Lectures on Propositional Calculi

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This book was initially published in 1984 by Ossolineum,
The Publishing House of the Polish Academy of Sciences

Contents

Preface

The book is based on the lectures I delivered at Instituto de Matemática e Estatistica e Ciencia de Camputacao, Universidade Estadul de Campinas, during my stay in Brazil at summer 1981; some of them were repeated at Instituto de Matemática e Estatistica, Universidade de Sao Paulo.

While the general topic of may lectures was the theory of propositional calculi, I concentrated myself on results obtained by Polish logicians, especially those grouped around the Section of Logic of the Institute of Philosophy and Sociology of the Polish Academy of Sciences. Consequently, the results mentioned above play the central role in this volume too. Now, as I explain below, many of them are of specific origin.

As early as in 1976, at one of the workshops organized annually by the Section of Logic of the Polish Academy of Science there was put forward idea to write a book on the theory (methodology) of propositional calculi. As a part of the project, various open problems in the area have been attacked and eventually many of them solved. The results obtained by J. Czelakowski, Z. Dywan, W. Dziobak, J. Hawranek, M. Maduch, G. Malinowski, T. Prucnal, W. Sachwanowicz, R. Suszko, M. Tokarz, P. Wojtylak, A. Wroński, J. Zygmunt, to mention as least some names have been often both of crucial importance for the project and of considerable theoretical significance by themselves. As one may expect, I largely exploited them in my Brazilian lectures.

The book we have planned is still under preparations. It is to consist of two parts: *The Theory of Propositional Calculi – An introduction* by the author of this volume and *The Theory of Propositional Calculi – Selected Topics by J. Czelakowski and W. Dziobak.* Clearly this volume covers some of the topics to be discussed in *The Theory of Propositional Calculi*, especially in its first part *An Introduction* but their exposition differs, often considerably, from that to be found in the book prepared. In this sense this publication is independent from the planned one.

Although the work on this book was concluded only after my return to Poland, the substantial part of it have been done during my stay in Brazil sponsored by FAPESP (Fundacao de Amparo a Pasquisa do Estado de Sao Paulo, Brazil, grant no 80/1188-8). I benefited a great deal and in various ways from the opportunity to have scientific context with my Brazilian colleagues and friends. My greatest debt has been to Prof. Ayda I. Arruda, at that time the Director of Instituto de Matematica e Estatistica e Ciencia de Computacao, Universidade Estadul de Campinas both for the care she took for creating me excellent conditions for work and for her keen and penetrating interest in the ideas I discussed in my lectures. Also I own a special debt to Prof. Newton C. A. da Costa for his invitation to Instituto de Matematica e Estatistica, Universidade de Sao Paulo and stimulating discussions we held, and to Prof. Elias Alves for his introducing me to people from the Logical Center of UNICAMP and his assisting me on many occasions. The complete list of Brazilian logicians who deserve

my gratitude would be very large indeed. I hope that they will accept my collective thanks for everything I own them.

In conclusion I would like to thank to all who help me to prepare the final version of the manuscript. My special thanks are due to J. Czelakowski, W. Dziobak and other my collaborators and friends for their valuable criticism and suggestions, to R. Ładniak for his undertaking the arduous at of getting the manuscript ready for the publisher, to Alicja Dębska for preparing index, and to Mrs Jadwiga Krasnowska for her excellent typing of the manuscript.

Chapter 1

Logical Systems

1. Propositional languages

1.1. We shall not discern between propositional (i.e. 0-order) languages and their algebras of formulas. Thus if S is the set of all formulas (sentences) formed by means of propositional variables $p_1, p_2, \dots, p_i \dots$ and connectives $\S_1, \dots, \S_n$, the abstract algebra

$$\mathcal{S} = (S, \S_1, \dots, \S_n)$$

will be referred to as the *propositional language* determined by the symbols mentioned above. Each $\S_i$ has $\geqslant 0$ arguments. We assume that at least one of the connectives is not nullary.

If otherwise is not stated clearly, the number of propositional variables that a language involves will be assumed to be denumerable. The set of all variables of the language $\mathcal{S}$ will be denoted by $Var(S)$. More generally, given any set of formulas $X \subseteq S$, we define $Var(X)$ to be the set of all variables appearing in formulas in X.

1.2. The language determined by the familiar connectives $\wedge, \vee, \to, \neg$ will be called *standard*. The standard language will be denoted by $\mathcal{L}$ and the set of all its formulas by L. We shall refer to the formulas of $\mathcal{L}$ as *standard formulas*.

The formulas of the form $(\alpha \to \beta) \wedge (\beta \to \alpha)$ will be abbreviated as $\alpha \longleftrightarrow \beta$. This convention will be applied to all languages that involve $\wedge$ and $\to$, $\mathcal{L}$ in particular.

1.3. Given any propositional language $\mathcal{S}$ and any two formulas α, β of that language, we assume that they are identical $\alpha = \beta$, iff they are formed by exactly the same *elementary symbols*, i.e. propositional variables and connectives, in exactly the same manner. From the algebraic standpoint it means that propositional language is an abstract algebra free in the class of all algebras similar to it and freely generated by the set of all propositional variables.

1.4. Let α, β be formulas of $\mathcal{S}$. The formula α is said to be a *substitution instance* of β iff $\alpha = e\beta$, for some endomorphism e of $\mathcal{S}$. In what follows endomorphisms of propositional languages will be, as a rule, referred to as *substitutions*.

Observe that given any formula α and any substitution e, $e\alpha = e'\alpha$, for all e' such that $ep = e'p$ for all propositional variables p appearing in α. (Incidentally, the notion of a propositional variable appearing in α can be defined in terms of substitutions as follows. A variable p_1 appears in α iff $e\alpha \neq \alpha$, for any substitution e such that $ep = p$ for all $p \neq p_1$ and $ep_1 \neq p_1$).

The following familiar convention will be very useful. Given any formulas α, $\beta_1, \ldots, \beta_n$, and any propositional variables $p_1, \ldots, p_n$ we define

$$\alpha(\beta_1/p_1, \ldots, \beta_n/p_n) = e\alpha,$$

where e is the substitution defined by for all variables p.

$Sb(X)$ will denote the set of all substitution instances of formulas in X.

If $Sb(X) = X$, i.e. if X is closed under substitutions, the set X will be called *invariant*.

2. Logical calculi

2.1. By a logic we shall mean a system of inferences rather than formulas. In order to define this notion rigorously we need some preparations.

An inference in language $\mathcal{S}$ will be defined to be a couple (X, α), such that $X \subseteq \mathcal{S}$, $\alpha \in \mathcal{S}$. Inferences of the form $(\emptyset, \alpha)$ will be called *axiomatic*. If X is finite, the inference (X, α) will be said to be *finitary*.

Instead of (X, α), we shall write $X \vdash$ ($\vdash \alpha$, if $X = \emptyset$). Under this convention, $X \vdash \alpha$ is merely another notation for (X, α), and the symbol $\vdash$ does not represent any specific relation between X and α. Still, if a particular propositional logic is defined, we shall discern between inferences *valid* and *invalid* on the ground of that logic. Thus, $X \vdash \alpha$ can be treated as a metasentence that reads *"there is a logically valid inference from X to α"*. Observe that, if we think about $X \vdash \alpha$ as a metasentence then, albeit it can be asserted about any X and any α, it can be truly asserted only about some X and some α. To define a logic is to define which of the assertions of the form $X \vdash \alpha$ are true (valid) and which are not.

2.2. An operation C defined on sets of formulas of $\mathcal{S}$ is said to be a *consequence operation* (A. Tarski [1930a]) iff it satisfies the following conditions:

(T1) $X \subseteq C(X)$,

(T2) $C(C(X)) \subseteq C(X)$,

(T3) If $X \subseteq Y$ then $C(X) \equiv C(Y)$.

2.3. If moreover for all substitutions e,

(ŁS) $eC(X) \subseteq C(eX)$,

(J. Łoś and R. Suszko [1958]) the consequence will be called *structural.*

2.4. By a (*propositional*) *logic* we shall mean either a couple $(\mathcal{S}, C)$, $\mathcal{S}$ being a propositional language and C being a structural consequence on $\mathcal{S}$, or C itself. In the latter case we shall refer to C as a *logic* (*defined*) in $\mathcal{S}$.

2.5. Couples of the form $(\mathcal{S}, C)$, C being a consequence on $\mathcal{S}$ but not necessarily structural, will be referred to as *propositional calculi.* Thus each propositional logic is a propositional calculus but not vice versa.

2.6. If $\alpha \in C(X)$, the inference $X \vdash \alpha$ will be referred to as *valid* in C, or just as an *inference* of C.

The set of all inferences valid in C will be denoted by $\vdash_C$. Of course, we shall write $X \vdash_C \alpha$ rather than $X \vdash \alpha \in\ \vdash_C$. When viewed as a relation, $\vdash_C$ will be called the *consequence relation corresponding* to C.

2.7. If $\alpha \in C(\emptyset)$, $\vdash \alpha$ will be called a *theorem* of C.

Observe that if C is structural, both the set inferences of C and that of theorems of C are closed under substitutions, i.e.

a. $C(\emptyset) = Sb(C(\emptyset))$,

b. $X \vdash_C \alpha$ implies $eX \vdash_C e\alpha$, for all substitutions e.

If $C(\alpha) = S$, S being the set of all formulas of the language of C, the formula α is said to be *inconsistent* (or else consistent) with respect to C, *C-inconsistent* (*C-consistent*), for short. Similarly, if for some set of formulas X, $C(X) = S$, the set X is called C-inconsistent. Observe that, in general, a C-inconsistent set X need not involve any C-inconsistent formula. Note also that C-inconsistency need not be a property preserved under substitutions; e may happen to be consistent with respect to C even if α is not.

If for each X

(C) $C(X) = S$ implies $C(X_f) = C$, for some finite $X_f \subseteq X$,

the consequence$C$ is called l-*compact.*

2.8. If C satisfies the following condition:

(F) If $\alpha \in C(X)$ then $\alpha \in C(X_f)$ for some finite $X_f \subseteq X$, the consequence C is called *finitary*.

The consequences that are both structural and finitary are called *standard*. In general, standard consequences need not be l-compact.

2.9. NOTE. Conditions (T1)–(T3) do not exhaust the full list of axioms for consequence operation set by A. Tarski in [1930a] (see also [1935a]). In particular, we have not demanded the set of all formulas on which the consequence C is defined to be denumerable. The postulate of denumerability was dropped out already by J. Łoś [1953]. Though the languages with denumerably many formulas are of special importance, non[a]denumerable languages are quite legitimate objects of investigations.

Apart from (T1)–(T3) and the denumerability of the language, Tarski postulated each consequence to be finitary (i.e. to satisfy condition (F)) and l-*compact*. As a matter of fact, he required even more than l-compactness, since he postulated that

(C_T) $C(X) = S$ implies that $C(\alpha) = S$, for some $\alpha \in X$.

The set conditions we have selected to define the notion of consequence coincides with that accepted by J. Łoś and R. Suszko [1958].

Some logicians prefer to study consequence relation rather than consequence operation. Of course, as the consequence relation is viewed as a relation is viewed as a relation between a set of formulas and a single formula each consequence relation defines a consequence operation and vice versa. Following G. Gentzen [1934–5] one can view the consequence relation as a binary relation between sets of formulas. But then correspondence between consequence relations and consequence operations is no longer one to one. Here we shall not dwell on this issue, for details see D. Scott, 1974.

2.10. Two methods of defining a logic (in a given language $\mathcal{S}$) are of special importance: semantic that consists in postulating certain truth condition for the formulas of $\mathcal{S}$ and syntactical that consists in postulating logical validity of some selected inferences.

From the technical standpoint, to define truth-conditions for $\mathcal{S}$ amounts to define a set $\mathbb{H}$ of functions $v: S \to S\{0,1\}$ called *admissible valuations* for $\mathcal{S}$. If $v(\alpha) = 1$, for all $\alpha \in X$, we interpret X as a set of such formulas of $\mathcal{S}$ that are simultanously true under the truth conditions imposed on the language.

We trust the reader to verify that the operation $Cn_{\mathbb{H}}$ defined by

$$(*) \qquad \alpha \in Cn_{\mathbb{H}}(X) \text{ iff for all } v \in \mathbb{H},\ v(\alpha) = 1$$
$$\text{whenever } v(\beta) = 1,\ \text{for all } \beta \in X$$

is a consequence operation. It will be referred to as the *consequence preserving truth* (under the set of admissible valuations $\mathbb{H}$).

In general $Cn_{\mathbb{H}}$ need not be a logic, i.e. a structural consequence. Roughly speaking, in order for $Cn_{\mathbb{H}}$ to be a logic, the truth conditions that define $\mathbb{H}$ should be "structural" in the sense that they refer only to the "logical structure" of sentences defined by the occurrences of logical connectives, not to their "content", the latter being represented by propositional variables the sentences involve. Incidentally, the notion of a structural consequence (cf. 2.3) is meant to explicate this very idea of structurality. The conditions under which $Cn_{\mathbb{H}}$ is structural are stated in rigorous terms in Section 16.

The way in which the syntactical methods works is even simpler. To each set Σ of inferences if $\mathcal{S}$, there correspond a consequence Cl_{Σ}, being the weakest consequence among all consequences relative to which all inferences in Σ are valid. (For the definition of a consequence weaker than another one, and proof that for each Σ, there is a Cl_{Σ} see Section 4).

2.11 Classical two-valued propositional logic, denoted throughout this book as K, may serve as a typical example of a logic defined semanticly. The set of admissible valuations for (the language of) K is the set of truth-value assignment defined by the familiar two-valued truth tables. Of course, syntactical definition of K are available. However, both from the historical and intuitive standpoint, the definition of K in terms of truth-tables is a standard one, i.e. any adequate definition of K should be equivalent to it.

Intuitionistic propositional logic will be denoted by J. It provides an example of a logic that was originally (A. Heyting [1930], [1930a]) defined synactically with the help of *Modus Ponens* (i.e. the set of all inferences of the form $\alpha,\ \alpha \to \beta \vdash \beta$, cf. 11.5) and suitably selected axiomatic inferences.

2.12. Needless to say that the definition of the set of inferences that are to determine a logic may by given in semantic terms, while the set $\mathbb{H}$ of admissible valuations may be defined in purely syntactical terms. If this were the case, the methods that was called syntactical should be called semantic and vice versa. We just have assumed that the definition of $\mathbb{H}$ involves in an essential way the notion of truth, while no semantic considerations intervene explicitly in the definition of Σ.

Still, one must agree that the difference between syntax and semantics is much more of philosophical than technical nature. Note, for instance, that the analyzes carried out in terms of truth can be carried out in terms of theories; the admissible valuations need not be interpreted as functions that assign truth-values to sentences but as characteristic functions of certain theories (call them for instance "admissible"). The quest for truth, that seems to be the cornerstone of any scientific activity, can be replaced by the quest for a "good" theories. This seems to be the gist of program put forward by R. K. Meyer, 1978, who launched attack on Tarskian concept of truth, with an attempt to show that Tarskian style semantics may have workable alternatives based just on the notion of a theory.

We should mention that apart that from syntactical or semantic interpretation, the logical validity may be viewed as a pragmatical notion, defined in terms of rationality of behavior; the inference $X \vdash \alpha$ is logically valid if one who accepts all sentences in X must accept α unless be behaves irrationally. An illustration of how this idea can be conveyed in rigorous technical terms, provides pragmatical interpretation of Łukasiewicz Logic given by R. Giles [1974].

