STANDARD, EXCEPTION, PATHOLOGY

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ABSTRACT. We discuss the meanings of the concepts: *standard*, *exception* and *pathology* in mathematics. We take into account some pragmatic components of these meanings. Thus, standard and exceptional objects obtain their status from the point of view of an underlying theory and its applications. Pathological objects differ from exceptional ones – the latter may e.g. form a collection beyond some established classification while the former are in a sense unexpected or unwilling, according to some intuitive beliefs shared by mathematicians of the given epoch. Standard and pathology are – to a certain extent – flexible in the development of mathematical theories. Pathological objects may become „domesticated" and give rise to new mathematical domains. We add a few comments about the dynamics of mathematical intuition and the role of extremal axioms in the search of intended models of mathematical theories.

1 Mathematical savoir vivre

Contrary to that what a reader might expect from the title of this section we are not going to discuss the behavior of mathematicians themselves. Rather, we devote our attention to the following phrase which is very common in mathematical texts: *An object X is well behaving*. Here X may stand for a function, a topological space, an algebraic structure, etc.

It should be overtly stressed that *well behaving* of mathematical objects is always related to some investigated theory or its applications. There is nothing like *absolute well behavior* – properties of objects are evaluated from a pragmatic point of view. Thus, we meet declarations similar to the following, in specific domains of mathematics:

1. *Topology*. Hausdorff spaces behave better than general topological spaces.
2. *Set theory*. Borel sets behave better than arbitrary sets.
3. *Analysis*. Differentiable functions behave better than continuous functions. Analytic functions (i.e functions from the class C^ω) behave better than smooth functions (i.e functions from the class C^∞).
4. *Algebra*. Fields behave better than rings.
5. *Computation theory*. Recursive functions behave better than arbitrary functions.
6. *Geometry*. Platonic solids behave better than arbitrary polyhedra.

It should also be mentioned that *well behaving* has nothing to do with *to be in majority*. Analytic functions are rare among continuous functions, almost every function is not differentiable anywhere, there are only $\aleph_0$ recursive functions, etc. Platonic solids are very special convex polyhedra and are in minority, too. Thus, *well behaving* corresponds rather to a prototypic object, useful in a common practice. Such objects occurred as something like patterns at the beginning of the corresponding theory.

The concept in question is also related to the history of mathematics. Objects may become well behaving only when a theory underlying them is sufficiently developed. For example, complex numbers were viewed as well behaving only after proving that they form a field. Similarly for the negative numbers (which, together with natural numbers form a structure closed under certain operations). Further examples include e.g. fractals or purely random sequences.

It is worth to add that there exists a certain limitation to the expressive power of the language of mathematics. Each theory uses at most countably many expressions because mathematical formulas are finite strings of symbols from at most countable vocabulary. On the other hand, one proves the existence of many uncountable sets in many mathematical domains. Due to the mentioned limitation, only countably many of them can obtain *names* in the corresponding domain. One should strictly distinguish between *effective descriptions* and *definitions* of mathematical objects. For example, we can effectively describe only countably many ordinal numbers (using suitable systems of ordinal notation) while we can of course define also several uncountable ordinal (and cardinal) numbers.

Exposition of any mathematical theory usually starts with description of objects which do well behave with respect to this theory or its applications. Then the presentation is extended in order to point out to several *counterexamples*. There exist nice collections of counterexamples in main mathematical theories – cf. e.g.

Gelbaum, Olmsted 1990, 2003, Steen, Seebach 1995, Wise, Hall 1993. The role of counterexamples is discussed e.g. in the classical work Lakatos 1976.

We distinguish *monomathematics* from *polymathematics* (terms introduced in Tennant 2000). In monomathematics one deals with a specific mathematical structure – as e.g. natural or real numbers. In polymathematics, in turn, one speaks about classes of structures – as e.g. groups, rings, fields, topological spaces, Hilbert spaces, Banach spaces, graphs, etc. It seems that it is polymathematics where talking about well behaving objects makes any sense at all.

2 Standard

First of all, one should notice that the concept of a *standard* object is not limited to the objects *called* standard in mathematical terminology. If an object *is* called standard by mathematicians, then this fact expresses their view that the object is somehow typical in the considered domain. There may exist, however, many other objects which deserve a qualification of being typical, prototypic, normal, etc. In some cases the name *standard* is given to structures which are *intended* models of a given theory. We will say a few words about intended models in the last part of the paper.

One of the important procedures in mathematics is to represent objects from a given domain in so called *normal*, *standard*, *canonical*, etc. forms. It is often the case that only after such a representation one is able to prove theorems about the initial – allegedly diversified – class of objects. Examples of the transformation in question are, among many others:

1. *Logic*. Conjunctive and disjunctive normal forms, prefix normal form, Skolem normal form, etc.
2. *Set theory*. Cantor normal form (for ordinal numbers).
3. *Algebra*. Jordan normal form (for matrices over an algebraically closed field).
4. *Number theory*. Canonical representation of integers (as products of powers of prime numbers).
5. *Analysis*. Canonical differential forms.
6. *Formal languages*. Chomsky, Greibach, Kuroda normal forms.
7. *Recursion theory*. Kleene normal form.

Another procedure, in a sense similar to the one mentioned above, is based on *representation theorems* – a very powerful tool in many mathematical domains. Examples of such theorems are:

1. *Stone Representation Theorem*. Each Boolean algebra is isomorphic to a field of sets.
2. *Cayley Representation Theorem*. Each finite group is isomorphic to some group of permutations.
3. *Mostowski Contraction Lemma*. Every extensional well founded structure is isomorphic to a transitive structure.
4. *Riesz Representation Theorem*. Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Then for every linear functional y^* from the dual space H^* there exists exactly one element $y \in H$ such that

   $$y^*(x) = \langle x, y \rangle$$

   for all $x \in H$. The mapping $y^* \mapsto y$ is an anti-linear one-one isometry.
5. *Nash Theorem*. Every Riemann manifold can be isometrically embedded in some Euclidean space.
6. *Whitney Representation Theorem*. For any m-dimensional differential manifold there exists its embedding into a $2m$-dimensional Euclidean space ($m > 0$).
7. *Representation Theorem for Lie algebras*. Every Lie algebra over a field is isomorphic to a subalgebra of some associative algebra.
8. *Gödel Representability Theorem*. Every recursive function is representable in Peano arithmetic.

