Theories in W-languages

Jerzy Pogonowski

Zakład Logiki i Kognitywistyki UAM
www.kognitywistyka.amu.edu.pl
pogon@amu.edu.pl

2021

- Roman Suszko used W-languages in a formalization of the ontology of *Tractatus* and in his works on reification of situations.
- This presentation is based on Omyła 1986, 152–165.
- The reader may also consult Roman Suszko (1968) "Ontology in the *Tractatus* of L. Wittgenstein", *Notre Dame Journal of Formal Logic* 9: 7–33.
- The language J (SCI-language with quantifiers) contains sentential variables, truth-functional connectives and quantifiers binding sentential variables.

- Models of J are structures $\mathfrak{M} = (\mathbf{A}, \bigwedge^{\mathfrak{M}}, \bigvee^{\mathfrak{M}}, D)$ such that:
  1. $(\mathbf{A}, D)$ is a SCI-model
  2. $\bigwedge^{\mathfrak{M}}, \bigvee^{\mathfrak{M}}$ are functions with domains equal to the set of all operations determined by formulas of J and values in A. We assume that:
     $\bigwedge^{\mathfrak{M}} f \in D$ iff $f(t) \in D$ for each $t \in A$
     $\bigvee^{\mathfrak{M}} f \in D$ iff $f(t) \in D$ for some $t \in A$.
- Let $WTQ = Cn(Gn\{\alpha \equiv \beta : (\alpha \leftrightarrow \beta) \in Cn(\emptyset)\})$.
- Theories in J which contain WTQ are called WTQ-theories. They formalize the idea that logically equivalent sentences describe identical situations.

- The language J^+ is obtained from J by adding sentential constants 1, 0, one-argument connectives PW, RW, SF, and two-argument connectives $\leqslant$, $\prec$. D_0 is the set of following definitions:
  1. $1 \equiv \forall p\,(p \vee \neg p)$
  2. $0 \equiv \exists p\,(p \wedge \neg p)$
  3. $(p \leqslant q) \equiv ((p \to q) \equiv 1)$
  4. $(p \prec q) \equiv (q \leqslant p)$
  5. $PWp \equiv (\neg(p \equiv 0) \wedge \forall q\,(q \prec p \vee \neg q \prec p))$
  6. $RWp \equiv (p \wedge PWp)$
  7. $SFp \equiv \forall q\,(q \to q \prec p) \wedge \forall r\,(\forall q\,(q \to q \prec r) \to p \prec r)$.
- $H = \{\forall p\,\forall q\,(((p \equiv q) \equiv 1) \vee ((p \equiv q) \equiv 0))\}$.
- Every theory in J^+ which contains $WTQ \cup D_0 \cup H$ is called WHQ-theory. The theories WTQ and are expansions of WT and WH, respectively.

Theorems of WHQ include all Boolean equations and their generalization in J^+, as well as generalizations represented by the following schemes (α, β, γ are arbitrary formulas, $i \neq j$, p_i is not a free variable in γ):

1. $\alpha[p_i/\beta] \prec \forall p_i\ \alpha$
2. $\exists p_i\ \alpha \prec \alpha[p_i/\beta]$
3. $\forall p_j\ (\forall p_i\ (\gamma \prec p_j) \rightarrow (\forall p_i\ \gamma \prec p_j))$
4. $\forall p_j\ (\forall p_i\ (p_j \prec \gamma) \rightarrow (p_j \prec \exists p_i\ \gamma))$.

