Extremal Axioms in Mathematics

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The term *extremal axiom* was introduced in the paper (Carnap, Bachmann, 1936). These authors tried to develop a general approach to axioms which were supposed to characterize in a unique way intended models of some important mathematical theories. They took into account: Hilbert's completeness axiom from his *Grundlagen der Geometrie* (Hilbert, 1899), Peano's axiom of induction in arithmetic (Peano, 1889) and Fraenkel's axiom of restriction in set theory (Fraenkel, 1928). They worked in type theory and tried to express in that language the fact that an extremal axiom is either a *maximal axiom* (like Hilbert's completeness axiom, which was later replaced by the continuity axiom for real numbers) or a *minimal axiom* (like axioms of induction or restriction, mentioned above). It may be worth recalling here the Hilbert's axiom of completeness which stated that: *To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms*.

Some critical remarks concerning the project in question can be found in (Hintikka, 1986) and (Fraenkel, Bar-Hillel, Levy, 1973). An earlier work (Carnap, 1930) containing a flawed proof of his *Gabelbarkeitssatz* and in a sense related to the project under consideration is discussed in (Awodey, Carus, 2001). The work (Lindenbaum, Tarski, 1936) solves similar problems, among others.

Our talk will be devoted mainly to historical remarks concerning extremal axioms. Because extremal axioms are supposed to determine models uniquely, they are connected with such properties as *categoricity*, *categoricity in power* and *completeness*. Obviously, categoricity implies completeness, but the converse implication in general does not hold – cf. e.g. (Tarski, 1940).

Limitative theorems obtained in the 20th century (Löwenheim, Skolem, Gödel, Rosser, Tarski, Church, Turing) have shown possibilities and restrictions in the characterization of such fundamental notions as: consistency, completeness, categoricity, decidability. They also shed new light on the role of extremal axioms in mathematics – cf. e.g. (Awodey, Reck, 2002) and (Scanlan, 1991) where, among others, the influence of the American Postulate Theorists (like Veblen and Huntington) on the emergence of the concepts of categoricity and completeness is discussed.

Categorical characterizations of natural numbers (Peano), Euclidean geometry (Hilbert) and real numbers (as a unique completely ordered field – here the extremal axiom of continuity is essential, cf. (Dedekind, 1872)) are all well recognized in mathematics. A little bit different situation has emerged in the case of set theory. Here both kinds of extremal axioms: minimal as well as maximal have been taken into account. The already mentioned Fraenkel's axiom of restriction (roughly: there are no more sets than those whose existence follows from the axioms of set theory) belongs to the first category. The famous Gödel's axiom of constructibility is also an axiom of restriction, i.e. a minimal extremal axiom (Gödel, 1940). It should be stressed that Gödel himself advocated later a possibility of extending set theory by maximal axioms, in analogy with Hilbert's axiom of completeness. An interesting, but less known proposal of a precise formal explication of the axiom of restriction in set theory is the axiom of canonicity introduced in (Suszko, 1951). The monograph (Fraenkel, Bar-Hillel, Levy, 1973) contains a sharp critique of set theoretical restriction axioms and presents arguments (but to a high degree pragmatical ones!) in favor of maximal axioms in set theory.

They are axioms of the existence of large cardinal numbers – cf. e.g. (Kanamori, 1994). It seems that such axioms were for the first time proposed in (Zermelo, 1930) where the author argued that the spirit of set theory requires consideration of a transfinite hierarchy of strongly inaccessible cardinals.

We will also compare the role of extremal axioms with some important isomorphism theorems in algebra (Frobenius, Ostrowski, Hurwitz, Pontriagin) which characterize up to isomorphism some fundamental structures (like real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$). Finally, we will point to chosen results in modern model theory related to the dependencies between categoricity and completeness.

The terms *intended model* and *standard model* are sometimes used interchangeably in the literature. However, we would like to propose the following distinction. By the *intended model* we will mean the structure which has been investigated for its own sake, typically for a long time, so that we have collected a large amount of data and have proven many fundamental theorems about it. One could say that intended models are such structures which became *domesticated*, easily accessible cognitively and responsible for the basic mathematical intuitions. One can see that this is only an intuitive characterization of the term *intended model* which, in turn, also is an intuitive term. Natural number series (with arithmetical operations) may serve as an example of intended model understood in this way.

One can talk about the *standard model* only after one has obtained a developed formal theory, ultimately an axiomatic one. In such a case one can establish the class of all possible models of the theory in question. Then the standard model of such a theory is its model most closely related to intended one. It may happen that the standard model of a theory obtains a precise characterization, e.g. in terms of isomorphism or elementary equivalence; for instance, Tennenbaum's theorem says that the standard model of first order Peano arithmetic is the only recursive model of that theory. Examples of standard models in the proposed sense are: the standard model of first order Peano arithmetic, the completely ordered real field $\mathbb{R}$, the model of Euclidean geometry (in Hilbert's axiomatization).

A theory may obviously have also *non standard* models which do not resemble the intended model at all or resemble it only to a certain degree. For instance, the first order Peano arithmetic has a lot of non standard models; all its countable non standard models have the same order type $\omega + (\omega^* + \omega) \cdot \eta$ but they are differentiated with respect to the arithmetical structure.

In general, standard models can be determined neither syntactically nor semantically. It is our epistemic choice to call a given model the standard one. That choice is of course influenced by the research practice of professional mathematicians – accumulation of the results characterizing the intended model.

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