In a sense, each completeness theorem in logic is a kind of a representation theorem. Also algebraic characterization of elementary equivalence serves as an example of representation of semantic notions.

Further, also several *classification procedures* may be conceived of as bringing order to the domain in which we observe some regularities. Here are a few examples:

1. *Classification of finite simple groups*. Every finite simple group is isomorphic to one of the following groups:

(a) a cyclic group of prime order
(b) an alternating group of degree at least 5
(c) a simple group of Lie type (among which several further classifications are made)
(d) one of the 26 sporadic simple groups.

2. *Classification of simple Lie groups*. Each simple Lie group either belongs to one of the infinite series A_n, B_n, C_n, D_n, or else is isomorphic with one of the five exceptional simple Lie groups, i.e. with one of the: G_2, F_4, E_6, E_7, E_8. The series correspond to:

   (a) A_n series: special linear groups $SL(n+1)$
   (b) B_n series: special orthogonal groups $SO(2n+1)$
   (c) C_n series: symplectic groups $Sp(2n)$
   (d) D_n series: special orthogonal groups $SO(2n)$.

3. *Classification of surfaces*. Every closed compact connected surface without boundary is homeomorphic with one of the three following:

   (a) two-dimensional sphere
   (b) connected sum of g thori, $g \geqslant 1$
   (c) connected sum of k projective planes, $k \geqslant 1$.

4. *Classification of Riemann surfaces*. Each simply connected Riemann surface is conformally equivalent to one of the following surfaces:

   (a) *elliptic* – the Riemann sphere $\mathbb{C} \cup \{\infty\}$
   (b) *parabolic* – the complex plane $\mathbb{C}$
   (c) *hyperbolic* – the open disc $D = \{z \in \mathbf{C} : |z| < 1\}$.

5. *Frobenius Theorem*. Every associative algebra over the field of real numbers is isomorphic with one of the fields: $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers) or $\mathbb{H}$ (quaternions).

6. *Hurwitz Theorem*. Any normed algebra with division is isomorphic either with $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$ or $\mathbb{O}$ (octonions).

7. *Ostrowski Theorem*. Any field complete with respect to an Archimedean norm is isomorphic with either $\mathbb{R}$ or $\mathbb{C}$ and the norm is equivalent with the usual norm determined by the absolute value.

8. *Pontriagin Theorem*. Any connected locally compact topological field is isomorphic with either $\mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$.

Classifications are ubiquitous in mathematics. We form more and more abstract objects (e.g. quotient spaces, quotient algebras, etc.) starting from an initial universe and creating new objects as equivalence classes.

A very special standard structure in mathematics is the set of all natural numbers (with operation of addition and multiplication as well as with the successor operation and zero as a distinguished element). This structure precedes, both conceptually and historically, other number systems (rational numbers, integers, reals, complex numbers). There are two ways in which one can create number systems:

1. *Genetic*. One generalizes the concept of number, passing from one domain to another, wider domain. Operations which were not allowed in the starting domain (as e.g. division of integers) become possible in a wider domain. Older domains become special cases of new domains. The domains are interconnected by homomorphisms, preserving the arithmetical operations. This was the way arithmetic was developed up to the middle of XIX century.

2. *Axiomatic*. Modern approach to number systems is an axiomatic one: we specify the axioms which must be obeyed by (arithmetical) objects and operations on them. This approach originates with the works of Peano, Grassmann, Dedekind, Weber, Hilbert.

Before „the structural revolution" which took place in mathematics of the XIX century *non-standard* could mean rare from the point of mathematical practice or rejected on the basis of some intuitive beliefs. The Pythagoreans were shocked by non-commensurable magnitudes. Hence irrationality of, say, $\sqrt{2}$ meant the presence of an „alien intruder" in the harmonious world of rational numbers. Similarly, early algebraists thought of $\sqrt{-1}$ as something belonging to mathematical fiction and not mathematical reality. This is reflected in the name *imaginary number* used also today. The common belief that the Euclidean geometry describes accurately and adequately the physical world was responsible for the view that it is the only „true" geometry and as such it is the only standard.

The discovery of non-Euclidean geometries, the development of abstract algebra (groups, rings, fields, etc.), the arithmetization of analysis – these great achievements of mathematics in XIX century created new contexts in which the notion *standard* might be used. It seems that recently by *standard* (objects or structures) one understands the prototypic, most successful in applications, somehow central in the investigated domain objects or structures. To a certain extent this understanding is reflected in teaching of mathematics (at least below the university level).

The standard model of Peano arithmetic is uniquely (up to isomorphism) characterized in a second order language. Mathematicians themselves take for granted that it is a perfectly determined structure. The logicians, in turn, keep on recalling that the first-order Peano arithmetic is not complete (and not categorical in any infinite power). This theory has continuum distinct countable models. Moreover, it has 2^κ (the highest possible number of) different models in any infinite power κ. It is a *wild* theory. At the moment, non-standard models of arithmetic are not widely used in mathematics – with a notable exception of non-standard analysis, where they enable us to talk in a precise way of *infinitely small* (and *infinitely large*) numbers. Recall that these infinitesimals were accepted by Leibniz (though with only metaphysical and not mathematical explanation), then were forbidden in the time of Cauchy, Weierstrass and their contemporaries. It was only some sixty years ago when they obtained a precise mathematical formulation proposed by Abraham Robinson.