3. Closure bases

3.1. In general, by a *theory* we shall mean a set of formulas. But, given a consequence C, by a *theory of* C we shall mean a theory X closed under C, $X = C(X)$. The theory S will be referred to as the *trivial* theory of C. It was A. Tarski ([1930], [1930a], [1935a]) who started systematic investigations into properties of theories of propositional calculi or, as he called them, *deductive systems*.

The set of all theories of C will be denoted by Th_C.

3.2. Verify that:

a. For each consequence C, Th_C is a *closure system* (i.e. for each $\mathbb{X} \subseteq Th_C$, $\bigcap \mathbb{X} \in Th_C$), $C(\emptyset)$ and S being the least and the largest element of it, and hence it is a complete lattice.

b. For each closure system $\mathbb{X}$ on $\mathcal{S}$, there exists a consequence C such that $Th_C = \mathbb{X}$.

3.3. Let $\mathbb{X} = 2^S$. We say that $\mathbb{X}$ is a *closure base* for a consequence C iff for each X, $C(X) = \bigcap\{Y \in \mathbb{X} : X \subseteq Y\}$. Observe that

a. If all $\mathbb{X}_t$, $t \in T$, are closure bases for C, so is $\bigcap \mathbb{X}_t$

b. Th_C is the largest closure base for C.

In general, C need not have the smallest closure base. We shall turn back to this point later, cf. Section 22.

3.4. Lemma. (R. Suszko) The following conditions are equivalent:

(i) C is structural.

(ii) Th_C is closed under counterimages of substitutions, i.e. for each $X \in Th_C$, and for each substitution e, $\vec{e}X = \{\alpha : e\alpha \in X\} \in Th_C$.

(iii) There is a closure base $\mathbb{X}$ for C closed under counterimages of substitutions.

Proof. (i)→(ii). Assume that C is structural, $X = C(X)$, and e is a substitution. We have to show that $C(\vec{e}X) \subseteq \vec{e}(X)$ (by 2.2.(T1) the converse is valid for all X). Observe that we have

$$eC(\vec{e}X) \subseteq C(e\vec{e}X) \subseteq C(X) = X,$$

which yields $C(\vec{e}X) \subseteq \vec{e}X$, as desired.

(ii)→(iii). Obvious.

(iii)→(ii). Let $X = \bigcap \mathbb{Y}$, for some $\mathbb{Y} \subseteq \mathbb{X}$. Let e be a substitution. We have to show that there exists a $\mathbb{Y}' \subseteq \mathbb{X}$ such that $\vec{e}X = \bigcap \mathbb{Y}'$. Verify that $\mathbb{Y}' = \{\vec{e}Y : Y \in \mathbb{Y}\}$ has the desired property.

(ii)→(i). Of course $eX \subseteq C(eX)$. Consequently, $X \subseteq \vec{e}C(eX)$, and $C(X) \subseteq C(\vec{e}C(eX))$. But, by (ii) $C(\vec{e}C(eX)) = \vec{e}C(eX)$, and we arrive at $C(X) \subseteq \vec{e}C(eX)$, which yields Łoś-Suszko's condition $eC(X) \subseteq C(eX)$.■

The lemma, we have just proved, will turn out to be very useful; we shall exploit it on many occasions.

4. Consequence operations from a complete lattice

4.1. Given any propositional language $\mathcal{S}$, the consequence operations in $\mathcal{S}$ are ordered as to their "strength": we say that C_1 *is weaker than* C_2 (or, alternatively, C_2 *is stronger than* C_1) iff $C_1(X) \subseteq C_2(X)$, for all X.

The ordering relation will be denoted by $\leqslant$, and its proper part by $<$.

4.2. Verify that the following conditions are equivalent:

a. $C_1 \leqslant C_2$,

b. $Th_{C_2} \subseteq Th_{C_1}$,

c. $\vdash_{C_1} \;\subseteq\; \vdash_{C_2}$.

4.3. Lemma. For each set $\mathbb{Q}$ of consequences in $\mathcal{S}$, the consequence $C_{\mathbb{Q}}$ defined by

$$C_{\mathbb{Q}}(X) = \bigcap\{C(X) : C \in \mathbb{Q}\}, \tag{1}$$

is the greatest lower bound of $\mathbb{Q}$, inf $\mathbb{Q}$.

Proof. Apply 4.2b. ■

Now, observe that the operation S on sets of formulas defined by

$$S(X) = S \tag{2}$$

is the strongest consequence in $\mathcal{S}$. S will be referred to as the *inconsistent* (or trivial) consequence in, S all the others will be called *consistent*, or *non-trivial*.

As a corollary to what has been said above we have

4.4. THEOREM. Let $\mathbb{C}$ be the set of all consequences in $\mathcal{S}$. Then $(\mathbb{C}, \leqslant)$ is a complete lattice. ■

(The fact that closure operations from a complete lattice is well known, see e.g. P. M. Cohn, 1965. Consequence operations are just closure operators defined on algebras of a special kind, cf. 1.3)

4.5. The weakest consequence in $\mathcal{S}$ will be denoted by Id, and called the *idle consequence* in $\mathcal{S}$. Id is defined by

$$Id(X) = X.$$

4.6. LEMMA. Let $\mathbb{Q}$ be a set of consequences in $\mathcal{S}$. Then

$$Th_{\sup \mathbb{Q}} = \bigcap \{Th_C : C \in \mathbb{Q}\}.$$

PROOF. The intersection of any set of closure systems is easily seen to be a closure system. Hence, $\bigcap \{Th_C : C \in \mathbb{Q}\}$ is a closure system. Denote the consequence it determines, cf. 3.2b, as $C^{\mathbb{Q}}$. Since $\bigcap \{Th_C : C \in \mathbb{Q}\} \subseteq Th_c$, for all $C \in \mathbb{Q}$, then $C \leqslant C^{\mathbb{Q}}$ for all $C \in \mathbb{Q}$. Suppose that $C \leqslant C'$ for all $C \in \mathbb{Q}$. Then $Th_{C'} \subseteq Th_C$ for all $C \in \mathbb{Q}$, and thus

$$Th_{C'} \subseteq \bigcap \{Th_C : C \in \mathbb{Q}\},$$

and we conclude that $C^{\mathbb{Q}} = \sup \mathbb{Q}$. ■

4.7. THEOREM. Let C_0 be a set of all structural consequences in $\mathcal{S}$. Then $(\mathbb{C}_0, \leqslant)$ is a complete sublattice of the lattice of all consequences in $\mathcal{S}$.

PROOF. Let Q be a set of structural consequences. If all $C \in \mathbb{Q}$ are structural, so is $\inf \mathbb{Q}$. Let us show this. We have, cf. 4.3, $\inf \mathbb{Q} = C_{\mathbb{Q}}$. Now, $eC_{\mathbb{Q}}(X) = e \bigcap \{C(X) : C \in \mathbb{Q}\} \subseteq \bigcap \{eC(X) : C \in \mathbb{Q}\} \subseteq \bigcap \{C(eX) : C \in \mathbb{Q}\} = C_{\mathbb{Q}}(eX)$, as desired.

In view of 3.4, in order to show that $\sup \mathbb{Q}$ is structural it suffices to show that $Th_{\sup \mathbb{Q}}$ is closed under counterimages of endomorphisms. Let X be a theory of $\sup \mathbb{Q}$. Then, by 4.6, X is a theory of all $C \in \mathbb{Q}$. They are structural and hence, by 3.4, for all substitutions e, $\vec{e}X$ is a theory of all $C \in \mathbb{Q}$. But then, by 4.6, $\vec{e}X$ is a theory of $\sup \mathbb{Q}$. ■

5. A bit more about the lattice of structural consequence

5.1. LEMMA. Let Q be a set of standard consequences defined in a language $\mathcal{S}$. Then for each α and each X, $\alpha \in \sup \mathbb{Q}(X)$ iff there is a finite sequence $C_{i_1}, \ldots, C_{i_n}$ of consequences from $\mathbb{Q}$ such that

$$\alpha \in C_{i_1}(C_{i_2} \ldots (C_{i_n}(X)) \ldots).$$

PROOF. Straightforward. ■

5.2. Lemma. Denote by $\mathbb{C}_0^f$ the set of all standard consequences in $\mathcal{S}$.

(i) For each finite $\mathbb{Q} \subseteq \mathbb{C}_0^f$, $\inf \mathbb{Q} \in \mathbb{C}_0^f$,

(ii) For each $\mathbb{Q} \subseteq \mathbb{C}_0^f$, $\sup \mathbb{Q} \in \mathbb{C}_0^f$,

(iii) For some $\mathbb{Q} \subseteq \mathbb{C}_0^f$, $\inf \mathbb{Q} \notin \mathbb{C}_0^f$.

Proof (i). Apply lemma 4.3 to prove that $\inf \mathbb{Q}$ is finitary whenever $\mathbb{Q}$ is a finite set of finitary consequences. If moreover the consequences in $\mathbb{Q}$ are structural then, by 4.7, $\inf \mathbb{Q}$ is structural, and hence standard.

(ii) – by lemma 5.1.

(iii) – select any connective § of $\mathcal{S}$. For each $n \in \omega$, define

$$C_n(X) = \begin{cases} X, & \text{if no formula in } X \text{ involves more than } n \text{ occurrences of §}, \\ S, & \text{otherwise.} \end{cases}$$

Let $\alpha_0, \alpha_1, \ldots$ be a sequence of formulas of $\mathcal{S}$ such that each α_i involves exactly i occurrences of §. Let $C = \inf\{C_n : n \in \omega\}$. Verify that all C_n are standard and hence, by 4.7, C is structural. Apply 4.3 to show that $C(\{\alpha_n : n \in \omega\}) = S$, and $C(X_f) = X_f$ for any finite $X_f \subseteq \{\alpha_n : n \in \omega\}$. Select any formula β such that $\beta = \alpha_i$ for no $i \in \omega$. Then $\beta \in C(\{\alpha_n : n \in \omega\})$, but $\beta \notin C(X_f)$ for no finite $X_f \subseteq \{\alpha_n : n \in \omega\}$. Hence C is not standard. ■

5.3. Denote by $\leqslant_f$ the relation on consequences in $\mathcal{S}$ defined by

(f_1) $\qquad C_1 \leqslant_f C_2$ iff for all finite X, $C_1(X) \subseteq C_2(X)$.

Define also

(f_2) $\qquad C_1 =_f C_2$ iff both $C_1 \leqslant_f C_2$ and $C_2 \leqslant_f C_1$.

Of course, $C_1 =_f C_2$ reads: C_1 and C_2 coincide on finite sets of formulas. Note that

a. $\leqslant_f$ is a quasi-ordering relation.
b. The restriction of $\leqslant_f$ to any set $\mathbb{Q}$ of finitary consequence is an ordering relation on $\mathbb{Q}$. In particular, $\leqslant_f$ is an ordering in the set of standard consequences.
c. If C_1 is finitary, then $C_1 \leqslant_f C_2$ iff $C_1 \leqslant C_2$.

5.4. Theorem. Let $\mathbb{C}$, $\mathbb{C}_0$, and $\mathbb{C}_0^f$ be as already defined (cf. 4.4, 4.7, and 5.2). Then

(i) $(\mathbb{C}_0^f, \leqslant_f)$ is a sublattice of the lattice $(\mathbb{C}_0, \leqslant)$

and hence

(ii) $(\mathbb{C}_0^f, \leqslant_f)$ is a sublattice of the lattice $(\mathbb{C}, \leqslant)$.

Moreover

(iii) $\mathbb{C}_0^f, \leqslant_f)$ is a complete lattice, though it is a complete sublattice of neither $(\mathbb{C}_0, \leqslant)$ nor of $(\mathbb{C}, \leqslant)$.

PROOF. (i) is an immediate corollary to lemma 5.2, and (ii) follows from (i). From 5.2 (iii) it follows that $(\mathbb{C}_0^f, \leqslant_f)$ is not a complete sublattice of any of the lattices mentioned of condition (iii) of 5.4. Still, in view of 5.2 (ii) $(\mathbb{C}_0^f, \leqslant_f)$ is a complete lattice; given any $\mathbb{Q} \subseteq \mathbb{C}_0^f$, sup $\mathbb{Q}$ in the lattice $(\mathbb{C}, \leqslant)$ and that in $(\mathbb{C}, \leqslant_f)$ coincide, now define C^f by

$$C^f(X) = \bigcap\{\bigcup\{C(X_f) : X_f \text{ is a finite subset of } X\} : C \in \mathbb{Q}\},$$

then $C^f = \inf \mathbb{Q}$ in $(\mathbb{C}_0^f, \leqslant_f)$.

5.5. An element a of a lattice $(A, \leqslant)$ is said to be an *atom* of the lattice iff the following conditions are satisfied:

(i) The lattice $(A, \leqslant)$ contains a least element; we shall denote it by 0_A.

(ii) $0_A < a$, and

(iii) There is no $b \in A$, such that $0_A < b < a$.

Similarly if

(j) There is a greatest element in $(A, \leqslant)$; we shall denote it by 1_A,

(jj) $a < 1_A$, and

(jjj) There is no $b \in A$ such that $a < b < 1_A$,

then a will be called a *coatom* of $(A, \leqslant)$.

5.6. Let C be a structural consequence. Then, by $[C)_0$ we shall denote the filter in $(\mathbb{C}_o, \leqslant)$ generated by C, $\mathbb{C}_0$ being the set of all structural consequences in the language of C. Similarly if C is standard by $[C)_0^f$ we shall denote the filter in $(\mathbb{C}_0^f, \leqslant_f)$ generated by C; $\mathbb{C}_0^f$ being the set of all standard consequences in the language of C.

Of course, $[C)_0$ is the set of all logics stronger than C, and $[C)_0^f$ is the set of all standard logics stronger than C, and, of course, both $([C)_0, \leqslant)$ and $([C)_0^f, \leqslant_f)$ are lattices. Moreover, the former is a complete sublattice of the lattice $(\mathbb{C}_0, \leqslant)$ and the latter a complete sublattice of the lattice $(\mathbb{C}_0^f, \leqslant_f)$.

In the logical terminology the atoms of $([C)_0, \leqslant)$, if any, are called *direct* (structural) *successors* of C and the coatoms of that lattice, if any, are called structural *maximalizations* of C. The atoms of $([C)_0^f, \leqslant_f)$, if any, are called *direct standard successors* of C, and the coatoms of that lattice, if any, are called *standard maximalizations* of C.

A lattice $(A, \leqslant)$ is said to be *atomic* (*coatomic*) iff for each $a \in A$ there is an atom (coatom) $b \in A$ such that $b \leqslant a$, unless $a = 0_A$ ($a \leqslant b$, unless $a = 1_A$). We conjecture that

a. A lattice of the form $([C)_0, \leqslant)$ need be neither atomic nor coatomic. What is more,
b. $([C)_0, \leqslant)$ need have neither atoms nor coatoms.

The following example may be useful in dealing with the problem raised. Let $\mathcal{S}$ be a language that involves two unitary connectives $\Diamond, \Box$. Abbreviate $\Box\Box\ldots\Box\alpha$ as $\Box^i\alpha$ where i is the number of $\Box$ preceding α. Define C_0 to be the weakest structural consequence of all structural consequences C that satisfy the following conditions:

(1) $C(\Diamond\alpha) = S$, for all α
(2) $C(\{\Box^i\Diamond\alpha : i \geqslant 1\}) = S$, for all α
(3) $\Box^i\Diamond\alpha \in C(\emptyset)$, for all $i \geqslant 1$ and all α such that the cardinality of the set of all subformulas of $\alpha \geqslant i$.