A model $\mathfrak{M} = (\mathbf{A}, \bigwedge^{\mathfrak{M}}, \bigvee^{\mathfrak{M}}, D)$ of J^+ is called a WHQ-model, when $WHQ \subseteq TR(\mathfrak{M})$. WHQ-models have the following properties:

1. Algebra $\mathbf{A}$ is a Boolean algebra.
2. Sentential constants 1, 0 are interpreted in $\mathfrak{M}$ as, respectively, $1^{\mathbf{A}}$ and $0^{\mathbf{A}}$.
3. $\leqslant^{\mathbf{A}}$ is an interpretation of $\leqslant$ defined as follows: $a \leqslant^{\mathbf{A}} b$ iff $a \rightarrow^{\mathbf{A}} b = 1^{\mathbf{A}}$. It is a Boolean ordering in $\mathbf{A}$.
4. For any valuation h of variables of J in a model $\mathfrak{M}$:
    1. $||p \leqslant q, h|| = 1^{\mathbf{A}}$ or $||p \leqslant q, h|| = 0^{\mathbf{A}}$
    2. $||p \leqslant q, h|| = 1^{\mathbf{A}}$ iff $h(p) \leqslant^{\mathbf{A}} h(q)$.
5. The connective $\prec$ is interpreted in a model $\mathfrak{M}$ as a relation $\leqslant^{\mathbf{A}}_{*}$, which is the converse of $\leqslant^{\mathbf{A}}$.

- Algebra of any WHQ-model is treated as the algebra of situations existing in this model, and D is the set of facts of this model.
- $\leqslant_*^{\mathbf{A}}$ is the ordering in the algebra of situations. We read $p \prec q$ as: "situation p is included in situation q" (or "situation p holds in situation q").
- $1^{\mathbf{A}}$ (the empty situation, the improper fact in $\mathfrak{M}$) is included in any situation from $\mathbf{A}$, because $1 \prec p$ is a theorem of any WHQ-theory.
- $0^{\mathbf{A}}$ is the impossible (inconsistent) situation in a model $\mathfrak{M}$.
- Each WHQ-model contains exactly one improper fact and exactly one inconsistent situation.
- It follows from the axiom $\forall p\, \forall q\, (((p \equiv q) \equiv 1) \vee ((p \equiv q) \equiv 0))$ that to each equality from J there corresponds in any model of J either the empty situation of the inconsistent situation.

- $\sup(X)$ and $\inf(X)$ (where $X \subseteq A$) denote, respectively, the upper and lower bound of X w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$ (if they exist).
- **Theorem** (Omyła). A model $\mathfrak{M} = (\mathbf{A}, \bigwedge^{\mathfrak{M}}, \bigvee^{\mathfrak{M}}, D)$ of J^+ is a WHQ-model iff:
  1. $\mathbf{A}$ is a Henle algebra.
  2. For any function f determined by a formula of J^+:
     1. $\bigwedge^{\mathfrak{M}} f = \sup\{f(t) : t \in A\}$
     2. $\bigvee^{\mathfrak{M}} f = \inf\{f(t) : t \in A\}$.
  3. D is a Boolean ultrafilter such that $\bigvee^{\mathfrak{M}} f \in D$ iff $f(t) \in D$ for some $t \in A$. □

- **Theorem**. Algebra **A** of any WHQ-model $\mathfrak{M} = (\mathbf{A}, \bigwedge^{\mathfrak{M}}, \bigvee^{\mathfrak{M}}, D)$ uniquely determines the functions $\bigwedge^{\mathfrak{M}}$, $\bigvee^{\mathfrak{M}}$, which are interpretations of quantifiers binding sentential variables. Any WHQ-model can thus be denoted by $\mathfrak{M} = (\mathbf{A}, D)$. □
- **Theorem**. For any WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$ and any valuation h of J^+ in this model and for any formula α built from equalities, truth-functional connectives and quantifiers: $||\alpha, h|| = 0^{\mathbf{A}}$ or $||\alpha, h|| = 1^{\mathbf{A}}$. □
- Formula α built from equalities, truth-functional connectives and quantifiers are called special formulas.

- **Theorem**. For any special formula α of J^+, any WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$ and any valuation h of J^+ in this model:

  1. $\sup\{t \in A : ||\alpha, h_t^p|| \in D\} = ||\forall p\, (\alpha \to p), h||$
  2. $\inf\{t \in A : ||\alpha, h_t^p|| \in D\} = ||\exists p\, (\alpha \wedge p), h||$.