The problem of establishing what is *standard* is more complicated in set theory (more exactly, in Zermelo-Fraenkel first order set theory, which – as it is claimed – may serve as a basis for all the mathematics). The accepted linguistic usage requires to call a model $\mathfrak{M}$ of this theory *standard*, if the element relation $\in^{\mathfrak{M}}$ of the model is the actual element relation $\in$ restricted to the universe of the model $\mathfrak{M}$. Each transitive model is standard and each standard model is well founded. Assuming the existence of a standard model implies the existence of a minimal standard model which is contained in all standard models.

However, set theory is far from being categorical. The method of forcing (Cohen 1966) can be used for the construction of many diversified models of set theory. „Normal" mathematicians (i.e. these who are not working on set theory) seem to believe in the universe of true sets and do not worry about subtle incompleteness phenomena inseparable from set theory itself. It is not possible to predict the future of set theory. It may happen that it will be replaced by another (more fundamental?) theory. It may happen that we will think of set theory in a similar way we think currently about such branches of mathematics as algebra or topology (cf. Mostowski 1967).

3 Exception

What is an *exceptional* object in mathematics? First, we are not talking about exceptional objects in any *absolute* sense. Exceptions are always connected with some results established within a certain mathematical theory. The most typical situation occurs when one classifies objects (according to some chosen criteria) and a few objects do not fit into any of the classes.

1. *Sporadic (finite simple) groups*. The classification of all finite simple groups gives, besides a few infinite regular families of such groups also 26 groups not showing any predictable pattern. You may call them exceptional groups, but from a purely formal point of view there is nothing strange in the fact that some classes of the classification contain only one member.

2. *Special Lie groups*. The five groups G_2, F_4, E_6, E_7, E_8 are distinguished among all simple Lie groups, because they do not fit into any of the four patterns of all the remaining such groups. Properties of the groups from the four infinite series are well known. Many properties of the special groups are also known – in particular, there are numerous connections between these groups and concepts from theoretical physics, as it was shown in the last decades.

The term *exceptional* can be applied not only to objects that lack properties taken into account during classifications but also to such objects which are distinguished due to possession of regularities absent in all other objects from a given class.

1. *Platonic solids in n dimensions*. In three dimensions there are exactly five Platonic solids, as it is well known from antiquity. They are convex polyhedra with faces formed by regular polygons such that at each vertex there is the same number of edges meeting at it. The Platonic solids are exceptional between convex polyhedra in the sense that they possess a lot of nice symmetries. Such regularly symmetric cells exist also in higher dimensions – e.g. in the fourth dimension there are six of them.

2. *Standard model of Peano arithmetic*. On the one hand, natural numbers are the most standard objects in mathematics. On the other, the standard model $\mathfrak{N}_0$ of Peano arithmetic PA is a very exceptional structure in the realm of all countable models of this theory:

   (a) $\mathfrak{N}_0$ is the only well-ordered model of PA

   (b) $\mathfrak{N}_0$ is the only recursive model of PA (Tennenbaum theorem)

   (c) $\mathfrak{N}_0$ is a prime model of PA.

Other contexts in which one talks about exceptional objects include looking for *counterexamples*.

1. *Alexander horned sphere*. This topological object is homeomorphic with the sphere S^2. However, it divides the space $\mathbb{R}^3$ in a different way S^2 does: its

inside is homeomorphic with the inside of S^2 but its outside is not homeomorphic with the outside of S^2. Hence it provides a counterexample to the Jordan-Schönflies theorem in three dimensions.

2. *Mertens conjecture*. The Mertens function $M(n)$ is defined by:

$$M(n) = \sum_{1 \leqslant k \leqslant n} \mu(k),$$

where μ is the Möbius function. The Mertens conjecture states that for all $n > 1$:

$$|M(n)| < \sqrt{n}.$$

It was conjectured by Stieltjes (1885, published 1905) and Mertens (1897) and disproved by Odlyzko and te Riele (1985). The first counterexample to the conjecture, though not explicitly known today, lies somewhere between 10^{14} and $exp(1.59 \cdot 10^{40})$. In 2006 Kotnik and te Riele proved that there exist infinitely many numbers n such that $\frac{M(n)}{\sqrt{n}} > 1.2184$, but so far no examples of them are known.

Still another situation when mathematicians talk about exceptions (or even pathology) is the absence of a recognizable pattern in a class (or series) of objects for which they expected to find such a pattern. But not always lack of the pattern indicates pathology:

1. *Prime numbers*. The prime numbers form a very distinguished sequence of natural numbers. The concept of a prime number is very elementary – it is, as a matter of fact, recursive. There exist algorithms which establish whether a given natural number is prime or not. However, the distribution of primes remains one of the most mysterious problem of not only number theory but also some other branches of mathematics. One can prove that there are infinitely many primes of a given form, one can prove that there occur arbitrarily long arithmetic sequences consisting entirely of primes, one can estimate the number of primes below a given natural number, etc. However, one can not say that we fully recognize the pattern according to which primes are allegedly distributed among all natural numbers.

4 Pathology

The very term *pathological* immediately implies some negative associations. Mathematical objects named *pathological* (sometimes also: *paradoxical*) appear as unexpected and, moreover, unwilling.

There seem to be at least two typical situations in which one speaks about pathological objects in mathematics:

1. *A clash with established intuitions*. At a given epoch, mathematicians share intuitive views about the concepts they are dealing with. Discoveries of new kinds of objects may contradict these intuitions. But if the new objects appear fruitful in applications, if they are equipped with a sound theory, then the initial intuitions are forced to change.