(The notion of a subformula of a formula α is obvious. If a rigorous definition is desired the following will do β is a *subformula* of α iff there is a formula γ and a variable p such that $\alpha = \gamma(\beta/p)$.)

We shall show that there is no maximalization of C_0 in the set of structural consequences. Assume the contrary. Let $C_0 \leqslant C'$ and let C' be a maximal structural consequence. In view of condition (2) there must exist p such that $\Box^i\Diamond p \notin C'(\emptyset)$, for some i, for otherwise $C'(\emptyset) = S$. Assume that $\Box^n\Diamond p \notin C'(\emptyset)$ and consider the set $Sb(\Box^n\Diamond p)$. If $C'(Sb(\Box^n\Diamond p)) \neq S$ then C' is strictly weaker than c'' defined by the condition

$$C''(X) = C'(X \cup Sb(\Box^n\Diamond p)),$$

for $\Box^n\Diamond p \in C''(\emptyset)$. On the other hand $C''(Sb(\Box^n\Diamond p)) = C'(Sb(\Box^n\Diamond p)) \neq S$, hence C'' is not trivial. It would follow that C' is not maximal, contrary to the assumption we made. Hence $C'(Sb(\Box^n\Diamond p)) = S$, which implies that

$$\Diamond q \in C'(Sb(\Box^n\Diamond p))$$

Let e be the substitution defined by the condition $er = \Diamond^n r$, for each propositional variable r. Clearly we have

$$e\Diamond q = \Diamond^{n+1}q \in C'(e\,Sb(\Box^n\Diamond p))$$

But, in view of condition (3) imposed on C_0 and in view of the fact that $C_0 \leqslant C'$, we have

$$e\,Sb\Box^n\Diamond p \in C'(\emptyset).$$

Hence $\Diamond^{n+1}q \in C'(\emptyset)$ which, by condition (1), yields that C' is inconsistent. Being trivial, C' is not maximal contrary to the assumption we made.

The argument we have presented shows that the consequence C_0 defined above admits no structural maximilization, thus the lattice $([C)_0, \leqslant)$ has no coatoms.

5.7. THEOREM. For each standard C, the lattice $([C)_0^f, \leqslant_f)$ is coatomic, i.e. for each standard C there is a standard maximization of C.

PROOF. Observe that a structural consequence C' is trivial iff $p \in C'(\emptyset)$. Indeed, if C' is a structural then $p \in C'(\emptyset)$ implies $ep \in C'(\emptyset)$ for all substitutions e, and thus $S \subseteq C'(\emptyset)$. Let C be a standard consistent logic. Consider any chain $\zeta \subseteq [C)_0^f$ of consistent logic. By lemma 5.2 (ii) sup $\zeta \in [C)_0^f$. Now $p \notin \sup \zeta(\emptyset)$, for $p \notin C'(\emptyset)$ for any $C' \in \zeta$. Indeed, suppose that $p \in \sup \zeta(\emptyset)$, then by lemma 5.1, $p \in C_{i_1}(C_{i_2} \ldots (C_{i_n}(\emptyset) \ldots)$ for some $C_{i_1}, \ldots, C_{i_n} \in \zeta$. Since ζ is a chain there is a strongest consequence among $C_{i_1}, \ldots, C_{i_n}$. Let it be C_{i_k}. Then $p \in C_{i_k}(\emptyset)$, contrary to the assumption that C_{i_k} is consistent.

Since, as we have shown, sup ζ is both standard and consistent, each chain ζ of consistent logics in $[C)_0^f$ has an upper bound. Hence, by Zorn's lemma, the set of all consistent logics in $[C)_0^f$ contains some maximal elements. Of course, any such element is a standard maximalization of C, and thus a coatom of $([C)_0^f, \leqslant^f)$. Moreover, the lattice $([C)_0^f, \leqslant^f)$ is coatomic, since for each $C' \in [C)_0^f$, the maximalization of C' is a maximalization of C.

5.8. A logic that is maximal in the lattice $([C)_0^f, \leqslant_f)$, i.e. a standard maximalization of C need not be maximal in $([C)_0, \leqslant)$. This remark is quite obvious, still it is not easy to produce a simple example that would prove it. Anyway, the one given below is rather involved.

Let $\Diamond, \Box$ be connectives of $\mathcal{S}$. Define:

$$T_1 = \{\Diamond^k \Box^m \Diamond^n \alpha : k \geqslant 0, m \geqslant 1, n \geqslant k, \alpha \in S\},$$
$$T_2 = \{\Diamond \Box^m \alpha : m \geqslant 0, \alpha \in S\},$$
$$T = T_1 \cup T_2,$$
$$N = \{\Diamond^k \Box^m \Diamond^n \Box^l \alpha : k > n \geqslant 1,\ m, l \geqslant 1, \alpha \in S\}.$$

In turn, define two consequences C and C^+ in $\mathcal{S}$ by

(C) $\alpha \in C(X)$ if and only if there is a finite subset $Y \subseteq X$ such that for each substitution e, if $eY \subseteq T$ then also $e\alpha \in T$.

(C^+) $\alpha \in C^+(X)$ if and only if there is a finite or infinite subset $Y \subseteq X$ such that for each substitution e, if $eY \subseteq T$ then also $e\alpha \in T$.

We shall leave to the reader the proof of the following:

(i) Both T and N are invariant sets.

(ii) C is a standard consequence.

(iii) C^+ is a structural consequence.

(iv) Both C and C^+ are consistent.

(v) If $\alpha \notin T$ then for some substitution e, $e\alpha \in N$.

(vi) If $\alpha \in N$ then $C(\alpha) = S$.

(vii) $T \subseteq C(\emptyset)$.

(viii) $C \leqslant C^+$.

Suppose that C' is a standard consequence strictly stronger than C, then for some X and α, $\alpha \notin C(X)$ though $\alpha \in C'(X)$. We may assume that X is finite and for some substitution e, $eX \subseteq T$ and $e\alpha \notin T$. By (v) the latter gives $e'e\alpha \in N$, for some substitution e'. At the same time, by (i) we know that $e'eX \subseteq T$. This by (vii) yields $C(e'eX) \subseteq C(\emptyset) \subseteq C'(\emptyset)$. By structurality of C' we have $e'e\alpha \in C'(e'eX) \subseteq C'(\emptyset)$ and this, by the assumption of the proof and (vi), gives $C'(\emptyset) = S$, i.e. c' is inconsistent. This proves that C is maximal in the lattice $[C)_0^f$.

It is easily seen that $q \in C^+(\Diamond p, \Diamond\Diamond p, \ldots)$, where p and q are variables, though $q \notin C(\Diamond p, \Diamond\Diamond p, \ldots)$. Thus C^+ is strictly stronger than C.

5.9. Note. The results in this section are based mainly on R. Wójcicki [1971].

Chapter 2

Are Logics Determined by Logical Theorems?

6. Structural completeness

6.1. The term ‘logic’ is very often applied in the sense of logical theory or system of logical theorems. Under the terminology to which we subscribe here, to define a logic in the sense above amounts to define the set $C(\emptyset)$, where C is a logic.

It is quite obvious that $C(\emptyset)$ does not determine C uniquely, except for the case when $C(\emptyset) = S$, S being the set of all formulas of the language of C. This remark is true, however, if C is meant to be any structural consequence, whatsoever. But, of course, if one imposes on C some additional postulates it may turn out that, at least in some cases, the correspondence between C and $C(\emptyset)$ is one-to-one.

6.2. Suppose that, given any system of logical theorems L, by the logic properly corresponding to L one means the strongest logic that preserves logical truth, i.e. the strongest logic in the set of all logics C such that $C(L) = L$. It is a matter of proof to show that such a logic always exists.

6.3. THEOREM. Given any set of formulas $L \subseteq S$, let $\mathbb{Q}_L$ be the set of all structural consequences in $\mathcal{S}$ such that $C(L) = L$. Let $C_L = \sup \mathbb{Q}_L$. then

a. $C_L(L) = L$. Moreover,

b. If L is invariant then $C_L(\emptyset) = L$.

PROOF. Observe that $\mathbb{Q}_L$ is not empty since it involves at least the idle consequence Id, i.e. the consequence defined by $Id(X) = X$. Of course, Id is structural.

Since all consequences in $\mathbb{Q}_L$ are structural then, by 4.7, sup $\mathbb{Q}_L = C_L$ is structural. Now, by lemma 4.6, we have

$$C_L(L) = \bigcap\{X : L \subseteq X \text{ and } C(X) = X, \text{ for all } C \in \mathbb{Q}_L\}$$

Hence, $C_L(L) = L$, and this concludes the proof of part a. of theorem.

In order to establish b. verify that if L is invariant, then the operation C_L^* defined by

$$C_L^*(X) - C_L(X \cup L)$$

is a structural consequence. By the definition $C_L \leqslant C_L^*$. On the other hand $C_L^*(L) = L$ and hence $C_L^* \leqslant C_L$. Thus the two consequences coincide, and since $C_L(\emptyset) = L$, we obtain $C_L(\emptyset) = L$, we obtain $C_L(\emptyset) = L$. ∎

6.4. Call a logic C *structurally complete with respect* to X iff $C(X) = X$ and C is the strongest logic with that property.

A logic is said to be *structurally complete* iff it is structurally complete with respect to $C(\emptyset)$.

6.5. The trouble with the solution suggested in 6.2 is that we expect the logic to allows us to derive not only logically true conclusions from logically true premisses but also true conclusions from true premisses. In view of 6.3 and 6.4 the strongest logic preserves L (L being an invariant set of all logically true sentences) is just the structurally complete logic C such that $C(\emptyset) = L$. Now, as we shall argue in Section 9, if a logic preserves truth, most likely, it is not structurally complete.

Before we examine the notion of structural completeness closer let us define some auxiliary notions of substantial methodological significance.

7. Rules of inference and inferential bases

7.1. A *rule of inference* can be viewed as a set of instructions of the form

(r) From X infer α,

or equivalently, from the formal point of view, as a set of inferences $X \vdash \alpha$. Of course, in all formal considerations the purely set-theoretical definition is preferable to the pragmatical one.

Write $r(X, \alpha)$, when $X \vdash \alpha$ is an instance of the rule r, i.e. $X \vdash \alpha \in r$. If all instances of r are finitary, the rule will be called *finitary*, if all them are axiomatic the rule will be called *axiomatic*.

7.2. A rule r that is closed under substitutions, i.e. $r(X, \alpha)$ implies that $r(eX, e\alpha)$, for all substitutions e, is called *structural*. By

(s) $\qquad X/\alpha$

we shall denote the rule that consist of all substitution instances of the inference $X \vdash \alpha$, i.e. $X/\alpha = \{eX \vdash e : e \text{ is a substitution}\}$. The rules of the form (s) will be called *sequential.* Of course, all sequential rules are structural. Furthermore, each structural rule is the set-theoretic union of some sequential rules.

A rule that both structural and finitary will be referred to as *standard.*

7.3. A rule r will be said to *preserve* X, and X will be said to be *closed* under r, iff for all X' and all α, if $X' \subseteq X$ and $r(X', \alpha)$ then $\alpha \in X$.

7.4. Given a consequence C we shall say that a rule r is *valid* with respect to C or, equivalently, r is a *rule of* C iff for all X and all α, $r(X, \alpha)$ implies that $\alpha \in C(X)$.

Verify that r is a rule of C iff r preserve all theories of C. In general a rule that preserves $C(\emptyset)$ need not be a rule of C.

7.5. It is rather obvious, anyway one can prove it easily, that each consequence operation is uniquely determined by its rules of inference. More rigorously, the following holds true. Let C_1, C_2 be consequence operations, then

a. $C_1 = C_2$ iff the set of all rules of C_1 coincides with that of C_2.

Verify also that:

b. If C_1, C_2 are structural then $C_1 = C_2$ iff the set of all structural rules of C_1 coincides with that of C_2.

c. If C_1, C_2 are finitary then $C_! = C_2$ iff the set of all finitary rules of C_1 coincides with that of C_2.

d. If C_1, C_2 are standard then $C_1 = C_2$ iff the set of all standard rules of C_1 coincides with that of C_2.

7.6. Let Q be a set of rules of inference. The weakest consequence in the set of all consequence C such that all rules in Q are rules of C will be denoted as Cl_Q and referred to as *determined by* Q. At the same time Q will be referred to as an *inferential base* for Cl_Q.

Let A be a set of formulas and Q a set of rules. A consequence C will be said to be *determined by* (A, Q) iff C is determined by $Q \cup \{r_A\}$, where r_A is the axiomatic rule defined by

$$r_A = \{\vdash \alpha : \alpha \in A\}.$$

In what follows (A, Q) will be treated merely as another notation for $Q \cup \{r_A\}$, and thus couples of the form (A, Q) will be called inferential bases (for the consequences they determine) on a part with inferential bases in the proper sense of word. The formulas in A will be referred to as the *axioms* of the inferential base (A, Q).

7.7. Given any consequence C and any set of rules Q, we shall say that C' is the *strengthening of* C *by means of* Q, and we shall denote C' as $C_{(+Q)}$, iff C' is the consequence determined by the set of rules of C enlarged by the rules in Q. If $C \leqslant C'$ and there is a set of formulas A such that $C(A \cup X) = C'(X)$, for all X, C' will be referred to as an *axiomatic strengthening* of C (by means of the set of axioms A), and will be denoted by $C_{(+A)}$.

Verify that the strengthening of C by A coincides with the strengthening of C by means of the rule r_A defined as in 7.6.

A straightforward but very useful is the following

7.8. THEOREM. Let Q_t, $t \in T$ be consequence operations and for each t, let Q_t be an inferential base for C_t. Let $C = \sup \{C_t : t \in T\}$. Then $Q = \bigcup\{Q_t : t \in T\}$ is an inferential base for C. ■

8. More on structural completeness

8.0. Denote by K_H the logic defined in the $\{\to, \neg\}$-fragment $(\mathcal{L} \restriction \{\to, \neg\})$ of $\mathcal{L}$ by an inferential base defined by the following schemata:

(H1) $\vdash \neg(\alpha \to \beta) \to \alpha$

(H2) $\vdash \neg(\alpha \to \beta) \to \neg\beta$

(H3) $\alpha \to \beta,\ \beta \to \gamma \vdash \alpha \to \gamma$

(H4) $\alpha \to (\beta \to \gamma),\ \alpha \to \beta \vdash \alpha \to \gamma$

(H5) $\neg\alpha \to \beta,\ \neg\alpha \to \neg\beta \vdash \alpha$

With the help of truth tables one verifies quite easily that both the axioms and rules of the base are classically valid (valid in K). Thus K_H is weaker than K. (Since in the classical logic $\wedge$ and $\vee$ are definable in terms of $\to$ and $\neg$, we may apply the same symbol to denote the classical logic defined in $\mathcal{L}$ and that defined on the reduct $\mathcal{L} \restriction \{\to, \neg\}$).

The logic K_H was defined by H. Hiż [1957], who also proved that it has the following "strange" properties:

(a) $K_H(\emptyset) = K(\emptyset)$,

(b) Modus Ponens $(p, p \to q/q)$ is not a rule of K_H, and hence

(c) $K_H < K$.

What exactly is so strange about this result? R. Suszko [1961] p.204 quotes the following remark made by H. Hiż in a conversation "the opinion, sometimes expressed, that a complete system of tautologies constitutes an adequate characterization of valid inferences of the sentential kind is shown to be unjustified".

