive functions as emerging through practical activity in a social environment.
- Polya: strategies of problem solving.
- Fischbein: intuition as the analog of perception at the symbolic level.
- Schoenfeld: metacognitive control and sense making.
- Tall: three worlds of mathematics.

- Sierpińska: *Understanding in Mathematics* (1994).
  - Understanding of mathematical concepts, ideas, problems, etc.
  - Epistemological obstacles.

Tall: three worlds of mathematics



Natural language and language of mathematics

- Sets: danger of paradoxes, fuzzy sets, troubles with infinity, *definable* versus *describable*.
- Limits: beware of metaphors!
- Continuity: explanation of a property "exceeding" density.
- Calculus: logical complexity in the ε-δ formulation. Remedy: non-standard analysis (but it faces the mathematical complexity in the definition of hyperreals).
- Notation: Leśniewski's iconic notation.

- To which degree natural language is necessary in talking about mathematics?
- What are the limits of linguistic explanations?

Sensual reports

- Pictures, drawings, diagrams, schemas: the power of visual representations and dangers of too suggestive hints. Funny quarrels concerning Venn's diagrams.
- 3D models: István Lénárt's didactic tools.
- Movies: sphere eversion, Hopf fibration, a journey into higher dimensions, etc.
- Mathematical songs: mnemonic techniques.

- Colors in graphic representations of analytic functions.
- Estimation of distance (vertical and horizontal).

Lego audio video erro ergo disco.

István Lénárt's spheres

István Lénárt

and his spheres

www.gombigeometria.eoldal.hu

Inspirations from Nature

- Archimedes: calculations of area and volume using mechanical models.
- Robert Ghrist: linkages and manifolds. *Any smooth compact manifold is diffeomorphic to the configuration space of some planar linkage*
- Mark Levi: *The Mathematical Mechanic*.
- Kinematic analogies: geometry.
- Electromagnetic fields: Poincaré about Klein's proof.
- Fluid flow and heat flow: complex analysis.
- Variational problems: Nature's wisdom.

- Probability experiments: Buffon's needle, Bertrand paradox.
- Infinity and physics: are supertasks possible?

Human experience

- Topology: rubber-like objects and operations on them.
- Geometrical representations of natural numbers (triangular, etc.).
- Negative numbers: debts, temperature, floors, jumping frogs, etc.
- Tools: compass, straight edge, ruler, pantograph, Peaucellier-Lipkin linkage, etc.
- Games: Ehrenfeucht's games, axiom of determinacy.
- Computers: software, experiments, metaphors.

- *Filters and ideals*: big and small.
- *Almost everywhere*: in finite and infinite domains.
- *Model thinks*: figurative speech.

Coherence of mathematics

- Euclid: theory of proportions in the system of geometry.
- Descartes: the origin of analytic geometry.
- Modern: algebraic topology, branches of number theory.
- Algebraic logic: mathematical approach to semantics.
- Unification programs: Weierstrass & Co., Hilbert, Thurston, Laglands.

- Generalizations of arithmetic operations and relations to abstract domains.
- Generic models in set theory and transcendental field extensions.
- Axiom of completeness in set theory: pragmatic justification.

The horizon of imagination

- Negative numbers: a few hundreds years of domestication; now acceptable by most of the population.
- Complex numbers: a few hundreds years of domestication; now standard objects for professionals, but still hard to understand by an average human being.
- Higher dimensions: rapid domestication (by professionals); still a fairy tale for most of the population.

- Infinite dimensional linear spaces
- Wild topological facts in low dimensions
- Exotic structures

Our journey continues as long as the horizon is active.

Who knows better?

- Poincaré: many kinds of mathematical intuition.
- Hadamard: stages in mathematical discovery.
- Gödel: strong Platonic position.

- Lakatos: quasi-empiricism in mathematics.
- Davis, Hersh: intuition emerging from practice.
- Thurston: high mental levels of mathematical discovery.
- Tao: three stages of understanding; good and bad intuitions.

Research problem: how to reconstruct the intuitions of professionals from the source texts?

A few standpoints in the philosophy of mathematics

We are going to avoid philosophical discussion during this talk. References to philosopher's standpoints are included in the written version of the talk.

- Descartes: understanding of proofs.
- Kant: synthetic a priori statements.

- Parsons: intuition *of* and intuition *that*.
- Tieszen: phenomenological approach.
- Byers: decisive role of antinomies and paradoxes.

Creation or Discovery in Mathematics dilemma is a philosophical problem; its solution is independent of the research practice of professional mathematicians.

Quest for truth and ways to proof

- Publications: Gauss' and Euler's styles of writing.
- Explanatory proofs: essential properties, unified perspective.
- Mathematical proofs: orthodox views and innovations (collective and computer proofs).
- Private language in mathematics: the cases of Erdős and Ramanujan.

- Imagination: the role of counterfactual thinking.
- Insight: does intuition emerge as a result of handling with mental representations of mathematical objects?
- Generalization and abstraction: creating new contexts, widening the perspective.
- Analogy: structural resemblance.
- Induction: from observed regularity to hypotheses.

Teachers and students

Teachers: the need to be persuasive.

- Teachers and textbooks.
- Teaching as negotiation.
- Drama of still changing recommendations for teaching styles.
- Didactic goals and tools confronted with teacher's own knowledge and vision of mathematics.

Students: the need for understanding.

- From idiosyncrasy to unification.
- Initial creativity and school rigor.
- Intellectual development, emotions, social abilities.

How effective it is?

- Our teaching experience is very modest and restricted to the specific audience: adult students of humanities.
- It is thus possible that our remarks and reflections are of little significance for mathematical education.

- Dangers of oversimplification and misunderstanding caused by intuitive explanations.
- Empirical research in teaching: experiments and their evaluation.

Poincaré: *Science and Method*

How is it that there are so many minds that are incapable of understanding mathematics? Is there not something paradoxical in this? Here is a science which appeals only to the fundamental principles of logic, to the principle of contradiction, for instance, to what forms, so to speak, the skeleton of our understanding, to what we could not be deprived of without ceasing to think, and yet there are people who find it obscure, and actually are the majority. That they should be incapable of discovery we can understand, but that they should fail to understand the demonstrations expounded to them, that they should remain blind when they are shown a light that seems to us to shine with a pure brilliance, it is altogether miraculous. And yet one need have no great experience of examinations to know that these blind people are by no means exceptional beings. We have here a problem that is not easy of solution, but yet must engage the attention of all who wish to devote themselves to education.

Quotation after Sierpińska 1994, 112.

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