   (a) *Infinity*. During centuries of its evolution mathematics continuously escaped from the fear of infinity. Actual infinity was treated with suspicion in the ancient times, then it became more and more domesticated, so to speak. We can see this process in the development of calculus, in the work with infinite series, in reaching the formally correct definition of a limit, etc. Some properties of infinite collections seemed paradoxical (e.g. contradicting the postulate of Euclid that a part can not be greater than the whole), but then a change of perspective took place: the numerical equivalence of a set with its proper subset was taken as a formal definition of an infinite set.

   (b) *Numbers*. Mathematicians gradually overcame the resistance against new kinds of numbers which naturally arose in arithmetical and algebraic problems. Thus, irrational, negative and imaginary numbers entered the realm of mathematics as fully accepted objects only after it was shown that they behave properly – form structures closed with respect to prescribed operations.

2. *New definitions of concepts formerly understood in an intuitive way*. This situation arose e.g. in the case of a general definition of *function*. Formerly, by a function one understood a recipe according to which one value could be obtained as depending on other values. After accepting the most general definition – a function as a set of ordered pairs – a plenty of *monsters* entered the stage. The first examples of continuous though nowhere differentiable functions were given by Bolzano and Weierstrass. „Space filling" functions were defined by Peano and Hilbert. All these monstra contradicted some intuitive beliefs about that how (the diagram of) a function should look like. This was the price for replacing intuitive views (based e.g. on concepts from outside mathematics, such as motion) by precise formal definition. Without any doubt this was a just solution: the development of mathematics is impossible without intuitions, but they should be nevertheless always abandoned when the deductive work begins.

A prototypic example of an object originally thought of as a pathological one and then becoming normal, standard, „domesticated" is the Cantor set. Recently no one among professional mathematicians considers it as a pathology. This is due to its fundamental role in e.g. topology. Only in popular books authors try to frighten the innocent readers with devilish mysteries of the Cantor set and other fractal sets.

Traditionally, when speaking of paradoxical objects and constructions one recalls the Banach-Tarski theorem which – as it is sometimes claimed – strongly violates our intuitions concerning volume and, in general, measure. It should be kept in mind, however, that the proof of this theorem is highly ineffective (axiom of choice is used) and the „pieces" into which the original ball is „cut" are non-measurable in the sense of Lebesgue. Hence, if one wants to talk about violation of intuitions in this case, then one should refer rather to the proof, and not to the theorem itself.

It is ridiculous – in our opinion at least – to expect that our common intuitions connected with everyday experience should be respected at any stage of sophisticated mathematical constructions. At any rate, there are cases in which operating with well known objects, using perfectly natural proof techniques one arrives at results which are strongly divergent from common intuitions:

1. *Smale's theorem*. This theorem states that it is possible to turn a sphere S^2 inside out in a three-dimensional space $\mathbb{R}^3$ (with possible self-intersections) without creating any crease. One speaks also of *eversion* of the sphere in this context. The proof of the theorem itself does not involve very sophisticated tools: one simply shows that there exists a regular homotopy of two immersions of S^2 into $\mathbb{R}^3$. Moreover, some physical models illustrating the eversion in question were elaborated – one can find in the net several movies showing how the eversion obtains.

2. *Exotic spheres*. We recall that an exotic sphere is a differentiable manifold M that is homeomorphic but not diffeomorphic to the standard Euclidean n-sphere. This means that M is topologically indistinguishable from Euclidean sphere, but admits a smooth structure which is essentially different from the standard such structure. The first exotic spheres were discovered by John Milnor: he proved that there exist distinct smooth structures on S^7 (there are 28 such structures on S^7). In some dimensions there are no exotic spheres (e.g. in dimensions: 1, 2, 3, 5, 6, 12), in other there exist thousands of them. Should we call exotic spheres pathological? If so, then we should admit that we have deep understanding of differential structures in very high dimensions which not seem to be the case today. The problem whether there exist exotic spheres in the fourth dimension remains open. If such an exotic

4-sphere existed, then it would be a counterexample to the smooth generalized Poincaré conjecture in dimension 4.

3. *Exotic* $\mathbb{R}^4$. We recall that an exotic $\mathbb{R}^4$ is a differentiable manifold that is homeomorphic to the Euclidean space $\mathbb{R}^4$, but not diffeomorphic to it. Let us also recall that dimension 4 is exceptional in this respect: exotic structures do not occur on $\mathbb{R}^n$ for $n \neq 4$. But $\mathbb{R}^4$ itself admits a continuum of distinct smooth structures on it. Again, we see that dimension 4 remains very mysterious.

We do hope that the above examples show that the concept of pathology in mathematics is pragmatically biased and relative to the development of mathematical theories.

1. *Surprising results*. Sometimes it is difficult to draw a boundary between unexpected surprises and pathologies in mathematics. For example, the existence of *pathological satisfaction classes* was, one may justly assume, a surprise. So on the one hand it is perhaps surprising that there are so many such classes and, on the other, that their properties contradict our intuitions concerning the relation of satisfiability.

2. *Pathology versus mystery*. One should remember that the non-existence of a recognizable pattern need not mean that one is surrounded by pathologies. It is not the case that everything we do not understand yet is automatically paradoxical or pathological. Mathematical knowledge does not form a closed system.

5 Dynamic character of mathematical intuition

Like any science, mathematics has its *context of justification* and *context of discovery*. The first is based solely on deduction – mathematical statements are accepted only when their proofs are given. It should be emphasized that proof in logic and in mathematics are different concepts – the former is defined purely formally, while the latter contains a pragmatic component. Mathematical theorems have to be *accepted* by the community of mathematicians and hence proof in mathematics depends also on psychological and sociological criteria. It is believed, however, that any mathematical proof can be transformed (at least in theory) to a proof satisfying criteria of logical correctness.