Indeed, it is often believed that logical inferences are determined by logical theorems. This to some extent explains why logicians are so often preoccupied with certain systems of logical theorems paying little attention to logically valid rules of inference.

But perhaps a logically valid rule of inferences is just a structural rule that preserves the set of logical theorems? Unfortunately this idea brings us back to the condition of structural completeness we have already discussed.

8.1. THEOREM. A logic C is structurally complete iff each structural rule that preserves $C(\emptyset)$ is a rule of C.

PROOF. Assume that C is structurally complete and assume that r preserves $C(\emptyset)$. Then $Cl_r(C(\emptyset)) = C(\emptyset)$. Hence $Cl_r \leqslant C$. This implies that each inference of Cl_r is an inference of C, and consequently r is a rule of C.

Now suppose that C is not structurally complete. Let C' be a structural consequence operation such that $C(\emptyset) = C'(\emptyset)$ and $C < C'$. Of course all structural rules of C are rules of C', but not vice versa. Let r be a structural rule characteristic of C' i.e. such that it is not a rule of C. Since $C'(\emptyset) = C(\emptyset)$, r preserves $C(\emptyset)$. ■

8.2. As a matter of fact it is just Theorem 8.2 that usually serves as a definition of structural completeness. Then the definition given in 6.4 becomes a theorem (cf. R. Wójcicki [1973]; D. Makinson [1976]).

The notion of structural completeness was introduced to the theory of propositional calculi by W. A. Pogorzelski [1971]. He was also the first to start systematic investigations of this property. It should by mentioned that the original definition of structural completeness was not equivalent to the one given here. A logic C was defined by W. A. Pogorzelski to be structurally complete iff all standard (thus both structural and finitary, not merely structural) rules that preserve $C(\emptyset)$ were rules of C.

It should also be noted that the notion of structural completeness need to be restricted to logics only, and can be applied to all propositional calculi (cf. W. A. Pogorzelski [1971]). In order to obtain such a general definition just replace the word 'logic' in 8.2 by 'consequence'. In connection with the last remark let us notice that some logics become structurally complete when strengthened by the substitution rule (from α infer all substitution instances of α). If C is the original logic, denote the strengthening by $\bar{C}$. The consequences of the form $\bar{C}$ are sometimes called *quasi-structural*. In general they are not structural. In each language $\mathcal{S}$ there are only two consequences which are structural and quasi-structural at the same time. These are the inconsistent consequence S (cf. 4.3) and the *almost-inconsistent consequence* $S_\emptyset$ defined by $S\emptyset(\emptyset) = \emptyset$ and $S_\emptyset(X) = S$, for all non-empty X. Łukasiewicz logics may serve as an example of logics that become structurally complete when transformed to quasi-structural form, cf. M. Tokarz [1972].

8.3. A consistent logic C such that no consistent logic is properly stronger than C is referred to as *maximal* (i.e. a logic is maximal iff it is maximal in the set of all non-trivial logics of a given language).

8.4. A consistent logic C such that no logic that is an axiomatic strengthening of C, but C itself, is consistent is referred to as *Post-complete*.

From the two definitions given above it follows almost immediately:

8.5. THEOREM. A consistent logic is maximal iff it is both structurally complete and Post-complete. ■

(A paper by M. Tokarz [1973] is a useful survey of various notions of completeness.)

8.6. Classical two-valued propositional logic is an obvious example of a logic that is maximal and thus structurally complete.

In a outline, this can be shown as follows. The semantic interpretation for K is provided by well-known two-valued truth tables. If $\alpha \notin K(X)$ then one may assign truth-values to variables in X and α so that all β in X become true (take the value 1) and α becomes false (takes the value 0). Now substitute $p \to p$, for each variable q to which 1 was assigned, and substitute $\neg(p \to p)$, for each variable q' to which 0 was assigned. Let e be the substitution we have defined. Then $eX \subseteq K(\emptyset)$ and $K(e\alpha) = L$ (the logic K is assumed to be defined in the standard language $\mathcal{L}$).

In order to gave the logic stronger than K we must strengthen K by a structural rule that is not a rule of K, i.e. it must involve an inference $X \vdash \alpha$ such that $\alpha \notin K(X)$. But the argument we presented shows that such a strengthening must be inconsistent (and thus $K_{+X/\alpha}$-inconsistent as well); a K-inconsistent sentence $e\alpha$ would be derivable from $K(\emptyset)$. Thus K is maximal.

Of course, there are logics other than K that are structurally complete (cf. e.g. T. Prucnal [1973], [1975], [1976]; M. Tokarz [1972]), not to mention that each logic can be transformed to be structurally complete. Still, which is hardly surprising, not too many logics of established significance are structurally complete.

9. Some examples

9.1. EXAMPLE 1. *Łukasiewicz many-valued logics.* Two-valued truth-tables for K are extend to n-valued by adding $\frac{1}{n-1}, \ldots, \frac{n-2}{n-1}$ as intermediate truth values. ω-valued truth tables involve all fractions of the form $\frac{k}{n}$, $0 \leqslant k \leqslant n$, i.e. all rational numbers in the interval $[0, 1]$. The operations corresponding to $\wedge, \vee, \to, \neg$ defined by the truth tables can be numerically defined as follows:

(Ł$\wedge$) $x \wedge y = \min(x, y)$,

(Ł$\vee$) $x \vee y = \max(x, y)$,

(Ł$\to$) $x \to y = \min(1, 1 - x + y)$,

(Ł$\neg$) $\neg x = 1 - x$.

The customary definition of n-valued (ω-valued) Łukasiewicz logic as the set of all formulas that have the value 1 under all value assignments in n-valued (ω-valued) truth-tables is not very useful for us. But before we suggest a definition of a Łukasiewicz logic in the inferential sense let us define the truth-table interpretation in a bit more rigorous way.

Denote by $\mathcal{L}_n$ the set $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$, and by $\mathcal{L}_\omega$ the union of all $\mathcal{L}_n$, $n \geqslant 2$. Apply the same symbols to denote the algebras

$$(*) \qquad \mathcal{L}_\eta = (\mathcal{L}_\eta, \wedge, \vee, \to, \neg), \quad \eta = 2, 3, \dots, \omega$$

where the operations $\wedge, \vee, \to, \neg$ are defined by Łukasiewicz truth-tables. *Łukasiewicz truth-table algebras* we have just defined coincide wits the logical matrices for many-valued logics defined in J. Łukasiewicz and A. Tarski [1930].

Observe that the functions that are usually referred to as truth-value assignments in Łukasiewicz η-valued truth-tables are just homomorphism from $\mathcal{L}$ into $\mathcal{L}_\eta$. From now on, they will be referred to as valuations in $\mathcal{L}_\eta$.

Following Łukasiewicz we interpret 1 as “true”. In fact, valuations in $\mathcal{L}_\eta$ were intended to represent all “admissible” truth-value assignments, i.e. all truth-value assignments that conform to the intended meaning of the connectives of $\mathcal{L}$, under the assumption that η-valued logic is the one that is accepted. Though they are not exactly the same as admissible valuations in the sense of 2.10 (except from $\eta = 2$, they have more than two values), it is clear what is to be meant by Łukasiewicz η-valued truth preserving logic, Ł$_\eta$.

Verify that the operation Ł defined by:

$$\alpha \in \text{Ł}_\eta(X) \text{ iff for each valuation } \alpha \text{ in } \mathcal{L}_\eta,$$
$$\lambda\alpha = 1 \text{ whenever } \lambda\beta = 1 \text{ for all } \beta \in X,$$

is (a) the strongest truth preserving consequence under valuations in $\mathcal{L}_\eta$, and (b) it is structural.

Furthermore, verify that

$$(c) \qquad \text{Ł}_2 = K.$$

As we shall see later (cf. 13.6), the term "Łukasiewicz η-valued logic" admits more interpretations than the one we have defined. Incidentally, what is often means to be Łukasiewicz η-valued logic in inferential sense is the logic defined by Modus Ponens and the set of all inferences of the form $\vdash \alpha$, where $\lambda(\alpha) = 1$, for all valuations λ in $\mathcal{L}_\eta$, i.e. by Modus Ponens and the *content* $\zeta(\mathcal{L}_\eta)$ of $\mathcal{L}_\eta$ defined by

$$\zeta(\mathcal{L}_\eta) = \{\vdash \alpha : \lambda(\alpha) - 1, \text{ for all valuations } \lambda \text{ in } \mathcal{L}_\eta).$$

This seems to be an ad hoc way of defining Łukasiewicz logic, still it turns out (R. Wójcicki, 1976) that, except from $\eta = \omega$, the logics defined in the way described, coincide with Łukasiewicz truth-preserving logic $Ł_\eta$.

Now we are in a position to examine whether Łukasiewicz logics (read Łukasiewicz truth-preserving logics) are structurally complete. Observe that for no Łukasiewicz logic $Ł_\eta$, $\eta \geqslant 3$, that involves $\frac{1}{2}$ as a truth value $(Ł_3, Ł_5, \ldots, Ł_\eta)$ the following rule

$$(p \vee \neg p) \to (p \wedge \neg p)/q$$

is truth preserving. To see this assigne $\frac{1}{2}$ to p and 0 to q. With the help of R. McNaughton [1951] theorem concerning definability in Łukasiewicz matrices, one may prove that there is no formula α such that the truth value of α is $\frac{1}{2}$, under all truth assignments in n-valued matrices, $n = 2, 3, \ldots$. Since $(p \vee \neg p) \to (p \wedge \neg p)$ takes the value 1 only if $\frac{1}{2}$ is assigned to p, the fact that there is no such α implies that no substitution instance of $(p \vee \neg p) \to (p \wedge \neg p)$ is logically true in Łukasiewicz logics. If so, then the rule we have defined preserves logical truth, simply because it never applies to logically true premisses.

A suitable modification of the rule we have discussed allows to repeat the argument for any of the remaining logics. For instance in the case of $Ł_4$ one has to use

$$(p \vee \neg p) \to ((p \vee \neg p) \to (p \wedge \neg p))/q.$$

It follows from the argument presented that:

– Łukasiewicz logics (of the kind we have defined!) are not structurally complete,

– we should keep them as such unless we do not care about preserving truth.

9.2. Example 2. *Modal logics.* We shall adopt the following convention. Given a modal system that is customarily denoted by a symbol X, we shall denote it by M_X. Thus for instance Kripke's system K will be denoted M_K. $S4$ will be denoted M_{S4} etc.

Again, as in the case of Łukasiewicz's systems, modal systems are sets of formulas not consequences. They are defined with the help of some rules, but if, say, the necessitation rule $p/\Box p$ is applied to define M_K, it does mean that $p/\Box p$ must be the rule of the logical consequence corresponding to modal systems should be defined. We shall discuss the matter later.

Of course, if for a particular system there is a semantics that not only is a more or less convenient tool for formal analyses, but also presents us with some philosophically acceptable interpretation of that system (some modal systems, M_K, Feys –von Wright's M_T, M_{S4}, M_{S5}, among others, seem to have such semantics), then the consequence corresponding to the system can be defined, along the pattern we have already discussed, as the truth preserving consequence.

We should be prepared, however, that if we follow this approach, typically modal rules, such as $p \to q/\Box p \to \Box q$, will turn out not to be truth-preserving. And there is a little surprise in that. From the intuitive standpoint they are not truth-preserving either. From that $p \to q$ is true it does not follow by any means that $\Box p \to \Box q$ is true, (of course, $\Box$ is to be interpreted as it is necessary that). Observe that rar is the material implication, i.e. $p \to q$ is false only if p is true and q false. Thus, for instance, the implication

Paris is Paris $\to$ Paris is the capital of France

is true, but if the components of it are prefixed by it is necessary that it becomes false. Even more obvious is that $p/\Box p$ is not truth preserving. Thus we face the choice: modal logics cannot be at the same time truth-preserving and structurally complete. Of course, we face this dilemma only if we insist on viewing modal logics as logics in the inferential sense.

9.3. Example 3. *Intuitionistic propositional logic* J (cf. 2.11). Let us leave open the question under which interpretation J is a truth-preserving logics. Anyway it is not structurally complete. Not all rules that preserve $J(\emptyset)$ are rules of J. As an example may serve:

(KP) $$\neg p \to (q \vee r)/(\neg p \to q) \vee (\neg p \to r)$$

(cf. R. Harrop, 1960) or

(MI) $$((p \to q) \to (p \vee r))/((p \to q) \to p) \vee ((p \to q) \to r))$$

(cf. Minc [1972]).

As a matter of fact J is structural incomplete to a "very large degree". As it was proved by W. Dziobak [1980], there are $2^{2^{\lambda^{r_0}}}$ logics $J^+ \geqslant J$ such that $J^+(\emptyset) = J(\emptyset)$.

(Dziobak's result is only apparently inconsistent with the fact established by V. A. Jankov [1968]) that the cardinal number of all superintuitionistic logics is $2^{\alpha^{r_0}}$. Jankov's theorem concerns superintuitionistic logics meant to be systems of theorems that include $J(\emptyset)$, and are closed under both Modus Ponens and substitutions. Dziobak dealt with consequence operations.)

9.4. Example 5. *Relevant logics.* Perhaps A. R. Anderson and N. D. Belnap's [1958] system E if entailment, of the same authors – cf. N. D. Belnap [1967] – system R of relevant implication, and R. K. Meyer and J. M. Dunn's [1969] system RM are the best known and most important. They are again systems of formulas. Still, in the systems we mentioned, the implication connective is meant to represent the consequence operation: $\alpha \to \beta$ reads 'α entails β'. This allows us to assign in a natural way to each relevant system S the relevant consequence operation S by postulating that

$$(\to) \qquad \alpha \in \vec{S}(\alpha_1, \ldots, \alpha_n) \text{ iff } (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \alpha \in S.$$

Some further conditions that define $\vec{S}$ on $\emptyset$ and infinite sets of formulas should be added, see Section 15.

The approach we have mentioned seems to be the only acceptable from philosophical standpoint, still, what is usually meant by logical consequences corresponding to E, R, RM are consequences C_E, C_R, C_{RM} determined by the inferential bases $(E, AD, MP), (R, AD, MP), (RM, AD, MP)$ respectively, where MP is Modus Ponens and AD is Adjunction Rule

$$(AD) \qquad p, q/p \wedge q.$$

The systems we are discussing, as well as many other relevant systems, are just defined with the help of these rules. Though $\vec{E}, \vec{R}, \vec{RM}$ rather than C_R, C_R, C_{RM} deserve to be called relevant consequence, in what follows, just to conform to the usual practice, by relevant logics we shall mean the latters not the formers.

There is a very special reason why relevant consequences must be notoriously structurally incomplete, when defined in an adequate way, i.e. by condition $(\to)$. The relevant systems are expected to satisfy the *relevance principle* or, at least, the *weak relevance principle.* Now:

a. The relevance principle holds for X iff for all α, β if $\alpha \to \beta \in X$ then $Var(\alpha) \cap Var(\beta) \neq \emptyset$, i.e. α and β have at least one variable in common.
b. The weak relevance principle holds for X iff for all α, β, if $\alpha \to \beta \in X$, $Var(\alpha) \cap Var(\beta) = \emptyset$ then both $\neg\alpha$ and β belong to X.

On the other hand, $\alpha \to \alpha$ is a theorem in most of relevant systems.

From the two facts stated above it follows directly that the rule

$$(*) \qquad p \to p/q \to q$$

cannot be a rule of any "genuine" relevant consequence though, no doubt, all relevant logicians would agree that this rule preserves not only truth but also logical truth. However, it violates both the relevance and weak relevance principle.