  If $\alpha(p)$ is a special formula with one free variable p, then for any WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$:

  1. the upper bound of the set of all elements satisfying the formula $\alpha(p)$ in $\mathfrak{M}$ equals $||\forall p\, (\alpha(p) \to p)||_{\mathfrak{M}}$
  2. the lower bound of the set of all elements satisfying the formula $\alpha(p)$ in $\mathfrak{M}$ equals $||\exists p\, (\alpha(p) \wedge p)||_{\mathfrak{M}}$.

- **Proof**. It follows from the Omyła's theorem that:
  $||\forall p\, (\alpha \to p), h|| = \sup\{||(\alpha \to p), h|| : t \in A\}$.

- If α is special, then for every $t \in A$: $||\alpha, h_t^p|| = 1^{\mathbf{A}}$ or $||\alpha, h_t^p|| = 0^{\mathbf{A}}$.
  - If $||\alpha, h_t^p|| = 1^{\mathbf{A}}$, then $||(\alpha \to p), h_t^p|| = t$
  - If $||\alpha, h_t^p|| = 0^{\mathbf{A}}$, then $||(\alpha \to p), h_t^p|| = 1^{\mathbf{A}}$.
- Therefore $\sup\{t \in A : ||\alpha, h_t^p|| \in D\} = ||\forall p\, (\alpha \to p), h||$.
- In a similar way we prove that $\inf\{t \in A : ||\alpha, h_t^p|| \in D\} = ||\exists p\, (\alpha \wedge p), h||$. □

For any WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$ and any valuation h of J^+ in this model:

- An element $a \in A$ satisfies the formula PWp (meaning $||PWp, h_a^p|| \in D$) iff a is not the inconsistent situation and for every $b \in A$: b holds in a or $\neg^{\mathbf{A}} b$ holds in a. PWp reads: "a situation p is a possible world in a model $\mathfrak{M}$". We see that p is a possible world in a model $\mathfrak{M}$ iff a is an atom w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$ in $\mathbf{A}$. Possible worlds in a WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$ are thus maximal elements w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$ in the set $A - \{0^{\mathbf{A}}\}$.

- An element $a \in A$ satisfies RWp (meaning $||RWp, h_a^p|| \in D$) iff $a \in D$ and a satisfies PWp. RWp reads: "a situation p is the real world in the model $\mathfrak{M}$". Thus, a is the real world in a model $\mathfrak{M}$, when a is a possible world in $\mathfrak{M}$ and it is a fact in $\mathfrak{M}$.
  We see that a is the real world in $\mathfrak{M}$, when the set D of facts, being a Boolean ultrafilter, is generated by a. This means that a is a fact and contains (w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$) all facts which hold in $\mathfrak{M}$.
- An element $a \in A$ satisfies SFp iff a is the upper bound of D w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$. SFp reads: "a situation p is the upper bound of facts which hold in $\mathfrak{M}$" (or: "a situation p is the sum of facts which hold $\mathfrak{M}$").

- **Theorem**. In any WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$ there exist:
  1. the upper and lower bounds of the set of facts
  2. the upper and lower bounds of the set of possible worlds.
- **Proof**. The lower bound of the set of all facts in any WHQ-model is the improper fact in this model.
- We are going to show that the set of all facts in any WHQ-model has the upper bound in this model.
- Suppose that in some WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$ there does not exist the upper bound of the set of all facts.
- This means that the sentence $\neg\exists p\,(\forall q\,(q \rightarrow (q \prec p)) \wedge \forall r\,(\forall q\,(q \rightarrow (q \prec r)) \rightarrow (p \prec r)))$ is true in the model $\mathfrak{M}$.
- Therefore also the sentence $\forall p\,(\exists q\,(q \wedge \neg(q \prec p)) \vee \exists r\,(\forall q\,(q \rightarrow (q \prec r)) \wedge \neg(p \prec r)))$ is true in $\mathfrak{M}$.