The most important component of the context of discovery in mathematics is based on (mathematical) intuition. We think that it is unreasonable – at the present moment – to speculate about our cognitive abilities responsible for mathematical intuitions. Rather, we propose to understand mathematical intuition as a complex of *verbalized* (!) beliefs about the world of mathematics. It is probable that not everything concerning mathematical intuitions can be verbalized. However, only explicitly stated judgments about it can ever be put into investigation. We also prefer to talk about *intuition that* (something holds) rather than about *intuition of* (some object).

Mathematical intuition is presented in the axioms, which is evident: they are accepted without proof, on the basis of some intuitive beliefs alone. One should keep in mind, however, that in the most important mathematical domains axiomatic approach was preceded by a huge cumulation of knowledge about the domain in question. The literature on this subject is huge. Let us only mention that e.g. the paper Grzegorczyk 1962a discusses possible bases of justification for axioms of mathematical systems and the paper Feferman, Friedman, Maddy, Steel 2000 tries to answer the question of whether mathematics needs new axioms.

We meet mathematical intuition also in the everyday practice of mathematicians – in reasoning by analogy, in making generalizations, in several heuristic rules of thumb, in using inductive assumptions before formulation of a hypothesis to be proven, etc. Needless to say, the main body of mathematical education (especially teaching on the elementary level) is based on intuitive explanations.

Why should we talk about intuition when discussing standards, exceptions and pathologies? There are at least two reasons for that, we think:

1. What a standard object is, depends on our intuitive beliefs about what is normal, prototypic, etc. in a given domain.
2. As we have seen from the examples of the previous section, pathologies sometimes become „domesticated". This, in turn, forces changes in our mathematical intuitions.

Unlike the more-or-less stable intuitions connected with everyday experience, mathematical intuition is more dynamic. Major sources of changes of mathematical intuition seem to be:

1. *Antinomies and paradoxes*. Each time we find a contradiction in a piece of mathematical considerations we have to get rid of it which implies rethinking the assumptions leading to this contradiction. Thus, for example, an allegedly innocent assumption that every property determines a set led to the Russell paradox. In order to save set theory from such contradictions we have changed the formulation of the axiom of comprehension, which of course implied the corresponding change in our intuitions concerning sets. In

the case of some paradoxes, however, we do not literally change our intuitions but rather admit that the intuitions of the everyday experience no longer apply at a certain level of sophistication of a mathematical theory (cf. for instance sublime constructions in general topology (or algebraic topology, or differential topology) or theory of infinite sets).

2. *Scientific programs*. Changes of intuitions can be also made deliberately, as a result of conducting some program.

   (a) *Arithmetization of analysis*. The beginnings of differential and integral calculus were not soundly based (according not only to the current criteria of correctness, but also to those of that era). It was a great achievement of mathematicians of the XIX century to change this situation. Instead of thinking in terms of kinematic analogies (movement) and geometrical intuitions, one proposed to base analysis on arithmetical relations only.

   (b) *Algebraic geometry*. Not very long after first investigations in topology took place one started to describe topological properties using also algebraic tools. In many cases it is much more easier to characterize topological spaces via algebraic characteristics associated with them.

   (c) *Geometrization conjecture*. Thurston's geometrization program is an example of setting up new directions in mathematics. Out of the seven Millenium Problems proposed by the Clay Institute only one (Poincaré conjecture) has been solved quite recently. The problem itself is closely related to the Thurston's program.

   (d) *Large cardinal axioms*. The axioms of set theory (ZF first order set theory) characterize the concept of set and the relation $\in$ very weakly. The current work on set theory is directed partly by the search for new axioms which could change this situation. Typical examples of such new axioms are postulates requiring the existence of very large cardinal numbers. Not all problems of set theory can be solved with such axioms (e.g. the continuum hypothesis could not be resolved in this way) but, still, large cardinal axioms appeared fruitful e.g. in investigations of consistency strength of theories.

   (e) *Hilbert program*. The original Hilbert program required – roughly speaking – that we should prove the consistency of mathematics using finitary tools only and, besides, prove also the completeness of mathematical knowledge. Due to the well know incompleteness results the program in this setting can not be realized – it can be only realized partially. But the lesson obtained was fruitful: we have changed our intuitive beliefs about intimate connections between proof and truth in mathematics.

(f) *Hilbert problems*. Hilbert problems, stated by him in 1900, influenced the mathematics of the whole XX century. It is sufficient to look at the list of Fields medalists in XX century to see how many efforts of mathematicians of this period were concentrated around problems from Hilbert's list.

3. *New results*. Accumulation of new knowledge about the investigated domain is not without influence on our intuitions. It is sometimes claimed that intuitive beliefs lack justification (and hence do not form knowledge). This seems not be true in the case of mathematical intuitions. Probably each professional mathematician will agree that his intuitions become more and more deep, sublime and sophisticated as a result of his growing knowledge of the subject.

Consider a few examples of changes in intuition:

1. Up to the results of Ruffini and Abel in the first decades of the XIX century mathematicians believed that algebraic equations of any degree can be solved by radicals. Now we know that this is possible only for equations of degree less than 5.

2. Even the great Cauchy believed once that the pointwise limit of a sequence of continuous functions is a continuous function, too. Now we know that we should assume uniform continuity in order to obtain a limit which will be continuous.

3. The gradual expansions of number domains doubtlessly evoked changes in our intuitions concerning numbers.

4. The discovery of non-Euclidean geometries changed mathematicians' beliefs supporting the view that the Euclidean geometry is the only, true, empirically confirmed geometry.

Mathematical intuition is influenced by (among others):

1. *Aesthetic values*. Professional mathematicians very often declare that their intuitions strongly depend on aesthetic values – theories, theorems, proofs, constructions should be beatiful, in addition to the self-evident criteria of correctness.