This remark does not settle the question of whether C_E, C_R, C_{RM} are structurally complete. The rule $p \to p/q \to q$ is the rule of all these consequences. But for example

$$(IN) \qquad p, \neg p/q$$

already is not. Though, of course, it preserves logical truth. All systems E, R, RM are closed under it.

9.5. EXAMPLE 6. *Logics tolerating inconsistency.* As useful survey of *paraconsistent logics*, as the logics tolerating inconsistency are called, was given by A. I. Arruda [1980]. Perhaps the most representative for this group of logics are S. Jaśkowski's discussive (or discursive) calculi (cf. S. Jaśkowski [1948], [1969]) and N. C. A. da Costa logic C_n, $1 \leqslant n < \omega$ (cf. N. C. A. da Costa [1958]).

By very definition of a paraconsistent logic, in no paraconsistent logic rule IN, we have just discussed, is valid. Similarly as relevant logics, paraconsistent logics do not involve contradictory theorems.

9.6. The idea of the logic properly corresponding to a system of logical theorems has very little to do with the notion of structural completeness.

Our discussion was motivated by heuristic reason. We went in the wrong direction on purpose, just, in order to define the problem in a clear way.

If there is any possibility to indicate among all structural consequence operations that share a system of logical theorems the one (as we have put it), properly corresponding to that system, the possibility must involve making use of some form of the Deduction Theorem. We shall make this idea precise in the next chapter. But before we reexamine our problem from the new angle, let us swell on inferential bases for a while.

10. More on inferential bases. Proofs

10.1. Let Q be a set of rules of inference. A sentence α is said to be *provable from X by means* of (rules in) Q iff there exists a finite sequence of formulas (called a *proof* of α from X by means of Q)

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

such that

(i) $\alpha = \alpha_n$,
(ii) for each α_i, $i = 1, \ldots, n$, either $\alpha_i \in X$ or for some $Y \subseteq \{\alpha_1, \ldots, \alpha_{i-1}\}$, $Y \vdash \alpha_i$ is an instance of a rule in Q.

One easily verifies that, if α is provable from X by means of Q, then $\alpha \in Cl_Q(X)$. The converse need not hold true. Let us examine the matter closer.

With the help of 4.3 one easily verifies that the following is valid

10.2. Lemma. Let Q be a set of rules. For each set of formulas X, $Cl_Q(X)$ is the least superset of X closed under all rules in Q. ∎

We shall often refer to sets of the form $Cl_Q(X)$ as the *closure of X under Q*.

10.3. Theorem. If all rules in Q are finitary, then for all α, and all X, $\alpha \in Cl_Q(X)$ iff α is provable from X means of Q.

Proof. Let α be provable from X by means of Q. Then α belongs to each superset Y of X closed under Q, as one may easily show by induction with respect to the length of the proof. Hence, by 10.2, $\alpha \in Cl_Q(X)$.

Now suppose that $\alpha \in Cl_Q(X)$ and all rules in Q are finitary. Put $Cl_Q^0(X) = X$, and for each $i > 0$ define $\beta \in Cl_Q^i(X)$ iff either

(i) $\beta \in Cl_Q^{i-1}(X)$, or

(ii) for some $Y \subseteq Cl_Q^{i-1}(X)$, $Y \vdash \beta$ is an instance of a rule in Q.

The union $\bigcup Cl_Q^i(X) : i \geqslant 0$ is closed under the rules in Q, and since it is contained in all supersets Y of X that are closed under Q it is the least of all such supersets, and consequently, by 10.2, we obtain

$$Cl_Q(X) = \bigcup \{Cl_Q^i(X) : i \geqslant 0\}.$$

If $\alpha \in Cl_Q(X)$ then, for some $i \geqslant 0$, $\alpha \in Cl_Q^i(X)$. If $i = 0$, then $\alpha \in X$ and, of course it is provable from X. Applying a recursive argument, verify that the same holds true for all i. ∎

(Observe that, as follows immediately from the definition of a proof, α is provable from X by means of Q iff α is provable from X by means of finitary rules in Q. Thus, in fact, it would be natural to restrict the notion of a proof to finitary rules only.)

10.4. Theorem. Let $\mathcal{S}$ be a propositional language and C a unary operation defined on sets of formulas of $\mathcal{S}$. Then the following conditions are satisfied:

(i) C is a consequence operation iff there exists a set of rules of inference Q such that $C = Cl_Q$.

(ii) C is a structural consequence iff there exists a set of structural rules Q such that $C = Cl_Q$.

(iii) C is a finitary consequence iff there exists a set of finitary rules Q such that $C = Cl_Q$.

(iv) C is a standard consequence iff there exists a set of standard rules Q such that $C = Cl_Q$.

PROOF. (i) We have to prove only the implication from the left to right. Verify that if C is a consequence and Q is the set of all rules of C then $C = Cl_Q$.

(ii) First verify that if C is structural and Q is the set of all structural rules of C then $C = Cl_Q$. Second, assume that $C = Cl_Q$ and all rules in Q are structural. Verify that each set of the form $\vec{e}Cl_Q(X)$, e being a substitution, is closed under rules in Q and hence is a theorem of Cl_Q. This by Lemma 3.4, implies the structurality of Cl_Q.

(iii) If C is finitary and Q is the set of all finitary rules of C, then $C = Cl_Q$. Conversely, if for some set Q of finitary rules $C = Cl_Q$, then, by 10.3, $\alpha \in C(X)$ iff α is provable from X by means of Q. But proofs are finite sequences of formulas, and hence $\alpha \in C(X)$ implies that $\alpha \in C(X')$ for some finite X', which is exactly condition (F), cf. 2.8, defining finitary consequences.

(iv) Verify that if C is standard and Q is the set of all standard rules of C then $C = Cl_Q$. Conversely, if $C = Cl_Q$ for some standard set of rules Q then, by (ii) and (iii), C is both structural and finitary, hence standard.■

10.5. Let us conclude this section with a few definitions.

a. Let Q, Q' be inferential bases. If $Cl_Q = Cl_{Q'}$ we say that Q and Q' are *equivalent*.

b. If Q and $Q \cup \{r\}$ are equivalent, the rule r is said to be *derivable* from Q. The rules in Q are often referred to as the *primitive rules* of the base Q.

10.6. NOTE. Theorems presented in this section are already "classical". Theorem 10.3 should be credited to A. Tarski [1935], while 10.4 to J. Łoś and R. Suszko [1958].

Chapter 3

Well-Determined Logics

11. Basic definitions and theorems

11.1. Let C be a logic that involves $\to$ and $\wedge$. We shall say that C is *well-determined* (*in terms of* $\to$ *and* $\wedge$) iff for all α, β,

(W1) C is standard,

(W2) $C(\alpha \wedge \beta) = C(\alpha, \beta)$,

(W3) $\alpha \in C(\beta)$ iff $\beta \to \alpha \in C(\emptyset)$.

11.2. Let us briefly comment on the definition given above. Any binary connective $\wedge$ (in this remark, consider $\wedge$ to be a variable representing connectives rather than any specific connective) that satisfies W2 is called *conjunction* with respect to C.

It is perhaps worthwhile mentioning that a connective $\vee$ is said to be *disjunction* with respect to C iff the language of C involves $\vee$ and the following holds true

$$(\vee) \qquad C(X, \alpha \vee \beta) = C(X, \alpha) \cap C(X, \beta).$$

Let us adopt the convention under which $\alpha_1 \wedge \ldots \wedge \alpha_n$ will be treated as an abbreviation for $(\alpha_1 \wedge (\alpha_2 \wedge \ldots (\alpha_{n-1} \wedge \beta))$. A similar convention will be adopted for disjunction. In many logics (but of course not in all) the order of conjuncts as well as disjuncts does not matter, in the sense that $C(\alpha) = C(\beta)$ whenever α, β differ only as to the order of conjuncts of the conjunctions (disjuncts of the disjunctions resp.) they involve.

From the definition 11.1 it follows immediately that:

11.3. THEOREM. If C is a well-determined logic then $C(\emptyset)$ defines C uniquely.■

What are the conditions to be satisfied by $C(\emptyset)$ in order for C to be well-determined?

11.4. Call a set of sentences L *deductive* iff there is a well-determined logic C such that $C(\emptyset) = L$.

11.5. THEOREM. L is deductive iff the following conditions are satisfied.

(D1) L is closed under substitutions,

(D2) For all α, $\alpha \to \alpha \in L$,

(D3) L is closed under the following rules of inference:
Rearrangement of Antecedent (RE) $\gamma, \gamma' : \gamma \to p/\gamma' \to p$, where γ, γ' are any formulas such that $Var(\gamma) = Var(\gamma')$ and the only connective appearing in both γ and γ' is $\wedge$. (Observe that (RE) is a class of sequential rules not a single rule).
Enlargement of Antecedent (EA): $p \to q/(p \wedge r) \to q$,
Composition (CM): $p_1 \to q_1$, $p_2 \to q_2/(p_1 \wedge p_2) \to (q_1 \wedge q_2)$,
Transitivity (TR): $p \to q$, $q \to r/p \to r$,
Modus Ponens (MP): p, $p \to q/q$,
Cancellation of a Valid Conjunct (CV): p, $(p \wedge q) \to r/q \to r$.

PROOF. ($\to$). This part of proof is straightforward and we shall leave it to the reader.

($\longleftarrow$) Assume that L satisfies (D1), (D2), (D3), and put

$$\alpha \in \vec{L}(X) \text{ iff } (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \alpha \in L,$$

$$\text{for some } \alpha_1, \ldots, \alpha_n \in \cup L.$$

Observe, that condition W3 can by replaced by

$$(W3') \qquad \alpha \in C(\alpha_1, \ldots, \alpha_n) \text{ iff } (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \alpha \in C(\emptyset).$$

The resulting set of conditions will be equivalent to the original one. We shall show that under the assumption made, W1, W2, W3 hold true.

Apply (D2) and make use of the rules CM and TR to show that $\vec{L}$ satisfies Tarski's conditions (T1) – (T3) (cf. 2.2), and hence is a consequence operation. Since L is closed under substitutions, $\vec{L}$ is structural, and it follows directly from the definition of $\vec{L}$ that $\vec{L}$ is finitary. Hence it is standard, as required by (W1).

Apply the rules RE, EA and D2 to show that $\vec{L}$ satisfies (W2).

Now, in order to establish (W3), assume that $\alpha \in \vec{L}(\beta_1, \ldots, \beta_n)$, $n \geqslant 1$. Hence for some $\beta'_1, \ldots, \beta'_m \in \vec{L} \cup \{\beta_1, \ldots, \beta_m\}$ we have

$$(1) \qquad \beta_1 \wedge \ldots \wedge \beta'_m \to \alpha \in L.$$

But, clearly, we have also

$$(2) \qquad (\beta'_1 \wedge \ldots \wedge \beta'_m \wedge \beta_1 \wedge \ldots \wedge \beta_n) \to \alpha \in L,$$

for (2) is derivable from (1) by successive applications of EA. What we have to do now is to apply successively either RE or CV in order to get rid of all $\beta'_1, \ldots, \beta'_m$; observe that each β'_i either belongs to L, or doubles some of β_j, $j = 1, \ldots, n$. In this way we arrive at

$$(3) \qquad \beta_1 \wedge \ldots \wedge \beta_n \to \alpha \in L \subseteq \vec{L}(\emptyset),$$

exactly as wanted. (The argument covers the case $n = 1$).

Suppose, in turn, that

$$(4) \qquad \beta_1 \wedge \ldots \wedge \beta_n \to \alpha \in \vec{L}(\emptyset).$$

Then, for some $\gamma_1, \ldots, \gamma_m \in L$,

$$(5) \qquad (\gamma_1 \wedge \ldots \wedge \gamma_m) \to ((\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha) \in L.$$

Apply successively CV in order to get

$$(6) \qquad \gamma_m \to ((\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha) \in L,$$

and then apply MP to get

$$(7) \qquad (\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha \in L$$

thus concluding the proof. ■

11.6. There are many logics that are well-determined. Observe, that to this category belongs each logic C that is standard involves conjunction (cf. 11.2) and for which Deduction Theorem holds true, i.e.

$$(DT) \qquad \alpha \in C(X, \beta) \text{ iff } \beta \to \alpha \in C(X).$$

The classical two-valued propositional logic K and the intuitionistic logic J are obvious examples of well-determined logics.

12. Deductive systems and well-determined logics

12.1. In what follows, given any set of formulas L being a deductive system, by $\vec{L}$ we shall denote the consequence operation defined just as in 11.5, and we shall refer to $\vec{L}$ as to *well-determined logic based on L.*

There is a very special reason why, if not always then at least considerably often, $\vec{L}$ fully deserves to be treated as the consequence "properly corresponding" to L, (cf. 6.2) and thus to $\vec{L}(\emptyset)$. It seems very natural to postulate that in order for $\to$, and C to be properly defined they must be linked by the condition

$$(\to) \qquad \beta \in C(\alpha) \text{ iff } \alpha \to \beta \in C(\emptyset).$$

The meaning of $\alpha \to \beta$ is often claimed to be best conveyed by saying that α entails β. Now, if α logically entails β (i.e. $\beta \in C(\alpha)$) then $\alpha \to \beta$ is logically valid ($\alpha \to \beta \in C(\emptyset)$), and vice versa. But, even if we insist upon the "material" interpretation of $\to$ ($\alpha \to \beta$ is true unless α is true and β false), if we keep interpreting C as logical derivability (or logical entailment), $(\to)$ remains valid.

Now, if the language of C involves $\wedge$, and the condition (again a very natural one)

$$(\wedge) \qquad C(\alpha \wedge \beta) = C(\alpha, \beta)$$

holds true, then $(\to)$ and $(\wedge)$ imply

$$\alpha \in C(\beta_1, \ldots, \beta_n) \text{ iff } (\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha \in C(\emptyset),$$

i.e. when C is well-determined, just the condition that defines $C(X)$ in terms of $\to$ and $\wedge$ for all finite X.

12.2. The argument presented above is, I hope, a good one to the effect that $\vec{L}$ should be treated as distinguished among all logics C such that $C(\emptyset) = L$. We must realize however, that if one defines a logic L as certain deductive set of formulas one need not necessarily view $\vec{L}$ to be the logical consequence corresponding to L. What one who invented the system L means by the derivability based on L, need not coincide with what we have defined to be the well-determined logic based on L. But after all the inventor of L may by so happy with "discovering" a new logical system (and just joining the Pantheon of founders of logic) that he may not bother with such a trifle as when and how one can use his "logic" as a logic, i.e. an instrument of deduction.

13. Are Łukasiewicz logics well-determined?

13.1. If by Łukasiewicz logics we mean the logics $Ł_3, Ł_4, \ldots, Ł_\omega$ defined in 9.1 then they are not. Indeed, in all these logics $\neg(\alpha \to \beta)$ is a consequence of $\alpha \wedge \neg\alpha$ but in none of these logics $(\alpha \wedge \neg\alpha) \to \neg(\alpha \to \beta)$ is a theorem. Thus none of these logics satisfies condition $(\to)$ (cf. 12.1). There is, however, more to be said about Łukasiewicz logics in this context.