- For $p = 0$ the following alternative $\exists q\,(q \wedge \neg(q \prec 0)) \vee \exists r\,(\forall q\,(q \rightarrow (q \prec r)) \wedge \neg(0 \prec r))$ is true in $\mathfrak{M}$.
- Because $q \prec 0$ for each q, the first component if this alternative is false in $\mathfrak{M}$.
- Hence $\exists r\,(\forall q\,(q \rightarrow (q \prec r)) \wedge \neg(0 \prec r))$ is true in $\mathfrak{M}$.
- Then there exists a valuation h in $\mathfrak{M}$ such that $||\forall q\,(q \rightarrow (q \prec r)) \wedge \neg(r \equiv 0), h|| \in D$.
- Either $h(r) \in D$ or $h(\neg r) \in D$, both not both.
  1. If $h(r) \in D$, then $||r \wedge \forall q\,(q \rightarrow (q \prec r)), h|| \in D$, which means that $h(r)$ is the upper bound of all facts (because it is a fact and contains all facts), and this contradict our supposition.
  2. Let us assume that $h(\neg r) \in D$. Then $||(\neg r \prec r) \wedge \neg(r \equiv q), h|| \in D$, and this is equivalent to $||(\neg r \equiv 1) \wedge \neg(r \equiv 0), h|| \in D$. But the formula $(\neg r \equiv 1) \wedge \neg(r \equiv 0)$ is satisfied in no WHQ-model.

Therefore we must reject the supposition that the upper bound of the set of all facts in $\mathfrak{M}$ does not exist.

- The upper bound of the set of all possible worlds in a WHQ-model $\mathfrak{M}$ can be defined as: $\sup\{t \in A : ||PWp, h^p_t|| \in D\}$, where h is a valuation of variables of J^+ in $\mathfrak{M}$.
- The lower bound of the set of all possible worlds in a WHQ-model $\mathfrak{M}$ can be defined as: $\inf\{t \in A : ||PWp, h^p_t|| \in D\}$, where h is a valuation of variables of J^+ in $\mathfrak{M}$.
- We remember that $||PWp, h||$ is the value of certain special formula. Hence:
- $\sup\{t \in A : ||PWp, h^p_t|| \in D\} = ||\forall p\, (PWp \rightarrow p)||$
- $\inf\{t \in A : ||PWp, h^p_t|| \in D\} = ||\exists p\, (p \wedge PWp)||$.
- Because the values on the right hand side of these equalities are well defined in $\mathfrak{M}$, the upper and lower bounds of the set of all possible worlds in $\mathfrak{M}$ exist in $\mathfrak{M}$. □

- **Corollary**. For any WHQ-model $\mathfrak{M} = (\mathbf{A}, D)$:
  1. $||\forall p\,(PWp \rightarrow p)||$ is the upper bound of the set of all possible worlds in $\mathfrak{M}$.
  2. $||\exists p\,(p \wedge PWp)||$ is the loer bound of the set of all possible worlds in $\mathfrak{M}$. □

- The mere existence of the bounds of the set of all facts follows from the fact that the algebra of situations is a Boolean algebra. The proof of the existence of bounds of the set of possible worlds requires reference to the interpretation of quantifiers in WHQ-models and the axiom H.
- A sufficient and necessary condition for the *non-existence* of situations being possible worlds is simply the case when the upper bound of the set of possible worlds is the empty situation (the improper fact) and the lower bound of this set is the inconsistent situation.
- There exists in $\mathfrak{M}$ a situation which is the real world iff the lower bound of the set of all possible worlds is a fact.