2. *Empirical experiments*. It is reasonable to assume that the origins of arithmetic lie in the processes of counting, the origins of geometry lie in measuring lengths, areas and volumes, the origins of differential and integral calculus are influenced by reflections on motion, change, velocity, etc. There always was an interconnection between research in physics and mathematics. It is astonishing, how many ideas of pure mathematics developed allegedly without any connection to the physical world have found – sooner or later – their applications in physics.

3. *Mathematical fashion*. This influence should not be underestimated. Each spectacular achievement in mathematics causes increase of interest in the domain in question. Also ambitious research programs attract mathematicians to focus their work on a chosen subject. The effects of these collective efforts direct the main streams of mathematics.

Two major sources of mathematical intuition seem to be:

1. *Our cognitive resources*. Most likely, our intuitions are determined by the senses we have at our disposal. It is sometimes stressed that mathematical intuitions are mostly visual, partly connected also with movement abilities. What are the basic mechanisms responsible for intuitive thinking still remains mysterious. The results of several experiments do not explain much in this respect. One could also speculate how would our mathematics look like, if we were – say – Rational Blots living in a completely liquid environment, without whatsoever access to any rigid bodies. Should we then start with general topological spaces, with some kind of proximity idea instead of developing Euclidean geometry? One may construct several thought experiments of this kind, but it seems to be a speculative frolic only. The genuine task is to explain really existing mathematics and our mathematical intuitions as responsible for the growth of mathematical knowledge.

2. *Symbolic violence in the school*. Teaching mathematics is not based on memorizing. Rather, it is an interactive process of common problem solving, analyzing examples, conducting constructions etc. At the beginning, a considerable role is played by teacher's persuasion techniques. Then, pupils are directed to think in a way approved by the teacher, who is training them, correcting their mistakes and showing the proper way to solution of the problems. It is often claimed that the main goal of teaching mathematics is just to evoke correct mathematical intuitions – the role of calculating abilities is only secondary.

6 Extremal axioms

The term *extremal axiom* (according to what we know used for the first time in Carnap, Bachmann 1936) has its origins in Hilbert's *Grundlagen der Geometrie* – Hilbert added to his system of axioms for geometry an extra axiom, the *axiom of completeness* which was supposed to express an idea that the underlying system of geometrical objects (points, lines and planes) is unable to any extension without violation of the remaining axioms. In this form, the axiom of completeness was not a statement of the object language of the theory, it belonged to metalanguage, as expressing some view about models of geometry. It has been replaced by the *axiom of continuity* in the later versions of *Grundlagen der Geometrie*. As a consequence, one can prove the categoricity of the system in question.

Categoricity (or categoricity in power) and completeness are among the desirable properties of mathematical theories, in a sense. Mathematicians identify isomorphic objects, i.e. objects which are indistinguishable from the point of view of their internal structure. Objects in mathematics are always classified „up to isomorphism". This kind of indistinguishability is thus based on algebraic criteria. A theory T is categorical in power κ, if all models of T of power κ are isomorphic.

A weaker kind of indistinguishability is based on semantic criteria. Two structures $\mathfrak{A}$, $\mathfrak{B}$ (of the same signature) are elementarily equivalent if the set of sentences true in $\mathfrak{A}$ coincides with the set of sentences true in $\mathfrak{B}$. Isomorphism implies elementary equivalence but not vice versa. If all models of a theory are elementarily equivalent, then the theory is *complete*: for any sentence ψ of its language, either ψ or $\neg\psi$ is a theorem of the theory in question.

Isomorphism and elementary equivalence enable us to classify models of theories. There are numerous theorems in model theory (classical and modern), concerning categoricity (in power) and completeness e.g. (we give only informal formulations, without technical details):

1. *Ryll Nardzewski theorem*. The following condition are equivalent for any (consistent) theory T:

   (a) T is $\aleph_0$-categorical.

   (b) For every natural number n, T has only finitely many n-types.

   (c) For every natural number n, every n-type is isolated.

   (d) For every natural number n, the Stone space $S_n(T)$ is finite.

2. *Łoś-Vaught theorem*. If a theory T (without finite models) is consistent and categorical in some infinite power, then it is complete.

3. *Morley theorem*. If a theory is categorical in one uncountable power, then it is categorical in all uncountable powers.

An important contribution to the investigation of categoricity is the Grzegorczyk's paper written half a century ago (Grzegorczyk 1962b). The main results of this paper are the following:

1. The author discusses several meanings connected with the concept of *categoricity*, which partly depend on the understanding of the term *model*.

2. Using some model-theoretic results, he shows a certain weakness of the notion of *categoricity in power*.

3. Then he discusses a notion of categoricity based on models with an absolute interpretation of set-theoretical concepts.

4. Grzegorczyk introduces a new, very important notion of a *constructive model*, using the well known procedure of Skolemization. He calls a theory *constructively categorical*, if any two of its constructive models are isomorphic. Then he proves the theorem about existence of constructive models (based on terms). He gives also a necessary and sufficient condition for a theory to be constructively categorical.

5. The author proves that:

   (a) Robinson's arithmetic has a constructively categorical axiom system (the standard natural numbers form its constructive model).

   (b) The complete system of the arithmetic of real numbers has a constructively categorical axiom system (its constructive model is isomorphic to the set of all real algebraic numbers). Also the complete system of Euclidean geometry is constructively categorical.

6. Grzegorczyk introduces two kinds non-categorical theories. A theory T is *essentially non-categorical* if and only if it has no consistent extension with a constructively categorical axiom system. A theory T is *weakly essentially non-categorical* if and only if it has no recursively enumerable consistent extension with a constructively categorical axiom system. He proves that the theory of a dense ordering with at least two elements is essentially non-categorical. He also proves some more general theorems concerning the notions in question.