Define $\alpha \to_n \beta$ recursively as follows $\alpha \to_1 \beta = \alpha \to_1 \beta$ and $\alpha \to_n \beta = \alpha \to (\alpha \to_{n-1} \beta)$. Now, for each Łukasiewicz logic $Ł_n$, n finite, the following is easily seen to be true

$$(1) \qquad Ł_n(\alpha \wedge \beta) = Ł_n(\alpha, \beta),$$

$$(2) \qquad \alpha \in Ł_n(\beta_1, \ldots, \beta_n) \text{ iff } (\beta_1 \wedge \ldots \wedge \beta_n) \to_{n-1} \alpha \in Ł_n(\emptyset).$$

Actually, we have not only (2) but also (cf. 29.4)

(2′) $$\alpha \in \text{Ł}_n(X,\beta) \text{ iff } \beta \rightarrow_{n^{a_1}} \alpha \in \text{Ł}_n(X).$$

"Not only" because (2′) combined with (1) implies (2), but (1) and (2) all by themselves do not suffice to derive (2′).

An interesting thing about $\rightarrow_{n-1}$ is that when we define $\neg_{n-1}$ by

$$\neg_{n-1}\alpha =_{\text{df}} \alpha \rightarrow_{n-1} \neg(\alpha \rightarrow \text{inf})$$

then $\wedge$, $\vee$, $\rightarrow_{n-1}$, $\neg_{n-1}$ fragment of Ł_n, $n = 3, 4, \ldots$ os an isomorphic copy of the classical logic K with $\rightarrow_{n-1}$ corresponding to classical implication $\rightarrow$ (i.e. $\rightarrow$ in the language of K) and $\neg_{n-1}$ corresponding to classical negation (cf. M. Tokarz [1971]).

As we shall see later (cf. 30.3), for each finite n, Ł_n is a finitary and thus it turns out that each Ł_n, though not well-determined in terms of Łukasiewicz implication and conjunction, is still well determined in terms of $\rightarrow_{n-1}$ and $\wedge$.

Ł is not finitary (cf. R. Wójcicki [1976]) and hence it is not well-determined in terms of any connectives.

We have then the following

13.2. Theorem.

a) None of the logics Ł_n, n finite $\geqslant 3$, is well determined in terms of $\rightarrow$, $\wedge$.
b) Each of the logics Ł_n, n finite $\geqslant 3$, is well determined in terms of $\rightarrow_{n-1}$, $\wedge$.
c) There are no connectives in terms of which $\text{Ł}\omega$ is well-determined.■

Let us, now, have a lock at Łukasiewicz logics from quite a different angle. Curiously enough we have

13.3. Theorem All systems $\zeta(\mathcal{L}_3), \ldots, \zeta(\mathcal{L}\omega)$ are deductive.

Proof. Verify that each of Łukasiewicz systems satisfies conditions (D1) – (D3) of Theorem 11.5. ■

Then, what are the logics $\overrightarrow{\zeta(\mathcal{L}_3)}, \ldots, \overrightarrow{\zeta(\mathcal{L}_\omega)}$? They certainly are not truth-preserving logics $\text{Ł}_3, \ldots, \text{Ł}_\omega$, we have consider thus far. We leave to the reader an easy task to verify that the following holds true.

13.4. Theorem. For each $\eta = 3, 4, \ldots, \omega$, each X and each α, the following two conditions are equivalent

(i) $\alpha \in \overrightarrow{\zeta(\mathcal{L}_\eta)}(X)$

(ii) For each valuation v in η-valued Łukasiewicz matrix

$$v(\alpha) \geqslant \inf(v(X))$$

(we define $\inf \emptyset = 1$). ■

Then, it turns out that $\overrightarrow{\zeta(\mathcal{L}_\eta)}$ preserve the degree of truth rather than truth; when applied to sentences whose truth-values are all greater than x, they yield conclusions whose truth-values are again greater than x.

From 13.4 we immediately have

13.5. Corollary. For each $\eta = 3, 4, \ldots$,

a. $\overrightarrow{\zeta(\mathcal{L}_\eta)}(\emptyset) = \text{Ł}_\eta(\emptyset)$

b. $\overrightarrow{\zeta(\mathcal{L}_\eta)} < \text{Ł}_\eta$.

13.6. Are $\overrightarrow{\zeta(\mathcal{L}_\eta)}$ Łukasiewicz logics? They surely deserve to be called that way. Thus there are at least two kinds of Łukasiewicz logics: *truth-preserving Łukasiewicz logics* $\text{Ł}_3, \ldots, \text{Ł}_\omega$ and *well[a] determined Łukasiewicz logics* $\overrightarrow{\zeta(\mathcal{L}_3)}, \ldots, \overrightarrow{\zeta(\mathcal{L}_\omega)}$. If not supplement with a suitable epithet, Łukasiewicz logic is an ambiguous term.

14. Well determined modal logics

14.1. By the *modal language* we shall mean the language $(\mathcal{L}, \Box)$, i.e. the standard language $\mathcal{L}$ extended by the familiar connective $\Box$ ('it is necessary that'). Instead of $(\mathcal{L}, \Box)$ we shall write $\mathcal{L}_\Box$. The formulas of $\mathcal{L}_\Box$. will be referred to as *modal formulas*, and their set will be denoted by $L_\Box$.

By a *modal logic* we shall mean any logic C defined in $\mathcal{L}_\Box$. If, moreover, all inferences of the classical logic K are valid in C, the logic C will be referred to as a *modal logic based on* K.

The following remark is in order here. It would not be true to say that if C is a modal logic based on K then $K \leqslant C$, for K and C are defined in different languages. But for any two languages $\mathcal{S}_1$, $\mathcal{S}_2$; $\mathcal{S}_1$ being a sublanguage of $\mathcal{S}_2$, and for any logic C_1 in $\mathcal{S}_1$ there exists a logic C_2 in $\mathcal{S}_2$ being a natural counterpart of C_1 in $\mathcal{S}_2$. This is the weakest logic in $\mathcal{S}_2$ in which all inferences C_1 are valid. In order words, C_2 results from C_1 by closing the inferences of C_1 under all substitutions in $\mathcal{S}_2$. The logic C_2 of the kind defined will be referred to as the *natural extension* (rather than natural counterpart) of C_1 onto $\mathcal{S}_2$.

Now if K_0 is such an extension of K onto $\mathcal{L}_\Box$ then, obviously, we have $K_0 \leqslant C$.

14.2. A set of modal formulas M will be said to be a *modal system based on* K, a *modal system* for short, iff

(M1) $K(\emptyset) \subseteq M$,

(M2) M is closed under substitution and Modus Ponens.

With the help of 11.5 one easily verifies that:

14.3. Theorem. All modal systems based on K are deductive, and for each such system M, $\overrightarrow{M}$ is a modal logic based on K. ■

Thus, for each modal system M, there exists a well[a]determined logic $\overrightarrow{M}$. Conversely, for each well-determined modal logic C based on K, $C(\emptyset)$ is a modal system. Hence, the correspondence between well-determined modal logics and modal systems (both being based on K) is one-to-one. Below, some examples of modal systems are given.

14.4. Example. A modal system M is said to be *classical* (cf. K. Segerberg [1971]) iff it is closed under the

$$\textit{Replacement Rule } (\mathrm{RR})\text{: } p \longleftrightarrow q/\Box p \longleftrightarrow \Box q.$$

The smallest classical modal system is usually denoted as E. Under the convention we adopted, cf. 9.2, it will be denoted by M_E.

14.5. Example. If M is classical, closed under the rule

$$p \to q/\Box p \to \Box q.$$

and moreover involves all formulas of the form

$$(\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta),$$

then the system M is called *regular*.

The least regular system is denoted by C — we shall denote it M_C.

14.6. Example. If M is regular and closed under

$$\textit{Necessitation Rule } (\mathrm{NR})\text{: } p/\Box p$$

it is called *normal*. The least normal system is known as *Kripke's logic*. Usually it is denoted by K, we shall denote it M_K.

17.7. Example. The smallest normal system that involves all formulas of the form

$$\Box\alpha \to \alpha$$

is known as *Feys-von Wrigh's system* T, thus M_T in our notation.

14.8. Example. The system M_T enlarged by all formulas of the form

$$\Box\alpha \to \Box\Box\alpha$$

is *Lewis'* system $S4$, thus M_{S4} in our notation.

14.9. Example. Again M_T, this time enlarged by all formulas of the form

$$\Diamond\Box\alpha \to \Box\alpha$$

where $\Diamond$ is defined by

$$\Diamond\alpha =_{\mathrm{df}} \neg\Box\neg\alpha,$$

is called *Brouwer's system* B, in our notation M_B.

14.10. Example. *Lewis' system* $S5$, in what follows denoted by M_{S5}, is the smallest modal system that contains both B and $S4$.

14.11. Example. A modal system based on K that contains all formulas of the form

$$\alpha \longleftrightarrow \Box\alpha$$

is called a *trivial modal system.* We shall denote it by M_{TR}. The connective $\Box$ in M_{TR} can be viewed as the classical two valued assertion connective defined by the truth-table

α	$\Box\alpha$
1	1
0	0

15. Surprisingly enough, relevant logics are not well-determined

15.1. This is somewhat surprising indeed, because under the intended interpretation the relevant implication, $\to$ means "entails". That is why Anderson–Belnap's E is called Entailment. Now the logics such as C_E, C_R, C_{RM} (cf. 9.4) are not well determined because the corresponding systems E, R, RM are not closed under CV (cf. 11.5). Indeed, both $p \to p$, and $((p \to p) \wedge q) \to (p \to p)$ are valid in these systems, but not $q \to (p \to p)$; the latter formula being derivable by CV from the former ones.

The paradox has a simple solution. As we have already noticed, cf. Section 9, consequences C_E, C_R, C_{RM}, etc. called pseudo-relevant. We shall not risk, however, modifying the terminology that, though wrong, is well established.

15.2. Given any relevant system S, the logic C properly corresponding to S should be defined to be the standard logic such that

(i) $C(\emptyset) = \emptyset$

(ii) $\alpha \in C(\beta_1, \ldots, \beta_n)$ iff $(\beta_1 \wedge \ldots \wedge \beta_n) \to \alpha \in S$.

We have already suggested to denote this logic by $\overrightarrow{S}$.

What is strange of the logics of that kind is that they are purely inferential: in view of (i) $\overrightarrow{S}(\emptyset) = \emptyset$. But is it strange indeed? An attempt to answer this question would lead us too far from what should be of our prime interest.

Chapter 4

Truth-valuations

16. Theories vs truth-valuations

16.1. By a binary *truth-valuations* for a language $\mathcal{S}$ we shall mean a mapping v: $S \to \{0, 1, \}$, with 0 and 1 interpreted as *falsity* and *truth* respectively. If the function V is defined only for a subset $X \subseteq S$, it will be referred to as a *partial-truth-valuation*.

Given an inference $X \vdash \alpha$ and valuation (possibly partial) V we shall say that V *satisfies* (or *verifies*) $X \vdash \alpha$, if either $v(\alpha) = 1$ or $v(\beta) = 0$, for some $\beta \in X$. If $v(\alpha) = 0$, and $v(\beta) = 1$ for all $\beta \in X$ we shall say that v *falsifies* $X \vdash \alpha$. The valuation v will be said to *satisfy* (*verify*) X, if $v(\alpha) = 1$, for all $\alpha \in X$, and it will be said to *falsify* X when $v(\alpha) = 0$, for some $\alpha \in X$. Accordingly, v *verifies* (*falsifies*) α if $v(\alpha) = 1$ ($v(\alpha) = 0$).

As we have already noticed, cf 2.10, the following is easily seen to hold true:

16.2. Theorem. For each set of truth valuations H for $\mathcal{S}$, the operation Cn_H on subsets of S defined by

$$\alpha \in Cn_H(X) \text{ iff all } v \in H \text{ satisfy } X \vdash \alpha,$$

is a consequence operation. ■

Cn_H is the consequence preserving truth under H. Occasionally it will be also referred to as the *consequence determined by* H.

16.3. As it is common, we shall apply the word "semantics" in a twofold manner.

Given a class σ we shall call it a *semantics in the generic sense* or σ-*semantics* iff to each $\sigma' \subseteq \sigma$ there has been assigned a consequence operation $Cn_{\sigma'}$. (Thus, rigorously speaking, σ-semantics is the couple (σ, Cn), where Cn is the operation that assigns to each $\sigma' \subseteq \sigma$ the consequence $Cn_{\sigma'}$).

On the other hand, given any σ-semantics, by a *semantics in the particular sense* (of the kind σ) we shall mean any subclass $\sigma' \subseteq \sigma$.

Correspondingly, there are two notion of (strong) completeness as well as two notions of weak completeness, both being applied in logical investigation

a. A consequence C is *complete* (*weakly complete*) with respect to a σ-semantics iff for some $\sigma' \subseteq \sigma$, $C = Cn_{\sigma'}$ ($C(\emptyset) = Cn_{\sigma'}(\emptyset)$, resp.)

b. Let σ be a semantics in the general sense. A consequence C is *complete* (*weakly complete*) with respect to a particular semantics $\sigma' \subseteq \sigma$ iff $Cn_{\sigma'} \leqslant C$ ($Cn_{\sigma'}(\emptyset) \subseteq c(\emptyset)$ resp.)

Recall that C is said to be *sound* (*weakly sound*) with respect to $\sigma' \subseteq \sigma$ if $C \leqslant Cn_{\sigma'}$ ($C(\emptyset) \subseteq Cn_{\sigma'}(\emptyset)$).

If C is both sound and complete (weakly sound and weakly complete) with the respect to a particular semantics σ', we shall say that σ' is *adequate* (*weakly adequate*) with the respect to C.

We trust the reader not to confuse the two notions of completeness. Still, when dealing with a semantics in the particular sense, we shall try to avoid the term completeness and, if possible, carry out our analyzes in terms of adequacy.

16.4. Theorem. Let $\mathbb{X}$ be a family of sets of formulas, and let $\mathbb{H}$ be a set of truth-valuations. For each set of formulas X define $\mathcal{X}_X$ to be the characteristic function of X (i.e. $\mathcal{X}_X(\alpha) = 1$ iff $\alpha \in X$), and for each valuation v define X_v to be the set of which v is the characteristic function. Then

a. $\mathbb{X}$ is a closure base for a consequence C iff
$\{\mathcal{X}_X : X \in \mathbb{X}\}$ is an adequate semantics for C.

b. $\mathbb{H}$ is an adequate semantics for a consequence C iff $\{X_v : v \in \mathbb{H}\}$ is a closure base for C.

Proof. Of course a. and b. are equivalent and hence it suffices to prove any of these conditions, say a. Let $\mathbb{X}$ be a closure base for C. Suppose that $\alpha \in C(X)$. If for some $Y \in \mathbb{X}$, $\mathcal{X}_Y$ verifies all $\beta \in X$ then $X \subseteq Y$ and hence, $C(X) \subseteq C(Y) = Y$. This yields $\mathcal{X}_Y(\alpha) = 1$. Now, suppose that $\alpha \notin C(X)$. Then for some $Y \in \mathbb{X}$, $X \subseteq Y$ and $\alpha \notin Y$, which yields $\mathcal{X}_Y(X) \subseteq q$ and $\mathcal{X}_Y(\alpha) = 0$. Thus, indeed $\{\mathcal{X}_X : X \in \mathbb{X}\}$ is adequate for C. In order to get the "only if" part of a. just reverse the argument. ■

Since for each C at least Th_C is a closure base for C, we have

16.5. Corollary. For each consequence C there exists a truth-valuational semantics $\mathbb{H}$ adequate for C, i.e. each consequence operation is complete with the respect to the truth-valuational semantics.

