Mathematical Therapy (for Adults)

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Motivation

- Lectures: *Puzzles* (2013–2015) for students of *cognitive science* (Adam Mickiewicz University).
- Text: *Odyssey of the Mathematical Mind* (in Polish); in preparation.
- English excerpts: *Entertaining Math Puzzles*, available on line at the web site of the *Group of Logic, Language and Information* (University of Opole).

- Main Goal: training in creative *problem solving* with special emphasis put on *paradox resolution*.
- Secondary Goal: analysis of *mistakes* caused by *common sense* intuitions.

History and resources

- Math puzzles sometimes give rise to new mathematical domains.
- Famous puzzle-makers: Archimedes, Alcuin of York, Fibonacci, Euler, Lucas, Loyd, Dudeney, Carroll, Gardner, Conway, Smullyan, Steinhaus.

- Hundreds of books with math puzzles, brain-teasers, conundrums, etc.
- Rich resources in the Internet.
- Collections of problems from mathematical competitions.
- Publications devoted to (mathematical) problem solving.

Methods

- Methodology of mathematical problems solving: Polya, Schoenefeld.
- Literature on problem solving strategies.
- Difference between a math puzzle and a standard exercise.

- Effective problem solving depends on many factors (conceptual and procedural knowledge, representation techniques, heuristic methods, metacognitive processes, attitudes, beliefs, emotion, motivation etc.).
- Comprehension of verbally formulated problems.

- Math anxiety and learning disabilities.
- Dysrationalia: the inability to think and behave rationally despite adequate intelligence.

Main Topics:

- The Infinite
- Numbers and magnitudes
- Motion and change
- Space and shape
- Orderings
- Patterns and structures
- Algorithms and computation
- Probability

- Logic puzzles
- Paradoxes
- Sophisms
- Puzzles in: physics, linguistics, philosophy.

Faith and proof

- Classic: Zeno's paradoxes (arrow, dichotomy, Achilles and Tortoise). Speedy fly (and Polish State Railways, Inc.). Hilbert's hotel.
- Smullyan's game: König's Lemma in action.

- Thomson's lamp is a device consisting of a lamp and a switch set on an electrical circuit. If the switch is *on*, then the lamp is lit, and if the switch is *off*, then the lamp is dim. Suppose that: at time $t = 0$ the switch is on, at $t = \frac{1}{2}$ it is off, at $t = \frac{3}{4}$ it is on, at $t = \frac{7}{8}$ it is off etc. What is the state of the lamp at time $t = 1$?
- Laugdogoitia's „beautiful supertask". Incompleteness of Newtonian mechanics and clash with general relativity theory.
- Hypercomputations. Is it possible to execute an infinite number of calculation steps in a finite time?

Lazy walk to infinity

- Ant on a rubber rope.
- Gabriel's horn.
- $H_n = \sum_{k=1}^{n} \frac{1}{k}$
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

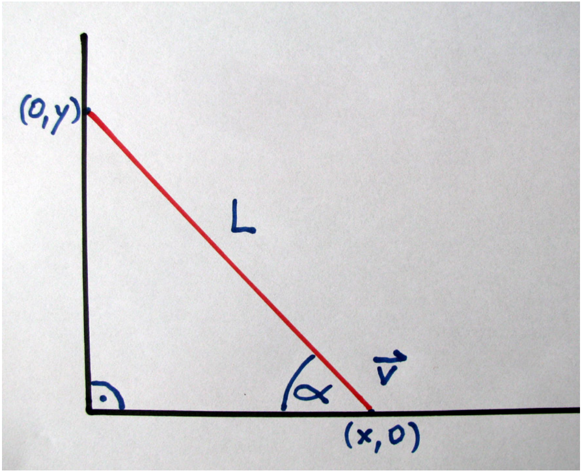
- Besicovitch's spiral: how an innocent girl can escape from a pervert.
- Jeep problem.
- Choice of the best candidate.
- Maximum hang-over.

Sum-Product Puzzle: Ignorance means strength!

- S knows only the sum and P knows only the product of two numbers x and y and they are both aware of these facts. They both know that $x > 1$, $y > 1$, $x + y \leqslant 100$. The following dialogue takes place:
  - P: I do not know the two numbers.
  - S: I knew that you do not know them.
  - P: Now I know these numbers.
  - S: Now I know them, too.
- Find these numbers x and y.

- The above statements (in this order) imply several arithmetic facts which enable us to find the (unique) solution.
- This is Freudenthal's puzzle from 1969, popularized later by Gardner.