7. Finally, the author considers still another notion of categoricity: a theory T is called μ-categorical if and only if it has a minimal model and any two of its minimal models are isomorphic. Here by a *minimal model* of a theory T one means any model $\mathfrak{M}$ of T such that if $\mathfrak{N}$ is a model of T and a submodel of $\mathfrak{M}$, then $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic. Grzegorczyk proves that:

   (a) The complete theory of dense ordering is μ-categorical.

   (b) Every minimal model is at most denumerable.

   (c) Any theory which has no finite models and has minimal models and is categorical in power $\aleph_0$ is μ-categorical.

   (d) The elementary arithmetic of natural numbers is μ-categorical.

Again, we should ask the question: what categoricity, completeness and extremal axioms have to do with the concepts of standard, exception and pathology? The answer may include, among others, the following:

1. As categoricity and completeness imply uniqueness of models (either in algebraic or in a semantic sense), they both could be partly responsible for determination what is standard according to a given theory.

2. Extremal axioms were thought of as a kind of warranty of the uniqueness of the investigated domain. Hence, they also could be responsible for forming ideas what a standard – according to a theory in question – should be.

The papers Awodey, Reck 2002a, 2002b discuss the historical context of the origins of the notions of categoricity and completeness (cf. also e.g. Corcoran 1980, 1981).

Extremal axioms are either *minimal* or *maximal* – they demand that the model should be either minimal or maximal, not merely with respect to the size of its universe but rather to its structural properties. The most famous extremal axioms are:

1. *Axiom of continuity*. The structure of the continuum has always puzzled mathematicians. What is it consisted of – indivisible individuals or continua „all the way down"? Do infinitely small magnitudes have any real existence? How is it possible to obtain an object with positive measure (e.g. an interval) from separate points which do not have any extension? These and many other similar questions have been waiting for centuries for precise and adequate answers.

(a) *Axiom of continuity in geometry*. Hilbert's *Axiom of Completeness* from his *Grudlagen der Geometrie* has the following form in editions 2–6:

> The elements (points, lines, planes) of geometry constitute a system of things which cannot be extended while maintaining simultaneously the cited axioms, i.e., it is not possible to add to this system of points, lines, and planes another system of things such that the system arising from this addition satisfies axioms AI–V1.

This axiom was later replaced by the *axiom of linear completeness* and finally by a form of the *axiom of continuity* for the system of real numbers. In this final form it belongs to the object language and not to the metalanguage of geometry. There are also other ways of expressing the completeness in question, notably the one proposed by Tarski – cf. e.g. Borsuk, Szmielew 1975.

(b) *Axiom of continuity in algebra*. The usual ordering of rational numbers is dense – taking any two of them one can find a number between them (actually, infinitely many such numbers). However, the density condition is not sufficient for a consistent linear ordering of both rational and irrational numbers. Here continuity of the real numbers enters the stage. There are many (more than a dozen) ways to express the condition of continuity in this case – the most commonly known are the proposals by Dedekind (real numbers as *cuts* in the ordering of rational numbers) and Cantor (real numbers as equivalence classes of sequences of rational numbers satisfying the Cauchy condition). In modern formulations, *The Axiom of Continuity* is added to the usual axioms of an ordered field $(R, +, \cdot, 0, 1, <)$ and may have e.g. one of the following forms (cf. Błaszczyk 2007, 306):

- For any cut (A, B) in $(R, <)$ either in A there exists the greatest element, or in B there exists the smallest element.
- Any non-empty bounded from above subset $A \subseteq R$ has the lowest upper bound in R.
- Any infinite and bounded subset $A \subseteq R$ has a limit point in R (in order topology).
- $(R, +, \cdot, 0, 1, <)$ is an Archimedean field and for any sequence $(a_n) \subseteq R$ there exists $a \in R$ such that $\lim\limits_{n\to\infty} a_n = a$.
- $(R, +, \cdot, 0, 1, <)$ is an Archimedean field and for any descending chain of closed intervals (A_n) we have $\bigcap\limits_n A_n \neq \emptyset$.

Let us recall the definition of a *real closed field* as an ordered field where the order relation is a maximal one compatible with the operations of the field. In this case we have several alternative (equivalent to the above) characterizations of real closed fields, e.g. it is an ordered field in which every positive element has the root and every polynomial of odd degree has a root. Thus, maximality condition (concerning order) can be replaced by an equivalent one without any explicit reference to models of the theory and hence without any necessity of quantifying over the space of models.

2. *Induction axiom in arithmetic*. Axiom of induction may be considered either as a single sentence in a second order language or as an axiom scheme in a first order language:

   Second order axiom:

$$0 \in X \wedge \forall x\ (x \in X \rightarrow s(x) \in X) \rightarrow \forall x\ (x \in X),$$

   where s is the symbol for successor.

   First order scheme:

$$(\psi(0) \wedge \forall x\ (\psi(x) \rightarrow \psi(s(x)))) \rightarrow \forall x\ \psi(x),$$

   where $\psi(x)$ is any formula with one free variable of the language of Peano arithmetic.

   In the first case, one can prove the categoricity of the underlying theory. In the second, due to the incompleteness theorem, the axiom scheme of induction is not sufficient for the elimination of *alien intruders* – non-standard natural numbers. There are several methods of constructing non-standard models of arithmetic – cf. e.g. Grzegorczyk 1971. PA is not finitely axiomatizable – we can not replace the infinite induction scheme by any finite set of formulas equivalent to it. Neither can we restrict the complexity of formulas in it and simultaneously keep the full force of PA. For more recent research on these topics see e.g. Kaye 1991 or Hájek, Pudlák 1993.

3. *Restriction axioms in set theory*. These axioms were supposed to express minimality conditions for the universe of all sets. Recently, this idea is abandoned – we rather try to impose some maximality conditions on this universe, allowing the existence of as many sets as possible.