With the help of Suszko's lemma 3.4 one can easily define sufficient and necessary conditions for $\mathbb{H}$ to determine a logic; of course, in general,

$Cn_{\mathbb{H}}$ need not be structural. As a matter of fact we shall need only the following

16.6. Lemma. Let $\mathbb{H}$ be an adequate semantics for C. If for each valuation $v \in \mathbb{H}$ and for each substitution e, the valuation v_e defined by

$$v_e(\alpha) = 1 \text{ iff } v(e\alpha) = 1.$$

is in $\mathbb{H}$, the consequence $Cn_{\mathbb{H}}$ determined by $\mathbb{H}$ is structural.

Proof. Apply Lemma 3.4. ∎

In what follows, if $Cn_{\mathbb{H}}$ is structural, the semantics $\mathbb{H}$ will be referred to as a *logical space of valuations*.

17. Epistemic valuations for $\mathcal{L}$

17.1. Let T a non-empty set and $\leqslant$ a partial ordering on T. The couples of the form $(T, \leqslant)$ are called *posets*. Here we shall refer to them as *epistemic frames*.

The elements of T will be referred to as *stages of investigation* (or just *points*), and $\leqslant$ as a *temporal succession*. For a philosophical interpretation of epistemic frames that motivates the terminology we adopted here, cf. A. Grzegorczyk [1964], [1968].

17.2. Let $(T, \leqslant)$ be an epistemic frame. A partial function $\epsilon : T \times L \to \{0, 1, \}$ will be said to be an *epistemic valuation for the standard language $\mathcal{L}$, defined relatively to* $(Y, \leqslant)$, iff for all α, β, and all $t \in T$, and all $t \in T$, it satisfies the following conditions:

(1) $\epsilon(t, \alpha) = 1$ implies $\epsilon(t', \alpha) = 1$, for all $t' \geqslant t$

($\wedge$) $\epsilon(t, \alpha \wedge \beta) = 1$ iff $\epsilon(t, \alpha) = \epsilon(t, be) = 1$

($\vee$) $\epsilon(t, \alpha \vee \beta) = 1$ iff either $\epsilon(t, \alpha) = 1$, or $\epsilon(t, \beta) = 1$, or $\epsilon(t, \alpha) = \epsilon(t, \beta) = 1$

($\to$) $\epsilon(t, \alpha \to \beta) = 1$ iff for all $t' \geqslant t$, $\epsilon(t', \beta) = 1$ whenever $\epsilon(t', \alpha) = 1$

($\neg$) $\epsilon(t, \neg\alpha) = 1$ iff for no $t' \geqslant t$, $\epsilon(t', \alpha) = 1$

(t' is assumed to run over T, $t' \geqslant t$ is an alternative notation for $t \leqslant t'$).

By producing a suitable example (e.g. take T to be a 1-element set, and define ϵ to satisfy the classical truth=functional conditions) one can show that the conditions of Definition 17.2 are consistent.

Apply an inductive argument in order to prove:

17.3. Lemma. Let $(T, \leqslant)$ be an epistemic frame. let ϵ_0 be a partial function from $T \times Var(L)$ into $\{0, 1\}$ such that for all $t, t' \in T$ and all $p \in Vat(L)$ if $t \leqslant t'$ then $\epsilon_0(t, p) = 1$ implies $\epsilon_0(t', p) = 1$.

a. For each such function ϵ_0 there exists an epistemic valuation ϵ for $\mathcal{L}$ defined relatively to $(T, \leqslant)$ such that $\epsilon \restriction Var(L) = \epsilon_0$.

b. If $\epsilon_1 \restriction Var(L) = \epsilon_2 \restriction Var(L)$, both ϵ_1 and ϵ_2 being epistemic valuations for $\mathcal{L}$ defined relatively to $(T, \leqslant)$, then for each α, and each $t \in T$, $\epsilon_1(t, \alpha) = 1$ iff $\epsilon_2(t, \alpha) = 1$. ■

Let $\mathbb{F}$ be a set of epistemic frames. Denote by $Cn_{\mathbb{F},\mathcal{L}}$ (or just by $Cn_{\mathbb{F}}$) the consequence operation that preserves truth under all epistemic valuations defined relatively to the frames in $\mathbb{F}$. More explicitly, though in a somewhat roundabout way, $Cn_{\mathbb{F},\mathcal{L}}$ can be defined as follows.

Denote by $H(\mathbb{F})$ the set of all truth-valuations of the form ϵ_t such that ϵ is an epistemic valuation with respect to a frame $(T, \leqslant) \in \mathbb{F}$, $t \in T$ and $\epsilon_t(\alpha) = \epsilon(t, \alpha)$, for each α. Then $Cn_{\mathbb{F},\mathcal{L}}$ is just the consequence in $\mathcal{L}$ preserving truth under $H(\mathbb{F})$, the valuations in $H(\mathbb{F})$ being treated as the admissible valuation for $\mathcal{L}$, cf 2.10.

Cf. 24.5 for the proof of the following

17.4. Adequacy theorem (S. Kripke [1965], A. Grzegorczyk [1964]). The class EFrame of all epistemic frames provides an adequate semantics for the intuitionistic propositional logic J (i.e. J is both sound and complete with the respect to EFrame)

Verify that:

17.5. Lemma. For each class $\mathbb{F}$ of epistemic frames $Cn_{\mathbb{F},\mathcal{L}}$ is a structural consequence.

Proof. Define $H(\mathbb{F})$ as in 17.3 and apply Lemma 16.6 to show that it is a logical space ■

From 17.4 and 17.5 is follows immediately that

17.6. Corollary. For each class $\mathbb{F}$ of epistemic frames $Cn_{\mathbb{F},}$ is a logic stronger than J. ■

18. Epistemic valuation for $\mathcal{L}_\sim$

18.1. Extend the language $\mathcal{L}$ by adding a new unary connective $\sim$, called *strong negation*. Denote the resulting language by $\mathcal{L}_\sim$ and by $L_\sim$ the set of formulas of that language.

18.2. Let $(T, \leqslant)$ be an epistemic frame. A partial function $\epsilon : T \times L_\sim \to \{0, 1\}$ will be referred to as an *epistemic valuation for $\mathcal{L}_\sim$, defined relatively to* $(T, \leqslant)$ iff for all α, β, all $t \in T$, it satisfies the following conditions

(0) $\epsilon(t, \alpha) = 0$ implies $\epsilon(t', \alpha) = 0$ for all $t' \geqslant t$,

(1) $\epsilon(t, \alpha) = 1$ implies $\epsilon(t', \alpha) = 1$ for all $t' \geqslant t$,

$(\wedge)_+$ $\epsilon(t, \alpha \wedge \beta) = 1$ iff $\epsilon(t, \alpha) = \epsilon(t, be) = 1$,

$(\wedge)_-$ $\epsilon(t, \alpha\wedge\beta) = 0$ iff either $\epsilon(t, \alpha) = 0$, or $\epsilon(t, be) = 0$ or $\epsilon(t, \alpha) = \epsilon(t, \beta) = 0$,

$(\vee)_+$ $\epsilon(t, \alpha\vee\beta) = 1$ iff either $\epsilon(t, \alpha) = 1$, or $\epsilon(t, \beta) = 1$, or $\epsilon(t, \alpha) = \epsilon(t, \beta) = 1$,

$(\vee)_-$ $\epsilon(t, \alpha \wedge \beta) = 0$ iff $\epsilon(t, \alpha) = \epsilon(t, be) = 0$,

$(\rightarrow)_+$ $\epsilon(t, \alpha \rightarrow \beta) = 1$ iff for all $t' \geqslant t$, $\epsilon(t', \beta) = 1$ whenever $\epsilon(t', \alpha) = 1$

$(\rightarrow)_-$ $\epsilon(t, \alpha \rightarrow \beta) = 0$ iff $\epsilon(t, \alpha) = 1$ and $\epsilon(t, \beta) = 0$,

$(\neg)_+$ $\epsilon(t, \neg\alpha) = 1$ iff for no $t' \geqslant t$, $\epsilon(t', \alpha) = 1$

$(\neg)_-$ $\epsilon(t, \neg\alpha) = 0$ iff $\epsilon(t, \alpha) = 1$

$(\sim)_+$ $\epsilon(t, \sim\alpha) = 1$ iff $\epsilon(t, \alpha) = 0$

$(\sim)_-$ $\epsilon(t, \sim\alpha) = 0$ iff $\epsilon(t, \alpha) = 1$

Again (cf. 17.2) one has to show that the conditions of 18.2 are consistent and, again, this can be easily done, incidentally by the same example as suggested in 17.2, with $\sim$ being interpreted in the same way as $\neg$.

Apply an inductive argument in order to show that

18.3. LEMMA. Let $(T, \leqslant)$ be an epistemic frame, and let ϵ_0 be a partial function from $T \times Var(L_\sim)$ such that for all $t, t' \in T$ and all $p \in Var(L_\sim)$, if $t \leqslant t'$ then both $\epsilon_0(t, p) = 1$ implies $\epsilon_0(t', p) = 1$ and $\epsilon_0(t, p) = 0$ implies $\epsilon_0(t', p) = 1$ and $\epsilon_0(t, p) = 0$ implies $\epsilon_0(t', p) = 0$.

a. For each such function ϵ_0 there exists an epistemic valuation ϵ for $\mathcal{L}$ defined relatively to $(T, \leqslant)$ such that $\epsilon \upharpoonright Var(L_\sim) = \epsilon_0$.

b. If $\epsilon_1 \upharpoonright Var(L_\sim) = \epsilon_2 \upharpoonright Var(L_\sim)$, both ϵ_1 and ϵ_2 being epistemic valuations for $\mathcal{L}_\sim$ defined relatively to $(T, \leqslant)$, then for each α, and each $t \in T$, $\epsilon_1(t, \alpha) = 1$ iff $\epsilon_2(t, \alpha) = 1$. ■

Given any class $\mathbb{F}$ of epistemic frames, denote by $Cn_{\mathbb{F},\mathcal{L}_\sim}$ (or just by $Cn_{\mathbb{F}}$) the consequence operation that preserves truth under all epistemic valuations for $\mathcal{L}_\sim$ defined relatively to the frames in $\mathbb{F}$ (i.e. under all truth-valuations of the form ϵ_t, ϵ being an epistemic valuation and t a reference point of the frame in $\mathbb{F}$ with respect to which ϵ is defined, cf. 17.3).

Verify that

18.4. LEMMA. For each class $\mathbb{F}$ of epistemic frames $Cn_{\mathbb{F},\mathcal{L}_\sim}$ is a structural consequence.

PROOF. Defined $H(\mathbb{F})$ as in 17.3 and apply Lemma 16.6 to show that it is a logical space. ■

The weakest of all logics of the form $Cn_{\mathbb{F},}$; i.e. the logic determined by all epistemic frames, to be denoted by N. As we shall prove in Section 24, N is the propositional part of the logic known as the *logic with constructive falsity*, or *D. Nelson's logic* (cf. D. Nelson [1949]). Independently of D. Nelson similar ideas were put forward by A. A. Markov [1950].

19. Neighborhood valuations

19.1. Let T be a non-empty set and let N be a function from T into the power set of the power set of T, i.e. for each $T \in T$, $N(t)$ is a family of subsets of T.

The pair (T, N) will be referred to as a *neighborhood* frame. The elements of T will be referred to as *possible worlds* or *points*.

A class $\mathbb{F}$ on neighborhood frames will be called a *neighborhood semantics*.

Before we define the notion of a neighborhood valuation (valuations of this kind provide interpretations for the modal language $\mathcal{L}_\Box$), let us define explicitly the notion of classically admissible truth valuation.

19.2. Let $\mathcal{L}$ be any language that involves standard connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$. A truth valuation v for $\mathcal{S}$ will be said to be *classically admissible* iff v satisfies the familiar conditions for the connectives of K:

($\wedge$) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = v(\beta) = 1$,
($\vee$) $v(\alpha \vee \beta) = 1$ iff either $v(\alpha) = 1$, or $v(\beta) = 1$,
($\rightarrow$) $v(\alpha \rightarrow \beta) = 1$ iff either $v(\alpha) = 0$, or $v(\beta) = 1$,
($\neg$) $v(\neg\alpha) = 1$ iff $v(\alpha) = 0$.

The logic determined in $\mathcal{L}$ by the set of classically admissible truth valuation is, of course, K.

19.3. Let (T, N) be a neighborhood frame. A function $\eta : T \times L_\Box \rightarrow \{0, 1\}$ will be said to be a *neighborhood valuation* defined relatively to the frame (T, N) iff for all α and all $t \in T$,

(i) the truth valuation η_t defined by $\eta_t(\alpha) = \eta(t, \alpha)$ is classically admissible
(ii) $\eta(t, \Box\alpha) = 1$ iff $\{t' \in T : \eta(t', \alpha) = 1\} \in N(t)$.

Verify that

19.4. Lemma. Let (T, N) be a neighborhood frame, and let $\eta_0 : T \times Var(L_\Box) \rightarrow \{0, 1, \}$. For each such function η_0 there exists exactly one neighborhood valuation η defined relatively to (T, N) such that $\eta \restriction Var(L_\Box) = \eta_0$. ■

Given any neighborhood semantics $\mathbb{F}$ we define both $H(\mathbb{F})$ and $Cn_\mathbb{F}$ in the familiar way. $H(\mathbb{F})$ denotes the set of all truth-valuations of the form η_t, where η is a truth-valuation defined relatively to a frame $(T, N) \in \mathbb{F}$, $t \in T$, and $Cn_\mathbb{F}$, referred to as the *consequence preserving truth under neighborhood valuations* defined with respect to frames in $\mathbb{F}$, is a shorthand for $Cn_{H(\mathbb{F})}$. Informally, $H(\mathbb{F})$ can be viewed as the set of admissible valuations for $\mathcal{L}_\Box$ under the interpretation of that language provided by frames in $\mathbb{F}$.

Denote by NFrame the class of all neighborhood frames.

19.5. THEOREM (Segerberg [1971]). The class NFrame of all neighborhood frames is an adequate semantics for M_E. The way in which the theorem can be proved is discussed in Section 27 (cf. 27.3). Apply 17.5 to verify that

19.6. LEMMA. For each class $\mathbb{F}$ of neighborhood frames the consequence $Cn_{\mathbb{F}}$ (defined in $\mathcal{L}_\Box$ is structural. ■

19.7. COROLLARY. For each class $\mathbb{F}$ of neighborhood frames the consequence $Cn_{\mathbb{F}}$ defined in $\mathcal{L}_\Box$ is a logic stronger than M_E.

20. Relational valuations

20.1. Let T be a non-empty set, $U \subseteq T$, and let R be a binary relation defined on T. The triple (T, R, U) is called a *relational frame*. As in the case of neighborhood frames, the elements of T are referred to as possible worlds or *points*. The relation R is often called an *accessibility* relation; if $t_1 R t_2$ the world t_2 is said to be *accessible* from t_1. The set U is referred to as the set of *non-normal worlds*. All elements of T/U are called *normal*. If $U = \emptyset$, the frame is called *normal*. Instead of $(T, R, \emptyset)$ we shall write just (T, R).

The relation frames as an instrument for semantic analyses of modal logics were defined by S. A. Kripke [1959], [1959a], [1963].

20.2. Let (T, R, U) be a relational frame. A function $\varrho : T \times L_\Box \to \{0, 1\}$ will be said to be a *relational valuation for $\mathcal{L}_\Box$ defined relatively to the frame* (T, R) iff for all α and all $t \in T$,

(i) the truth valuation ϱ_t defined by $\varrho_t(\alpha) = \varrho(t, \alpha)$ is classically admissible

(ii) $\varrho(t, \Box\alpha) = 1$ iff for all normal t' such that tRt', $\varrho(t', \alpha) = 1$.

