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Zermelo: a Well Founded Antiskolemism*

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In his response to the letter from Paul Bernays (with congratulations on the occasion of his 70th anniversary) Zermelo wrote (October 1, 1941):

> Man wird eben immer einsamer, ist aber umso dankbarer für jedes freundliche Gedanken. [...] Wo mein Name noch genannt wird, geschieht es immer *nur* in Verbindung mit dem 'Auswahlprinzip', auf das ich *niemals* Prioritätsanspruche gestellt habe. [...] Dabei erinnere ich mich, daß schon bei der Mathematiker-Tagung in Bad Elster mein Vortrag über Satz-Systeme durch eine Intrige der von Hahn und Gödel vertretenen Wiener Schule von der Diskussion ausgeschlossen wurde, und habe seitdem die Lust verloren, über Grundlagen vorzutragen. So geht es augenscheinlich jedem, der keine 'Schule' oder Klique hinter sich hat. Aber vielleicht kommt noch eine Zeit, wo auch meine Arbeiten wieder entdeckt und gelesen werden.[1]

Zermelo had in mind here his works listed below. They concern a foundational program formulated by him with special emphasis put on the *infinitary* (though always well founded) nature of mathematical proof. This idea is of course in sharp opposition to the (quite well established at that time) common uderstanding of the notion of finitary formal proof. Zermelo rejects what he himself calls *Skolemism* and *the finitary prejudices*: the views that set theory should be axiomatized in a first order language (which implies that quantification over propositional functions in the comprehension axiom would be out of question) and that mathematical theories in general should be codified solely in terms of finitary logic.

In the formulation of his program Zermelo makes an essential use of hierarchies of well founded domains described in his paper *1930*. Besides the distinction between *closed* and *open* domains this paper brings categorical characterizations of models of set theory (with respect to two numerical parameters — the number of urelements and the ordinal rank of the domain). It contains also an important observation that the hierarchies V_κ, where κ is strongly inaccessible, form natural models

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[1]Cf. Peckhaus 1990, p. 20.

for set theory (Zermelo works here in set theory with a second order axiom of comprehension, with a version of the replacement axiom, with the axiom of foundation, without the axiom of infinity and without the axiom of choice — the latter is assumed in the metatheory, as a logical principle).

Any mathematical theory is represented, according to Zermelo, by: an infinite *domain*, a collection of *fundamental relations* over that domain and a collection of *truth partitions* of the formulas of this theory (making its axioms true). Formulas may be treated as (well founded!) *sets*; infinite conjunctions and disjunctions are allowed. Zermelo's notion of *proof* corresponds, in a sense, to that what is now commonly understood by *logical consequence*. Incompleteness in his sense is different from that of Gödel. Zermelo believed in decidability of all mathematical problems; however, he was aware that there are systems in which some true *Zermelo's sentences* have *Zermelo's proofs* which lie beyond those systems themselves.

At the time when Zermelo originated his program, it had little, if any, chances to be fully developed. The standard of finitary first order logic was winning, due to the achievements of Skolem and Gödel. Two decades later the situation looked different: Tarski, Henkin and Karp started the investigation of infinitary languages and Mostowski introduced generalized quantifiers. In the sixties Lindström developed Mostowski's approach and in the early seventies Barwise proposed and elaborated a whole domain of research in infinitary logic, soft model theory and admissible structures. It seems that the systems of logic based on admissible set theory are the closest counterparts to the original ideas of Zermelo. *Also, doch ist die Zeit angekommen...*

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