Sliding ladder

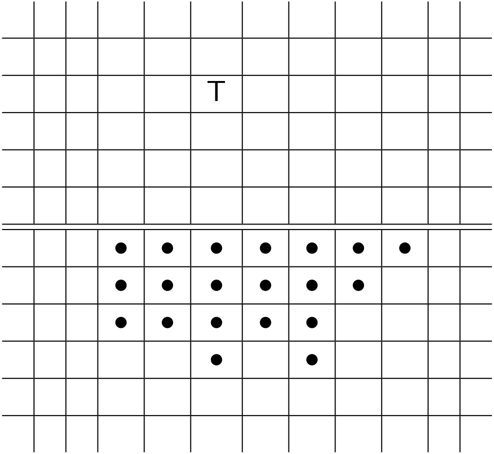


Incidentally created black hole

- A ladder of length L is leaning against a vertical wall. The bottom of the ladder is being pulled away from the wall horizontally at a uniform rate v. Determine the velocity with which the top of the ladder crashes to the floor. Bottom: $(x, 0)$, top: $(0, y)$.
  $x^2 + y^2 = L^2$ and hence $\frac{dy}{dt} = -v \cdot \frac{x}{y}$. Thus, $\frac{dy}{dt} \to \infty$ when $y \to 0$.
- Assuming that the top of the ladder maintains contact with the wall we obtain an absurdity: the velocity in question becomes infinite!

- Actually, at a certain moment the ladder looses contact with the wall. After that, the motion of the ladder is described by the pendulum equation.
- More accurate descriptions of this problem involve friction, pressure force, etc.

Example: the accessible fourth level

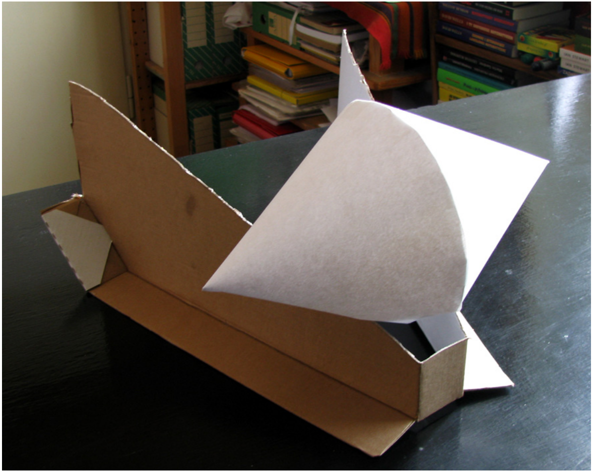


The inaccessible fifth level

- The game is played on an infinite board – just imagine the whole Euclidean plane divided into equal squares and with a horizontal border somewhere. You may gather your army of checkers below the border. The goal is to reach a specified line above the border. The checkers move only vertically or horizontally. Thus diagonal moves are excluded. As in the genuine checkers, your soldier jumps (horizontally or vertically) over a soldier on the very next square (which means that he kills him) provided that it lands on a non-occupied square next to the square occupied previously by the killed soldier.

- It is easy to show that one can reach the first, second, third and fourth line above the border. However, no finite amount of soldiers gathered below the border can ever reach (by at least one surviving soldier) the fifth line above the border!

Double cone rolling up

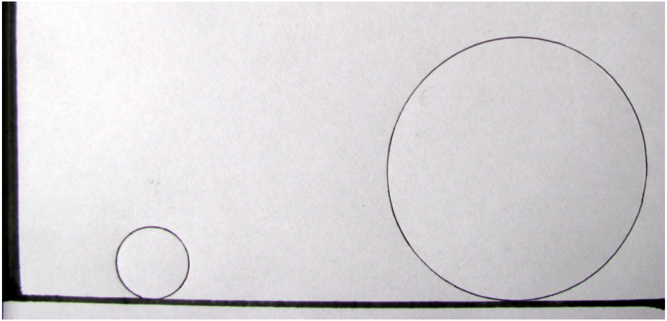


Defying gravity?!

- Imagine a double cone (two identical cones joined at their bases). It is put on the inclined rails which, in turn, are placed on a table. The rails have a common point and are diverging. The common point is the lowest point of the rails – they are directed upwards.
- Let the angle of inclination of the rails equals α, the angle between horizontal surface of the table and the up-going rails equals β and the angle at the apex of each cone equals γ. Finally, let the radius of the circle forming the common base of the cones equals r.