(a) *Fraenkel's axiom of restriction*. Fraenkel axiom says, roughly speaking, that there are no more sets than those whose existence follows from the axioms of set theory. Its formulation in Fraenkel 1928 runs as follows:

> *Axiom der Beschränkheit*. Außer den durch die Axiome II bis VII (bzw. VIII) gefordeten Mengen existieren keine weitere Mengen.

Judging from Fraenkel 1928 one may suppose that Fraenkel was interested in achieving some kind of uniqueness of the set-theoretical universe with his axiom of restriction. He discussed this axiom also in the context of whether all sets should be well founded (Fraenkel 1928, 354):

> Unsere Axiome enthalten nämlich keine Verfügung über die ursprünglichen „Bausteine" beliebiger Mengen, oder allgemein über die Natur (d.h. über die Elemente) derjenigen Mengen, die nicht schon durch die Axiome völlig gesichert sind.

But the axiom of restriction is not necessary to decide this problem – a proper axiom is the axiom of regularity. Georg Schiemer has pointed out that it is likely that Fraenkel got an inspiration for his Axiom of Restriction from Dedekind's considerations about natural numbers, and especially from his *Kettentheorie* (Schiemer 2010). The set of all natural numbers is a minimal chain; similarly the universe of all sets should be in some — to be made precise — sense *minimal*.

(b) *Gödel's axiom of constructibility*. Axiom of Constructibility was not conceived as a restriction axiom, though it has a form of axiom of minimality in set theory. The inner model of all constructible sets was devised in order to prove that if set theory ZF is consistent, then also ZF plus the axiom of choice and the Continuum Hypothesis is consistent. As everybody remembers (cf. Gödel 1940), at successor stages in building the constructible universe one makes use of the poorest powerset operation possible: the powerset of x contains only *definable* subsets of x. At limit stages we take of course unions of all stages constructed so far. The class of all constructible sets is a minimal countable transitive model of set theory containing all ordinal numbers. Kurt Gödel himself was against axioms of restriction in set theory and he overtly expressed his view in favour of axioms of maximality (Gödel 1964, quotation after CW II, 262–263):

> On the other hand, from an axiom in some sense opposite to this one [i.e. to the Axiom of Constructibility — JP], the negation of Cantor's conjecture could perhaps be derived. I am

> thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas axiom A [i.e. the Axiom of Constructibility — JP] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set [. . .]

It seems that nobody in the community of set theoreticians has ever seriously taken into account a possibility of adjoining the Axiom of Constructibility to the body of fundamental axioms of set theory. "Normal" mathematicians may have different opinion in this respect – cf. Friedman's judgment in Feferman, Maddy, Steel, Friedman 2000, 436–437. Nevertheless, the Axiom of Constructibility, taken as a working assumption, has many consequences of considerable interest, in combinatorics, algebra, model theory, theory of recursive functions, etc. However, the Axiom of Constructibility implies e.g. the nonexistence of measurable cardinals as well as the negation of Suslin hypothesis. The prize to be paid, if one accepts this axiom seems to be too high, compared with its alleged naturalness and evident economy. We prefer to stay in the *Cantor's Paradise*.

(c) *Suszko's axiom of canonicity*. Roman Suszko has written his *Canonic axiomatic systems* (Suszko 1951) with the goal of explicating the Skolem's (alleged) Paradox. He also stressed that his explication does not refer to the Löwenheim-Skolem theorem itself. Suszko claimed that his *axiom of canonicity* is a formal counterpart of Fraenkel'a axiom of restriction. Independently, some similar ideas were suggested by John Myhill (cf. Myhill 1952). As far as we know, nobody has developed Suszko's approach later.

(d) *Critique of restriction axioms*. Restriction axioms in set theory were criticized from the very beginning. They have been rejected already by von Neumann in 1925 and – though on different grounds – by Zermelo in 1930. The most destructive critique of minimal axioms is presented in Fraenkel, Bar-Hillel, Levy 1973. We are not going to report on that here – let us only add that the authors give some – based on pragmatic grounds – arguments for maximality axioms in set theory.

4. *Large cardinal axioms*. These axioms are supposed to express maximality conditions for the universe of all sets. There is already a huge literature on this subject – the interested reader may consult e.g. the monograph Kanamori 1994. One may of course ask whether such axioms are *natural*, whether they

justly express „the spirit” of set theory – cf. e.g. Bagaria 2005. At any rate, these axioms have appeared very fruitful in the development of modern set theory – cf. e.g. the discussion in Koellner 2010.

Extremal axioms expressed mathematicians’ dreams: a pursuit of intended models of their theories. First order logic, though equipped with many nice and convenient deductive properties is very poor as far as its *expressive* power is concerned. However, logical subtleties are usually not extremely interesting for working mathematicians. Jon Barwise has put it in the following way (Barwise 1985, 7):

> But if you think of logic as the mathematicians in the street, then the logic in a given concept is what it is, and if there is no set of rules which generate all the valid sentences, well, that is just a fact about the complexity of the concept that has to be lived with.

Intended models of mathematical theories are determined only on the metalinguistic level. The fact that theories themselves can not uniquely determine their intended models should not be surprising – finally, the very concept *intended model* contains an irremovable pragmatic component. Similarly for the concepts of *standard* and *pathology*, as we tried to show above.

7 A final word

We admit that all the above remarks bring no essentially new insights into the realm of mathematics. But we do hope that they at least clarify some pragmatic components involved in the way we speak about mathematical objects: standard ones, exceptional, and pathological. The mathematical examples chosen in the paper belong to the most typical though they were picked up in a sense randomly, without any claim to completeness. It might be an entertaining challenge to work on standards, exceptions and pathologies in separate branches of mathematics from a historical perspective.

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