- For any situation a from the algebra $\mathbf{A}$, different from $0^{\mathbf{A}}$, the situation a holds in some possible world iff $||\exists p\,(p \wedge PWp)|| = 1^{\mathbf{A}}$, which means that the lower bound of the set of possible worlds is the empty situation (the improper fact):

  1. Assume that $a \in A$, $a \neq 0^{\mathbf{A}}$ and for any b which is a possible world in $\mathfrak{M}$, a does not hold in b. Then $\neg^{\mathbf{A}} a$ holds in every possible world in $\mathfrak{M}$, and this in turn implies that $\neg^{\mathbf{A}} a$ is included in the situation being the lower bound of the set of all possible worlds. Hence we have: $1^{\mathbf{A}} \neq \neg^{\mathbf{A}} a \leqslant_*^{\mathbf{A}} ||\exists p\,(p \wedge PW_p)||$, and this means that the lower bound of the set of all possible worlds in $\mathfrak{M}$ is not the improper fact in $\mathfrak{M}$.
  2. Assume that $||\exists p\,(p \wedge PW_p)|| = a \neq 1^{\mathbf{A}}$. Then $\neg^{\mathbf{A}} a$ is not the inconsistent situation and since a holds in every possible world, the situation $\neg^{\mathbf{A}} a$ holds in none of them. This means that there exists a situation different from the inconsistent one and not holding in any possible world in $\mathfrak{M}$.

- Suszko proposed as a formalization of the ontology presented in *Tractatus* the theory WHQ with the additional axioms:
  1. $\forall p\,(\neg(p \equiv 0) \rightarrow \exists q\,(PWq \wedge (p \prec q)))$
     This axiom says that every situation different from the inconsistent situation holds in some possible world.
  2. $\exists p\,(p \wedge PWp)$
     This axiom asserts the existence of a situation being the real world.
  3. $\exists p\,(\forall q\,(\alpha(q) \rightarrow (q \prec p)) \wedge \forall r\,(\forall q\,(\alpha(q) \rightarrow (q \prec r)) \rightarrow (p \prec r)))$
     This axiom scheme says that for any formula α from J^{+} there exists the upper bound of the set of all elements which satisfy this formula.
  4. $\exists p\,(\forall q\,(\alpha(q) \rightarrow (p \prec q)) \wedge \forall r\,(\forall q\,(\alpha(q) \rightarrow (r \prec q)) \rightarrow (r \prec p)))$
     This axiom scheme says that for any formula α from J^{+} there exists the lower bound of the set of all elements which satisfy this formula.
- The theory WHQ augmented in this way is consistent, because it is satisfied in the Fregean model $(\mathbf{A}, D)$, where $\mathbf{A}$ is a two-element Henle algebra with domain $\{0, 1\}$ and $D = \{1\}$.

Suszko accepted the following assumptions concerning any model $\mathfrak{M} = (\mathbf{A}, D)$ of J^+ being a model of a theory of situations:

- The algebra **A** is a Henle algebra and it is atomic w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$.
- The algebra **A** is elementarily complete, which means that for any formula α from J^+ and for any valuation h there exist both bounds (w.r.t. the ordering $\leqslant_*^{\mathbf{A}}$) of the set $\{t \in A : ||\alpha, h_t^p|| \in D\}$.)
- The set D is a Boolean ultrafilter generated by some atom of the algebra **A**, which means that there exists a fact which includes all facts of the model $\mathfrak{M}$.

- Bloom, S. 1971. A completeness theorem for "Theories of kind *W*". *Studia Logica* 27, 43–55.
- Bloom, S., Suszko, R. 1972. Investigations into the sentential calculus with identity. *Notre Dame Journal of Formal Logic* 13 (3), 289–308.
- Omyła, M. 1986. *Zarys logiki niefregowskiej*. Państwowe Wydawnictwo Naukowe, Warszawa.
- Suszko, R. 1968. Ontology in the *Tractatus* of L. Wittgenstein. *Notre Dame Journal of Formal Logic* 9, 7–33.
- Suszko, R. 1975. Abolition of the Fregean axiom. *Lecture Notes in Mathematics* 453, 169–239.