- Puzzle: can we determine α, β, γ and r in such a way that the double cone will be rolling upwards on the rails?
- This can be done. Obviously, the center of gravity of the cone will move downwards, obeying the law of gravity.

Bouncing balls



Watch the video Billiard Balls Count π at YouTube.

Catch π

- Let the mass of two balls be M and m, respectively. Suppose that $M = 100^n m$ ($n \geqslant 0$). We roll the ball M towards ball m which is near the wall. Thus M hits m which bounces off the wall. How many times do the balls touch each other before the ball M changes direction?
- The answer depends on n, of course.

- A surprising fact is that the number of balls' collisions is equal *precisely* to the first $n+1$ digits of π. Actually, one can obtain $n+1$ first digits of any real number, suitably modifying the ratio $\frac{M}{m}$.
- It is worth noticing that the result is purely *deterministic* and not based on probability, as in the well-known Buffon's needle puzzle.

Vision, touch, imagination

- Sections of solids (e.g. Villarceau circles, three orthogonal cyllinders).
- Toroidal puzzles.
- Tilings of the plane.
- Space fillings.

- 4D intuitions.
- Concentration of measure.
- What-is-puzzles (dimension, knot, hole, etc.).

Tomorrow, and tomorrow, and tomorrow,. . .

- Many faces of rational numbers (e.g. Euclid's Orchard, Stern-Brocot tree, Wilf-Calkin tree, etc.). Does the operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}$ make any sense?
- Condorcet paradox: non-transitivity of global preferences.

- How to order all maximal paths in the full binary tree?
- Josephus' puzzle.
- Pair of uncles: is it possible that Paul is an uncle of David and at the same time David is an uncle of Paul? No incest permissible, of course.

What is regularity?

- Combinatorial puzzles (e.g. Einstein's puzzle).
- Symmetries.

- Classifications of polyhedra.
- Properties of algebraic operations.

- Pathological curves.
- Counterexamples in topology.

Recipes and calculations

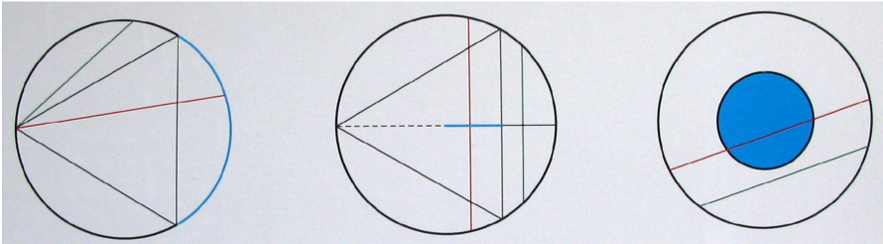
- Moser-Steinhaus notation.
- Knuth's arrow notation.

- Wolf, goat and cabbage.
- e^{π} and π^{e}.

- Many applications of Fibonacci numbers.
- Puzzles involving recurrence.

What is probability?

- A chord of a unit circle is chosen randomly. What is the probability that its length is greater than the length of the equilateral triangle inscribed in this circle?
- Possible answers: $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$. Probability depends on measure!



Lewis Carroll and Raymond Smullyan

- We use the excellent logic puzzles invented by Lewis Carroll and Raymond Smullyan.
- We recall also some well known paradoxes and sophisms presented in the literature.

- Recently translated into Polish: Raymond Smullyan *The Gödelian Puzzle Book: Puzzles, Paradoxes and Proofs*. Dover Publications, Inc., Mineola, New York, 2013.
- Currently under translation: Raymond Smullyan *A Beginner's Guide to Mathematical Logic*. Dover Publications, Inc., Mineola, New York, 2013.

Nothing as it seems

- Newton's bucket.
- Paradoxes in physics.

- Translations from an unknown language.
- Artificial languages.

- The horror of immortality.
- Yablo paradox.

Wissenschaft entsteht im Gespräch

- Students are active during classes and seem to be really interested in this enterprize. They are encouraged to use their imagination and to be creative. Besides, it is much more easier for them to acquire small, concise chunks of dissipated knowledge rather than to listen to lengthy expositions of entire theories (which they can get from textbooks anyway).
- Beware of didactic pitfalls! One has to be careful while presenting „bizarre" mathematical objects and „astounding" facts to the intellectually innocent students.

- Problem: How to convert the hate of math into admiration of math?
- Problem: Can we reasonably talk about *folk mathematics* understood as a *mathematical picture of the world* held by an average human (in a developed society)? What can math teachers do in order to improve this picture?

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