20.3. Again, given any relational frame semantics we define $Cn_{\mathbb{F}}$ to be the truth preserving consequence with respect to the relational valuations defined relatively to frames in $\mathbb{F}$.

Let RFrame be the class of all normal relational frames.

20.4. THEOREM (S. Kripke [1963]). The class RFrame of all normal relational frames is an adequate semantics for M_K.

For some comments on how to prove the theorem, see Section 27 (in particular 27.3).

Given any two semantics (they need not be of the same kind) call them *equivalent* iff they define exactly the same consequence operation.

20.5. *Theorem.* For each normal relational frame $\mathcal{F}^R$ there exists a neighborhood frame $\mathcal{F}^N$, such that the two frames are equivalent.

PROOF. Given a normal relational frame (T, R) define the neighborhood function N_R of T by

$$N_R(t) = \{T' \subseteq T : \{t' : tRt'\} \subseteq T'\}$$

and verify that (T, R) and (T, N_R) are equivalent. ■

From 20.5 and 19.2 it follows that

20.6. LEMMA. For each class $\mathbb{F}$ of relational frames, $Cn_{\mathbb{F}}$ is structural. ■

Of course, one can easily prove 20.6 directly.

20.7. COROLLARY. For each class $\mathbb{F}$ of normal relational frames the consequence $Cn_{\mathbb{F}}$ defined in $\mathcal{L}_{\Box}$ is a logic stronger than M_K.

Chapter 5

Henkin's Style Completeness Proofs — the Scope of the Method

21. Completeness lemma

21.1. Given a consequence operation C and truth-valuational semantics H, it is usually an easy thing to decide whether C is sound with respect to H, $C \leqslant Cn_H$, or not. As a rule, the proof of completeness or, perhaps, incompleteness of C with respect to H, is much more involved.

Now, suppose than C has the least closure base $\mathbb{X}$. Then there exists a least truth-valuational semantics adequate for C. It is (cf. 16.4)

$$H(\mathbb{X}) = \{\mathcal{X}_X : X \in \mathbb{H}\}, \tag{1}$$

$\mathcal{X}_X$ being the characteristic function of X. Since for any truth-valuational semantics H_1, H_2, $H_1 \subseteq H_2$ implies that $Cn_{H_2} \leqslant Cn_{H_1}$, if $H(\mathbb{X}) \subseteq H$ then $Cn_H \leqslant Cn_{H(\mathbb{X})} = C$, and thus C is complete with respect to H. We have proved that

21.2. Completeness Lemma. Let C be a consequence operation and let $\mathbb{X}$ be a least closure base for C. Then for each truth-valuational semantics H, C is complete with respect to H iff $H(\mathbb{X}) \subseteq H$. ■

What is usually referred to as Henkin's style completeness proofs are proofs based on Lemma 21.2.

21.3. It is obvious that in order for $\mathbb{X}$ to be the least closure base for C, $\mathbb{X}$ must involve all theories X of C such that no $\mathbb{Y} \subseteq Th_C/X$, $X = \bigcap \mathbb{Y}$, and, of course, it must involve only such theories. (If S is the set of all formulas, we put $\bigcap \emptyset = S$)

The theories of the kind we have just defined are called *irreducible*. Now, a theory $X \in Th_C$ is said to be *maximal relatively to* α iff $\alpha \notin X$ and for each β, either $C(X, \beta) = C(X)$ or $\alpha \in C(X, \beta)$. We shall say that $X \in Th_C$ is a *relatively maximal* theory of C iff it is a maximal relatively to a formula α.

The set of all relatively maximal theories of C is usually denoted by $\mathbb{X}_C$.

Observe that a straightforward argument yields

21.4. Lemma. For each $X \in Th_C$, X is relatively maximal iff X is irreducible and non-trivial.

Hence, we have

21.5. Theorem. Let $\mathbb{X}$ be a closure base for C. Then the following conditions are equivalent:

(i) $\mathbb{X}$ is the least closure base for C,

(ii) $\mathbb{X}$ is the set of all irreducible and non-trivial theories of C,

(iii) $\mathbb{X}$ is the set of all relatively maximal theories of C. ■

One may easily verify that the theorem can be supplemented by the following additional condition,

(iv) $\mathbb{X}$ is a minimal closure base for C.

What are the conditions under which C has the least closure base?

22. Consequence with Lindenbaum property

22.1. A consequence C is said to have *Lindenbaum property*, or just to be a *Lindenbaum consequence*, iff for each $X \subseteq Th_C$ and for each $\alpha \notin X$, there is $Y \in Th_C$ maximal relatively to α such that $X \subseteq Y$.

22.2. Theorem. Assume that C is a consequence or assume that C is a structural consequence. In both cases the following conditions:

(i) C is finitary.

(ii) The set of all maximal theories of C is a closure base for C.

(iii) C has the Lindenbaum property.

(iv) There exists a least closure base for C.

are related exactly in the way defined by the diagram below, i.e. no arrow (read 'implies') can be added.

$$\begin{tikzcd} \text{(i)} \arrow[dr] & & \\ & \text{(iii)} \arrow[r] & \text{(iv)} \\ \text{(ii)} \arrow[ur] & & \end{tikzcd}$$

Proof. In order to prove that (i) implies (iii) assume that $\alpha \in X \in Th_C$ and proceed further as follows.

Step 1. Arrange all formulas of the language of C into a sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots$$

Step 2. Define recursively X_i as follows. Put $X_0 = X$, and for each $i \geqslant 1$ put

$$X_i = \begin{cases} X_{i-1}, & \text{if } \alpha \in \ C(X_{i-1},\ \alpha_i) \\ C(X_{i-1},\ \alpha_i), & \text{otherwise.} \end{cases}$$

Step 3. From the union $\bigcup X_i$, and apply the assumption that C is finitary in order to show that $C(\bigcup X_i) = \bigcup X_i$, i.e. the union $\bigcup X_i$ is a theory of C. If so, then it is a maximal theory relatively to α and the proof of (i)→(iii) is concluded.

The implication from (i) to (iii) is often referred to as Lindenbaum Lemma (or Theorem). In fact the theorem in the version stated by A. Lindenbaum (cf. Tarski [1930]) differs from the one mentioned above in two respects. First, it applies only to the classical logic K; second, it concerns extensions to maximal not relatively maximal systems. But already J. Łoś [1953] (cf. also [1951]) noticed that Lindenbaum theorem can be extended onto all finitary consequences. The second of the two differences we have mentioned, between the original and the new version of the theorem is, at least in the case of K, inessential (cf. Suszko [1961]).

The proof given above is a version of the one given by J. Łoś [1953]. In A. Tarski [1930] the theorem is published without proof; A. Lindenbaum did not publish his result.

Perhaps it is worth-while to notice that the Lemma holds true even if the language of C is not denumerable (apply Zorn's lemma to prove it in this more general case).

Let us turn back to the proof of our theorem, though, as far as the positive part of the theorem is concerned not much can be added. The implications (ii)→(iii), and (iii)→(iv) are straightforward.

In order to conclude the proof we have to show that no arrow can be added to the diagram of the theorem. This is a more involved part of the proof.

(i)$\not\rightarrow$(ii). The intuitionistic logic J may serve here as a suitable example. As known, the maximal theories of J are the same as those of K, and the closure base they form is a closure base for K, not for J.

(ii)$\not\rightarrow$(i). In this case $Ł_\omega$ is a good example. Let v be a valuation in a ω-valued Łukasiewicz matrix. Then $X_v = \{\alpha : v(\alpha) = 1\}$ is, as one may prove, a maximal theory. In order to see this apply McNaughton

[1951] results to show that for each $x \in \mathcal{L}_\omega$ there exists a formula $\alpha(p)$ in one variable p such that for all $y \in \mathcal{L}$, $\alpha(p) = 1$ iff $y = x$ (pedantically: $v(\alpha(p)) = 1$ iff $v(p) = x$, for all valuations in $\mathcal{L}_\omega$). For instance: $p \longleftrightarrow \neg p$ is such a formula for $x = \frac{1}{2}$, $\neg p \longleftrightarrow (\neg p \to p)$ is such a formula for $x = \frac{1}{3}$, $p \longleftrightarrow (p \to \neg p)$ is such a formula for $x = \frac{2}{3}$, etc. Select any formula $\beta(p_1, \ldots, p_n)$ is the variables $p_1, \ldots, p_n$ that is not in X_v. Let $v(p_i) = x_i$ and let $\alpha_i(p_i)$ be a formula such that $\alpha_i(x) = 1$ iff $x = x_i$. All $\alpha_i(p_i)$ are in X_v, and of course $Ł_\omega(\beta, \alpha_1, \ldots, \alpha_i) = L$, Hence X_v is maximal. Since the set of all valuations forms an adequate semantics for $Ł_\omega$, the sets of the kind X_v form a closure base for $Ł_\omega$. In order to show this apply 16.4 making use of the fact that to each Łukasiewicz valuation v there corresponds in a one-to-one manner the truth-valuation v^+ defined by $v^+(\alpha) = 1$ iff $v(\alpha) = 1$, and the set of all v^+ forms a semantics equivalent to the original one.

The proof that $Ł_\omega$ is not finitary, was given in R. Wójcicki [1976].

(iii)$\not\to$(i). Once more $Ł_\omega$ will do.

(iii)$\not\to$(ii). The logic J is again useful as a suitable example.

(iv)$\not\to$(iii). At the first glance it seems that if the set of all relatively maximal theories is a closure base for C, and this is exactly the case when C has the least closure base, then C must have Lindenbaum property. It need not be so, because if $\alpha \notin C(X)$ and $C(X) = \bigcap \mathbb{X}$, $\mathbb{X}$ being a family of relatively maximal theories of C, still it may happen that no $Y \in \mathbb{X}$ is maximal relatively to α; all $Y \in \mathbb{X}$ may happen to be maximal with the respect to some formulas different from α.

The situation we have described is a very special one and it seems unlikely that any of the known and studied logics is of the kind for which we are looking. Thus we must resort to an artificial example, i.e. invented only for the purpose of the proof.

The language of the logic we are going to define will involve only two connectives, both of them unary. Let they be $\Diamond$ and $\Box$. By $\Diamond^m$ we shall denote m repetitions of $\Diamond$. The similar convention applies to $\Box$. Furthermore, the following abbreviations will be useful

$$\Diamond_{m,n}\alpha =_{\mathrm{df}} \Diamond^m \Box^n \Diamond\alpha, \quad m, n \geqslant 1$$

$$\Box_{m,n}\alpha =_{\mathrm{df}} \Box^m \Diamond^n \Box\alpha, \quad m, n \geqslant 1$$

Let us denote the logic to be defined by the set of all sequential rules of the following form:

$$\Box_{m,n}\alpha / \Box_{m,n}\alpha' \text{ -- for all } \alpha, \alpha' \text{ such that } Var(\alpha = Var(\alpha')$$

$$\Diamond_{m,n}\alpha / \Diamond_{m,n}\alpha' \text{ -- for all } \alpha, \alpha' \text{ such that } Var(\alpha = Var(\alpha')$$

$$\{\Diamond_{m,n}p / m \geqslant 1\} / \{\Diamond_{m',n}p\} / \Diamond_{m'n}p \text{ -- for all } n \geqslant 1,\ m' \geqslant 1$$

$$\{\Diamond_{m,n}\alpha : m \geqslant 1\}/\alpha \text{ -- for all } n \geqslant 2 \text{ and all } \alpha \in \{\Diamond_{m,n-1}\beta : m \geqslant 1\}$$

$$\Box_{m,n}p/\Diamond_{m-i,n}p \text{ -- for all } m \geqslant 2,\ n \geqslant 1,\ 0 \leqslant i \leqslant m-1$$

$$\Box_{m,n}p/\Diamond_{m,n+i}p \text{ -- for all } m, n \geqslant 1,\ \text{all } i \geqslant 0$$

$$\{\Diamond_{m,n+1}p : i \geqslant 1\}/\Box_{m,n} \text{ -- for all } m, n \geqslant 1$$

The thing is definitely for fans of logical puzzle and they will find time and energy necessary to prove that L is just as we want it to be, i.e. it has the least closure base but it does not have Lindenbaum property. Of course, L is structural, it is defined by sequential rules. Hence the example we have produced (the same remark applies to the examples given earlier) covers both the case when C is assumed to be a consequence without specifying whether it is structural or not and the case when C is assumed to be a structural consequence. ■

23. Inferential bases for K, H, $J_{\min}$, J, and some useful theorems

23.1. Given any logic C call it *derivational* (with the respect to $\to$) iff C has an inferential base of the form (A,MP), where A is a set of axioms, and MP is Modus Ponens ($p, p \to q/q$). Observe that this definition is equivalent to the following one: C is derivational iff ($C(\emptyset)$,MP) is an inferential base for C. Note also that in view of 10.4 all derivational logic are standard.

All logics we are going to deal with in this section, K in particular, are derivational.

23.2. An inferential base for K

I. Axioms for implication

A1 $\vdash \alpha \to (\beta\alpha)$,

A2 $\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

II Axioms for conjunction

A3 $\vdash (\alpha \wedge \beta) \to \alpha$,

A4 $\vdash (\alpha \wedge \beta) \to \beta$,

A5 $\vdash (\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma))$.

III Axioms for disjunction

A6 $\vdash \alpha \to (\alpha \vee \beta)$,

A7 $\vdash \beta \to (\alpha \vee \beta)$,

A8 $\vdash (\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$.

IV Axioms for negation

A9 $\vdash (\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$,

A10 $\vdash \neg(\alpha \to \alpha) \to \beta$.

V The Law of Excluded Middle

A11 $\vdash \alpha \vee \neg\alpha$.

VI Modus Ponens

MP $\alpha,\ \alpha \to \beta \vdash \beta$.

We have defined the inferential base for K, denote it as $\mathbb{B}(K)$, somewhat implicitly by means of schemata of valid inferences. The definition of $\mathbb{B}(K)$ in the proper form, i.e. as a set of rules is straightforward.

By $\mathbb{B}(J)$ we shall denote the inferential base that results from $\mathbb{B}(K)$ by removing A11. The intuitionistic logic J is (by stipulation!) the logic determined by $\mathbb{B}(J)$.

By $\mathbb{B}(J_{\min})$ we shall denote the inferential base that results from $\mathbb{B}(J)$ by removing A10. The logic determined by $\mathbb{B}(J_{\min})$ is known as *Johansson's minimal logic* (cf. I. Johansson [1937]).

Given any subset $\Phi \subseteq \{I, II, III, IV\}$ the axioms involved in Φ provide an axiom base for the fragment of J defined by the connectives appearing in Φ. Thus, e.g. I is an axiom base for the purely implicational fragment of the intuitionistic logic, and I∪II∪III for the positive fragment of it. The latter logic is known as *Hilbert's positive logic* (cf. D. Hilbert and W. Ackerman [1928]).

23.3. Adequacy Theorem. $\mathbb{B}(K)$ is an adequate inferential base for K, i.e. K and the logic $Cl_{\mathbb{B}(K)}$ determined by $\mathbb{B}(K)$ coincide.

For a short proof of this well known theorem, cf. 23.7. We have to precede it with some preparatory results.

23.4. Theorem. Let C be a derivational logic, and let A1, A2 be (schemata of) theorems of C. Then

DT. $$\beta \in C(X, \alpha) \text{ iff } \alpha \to \beta \in C(X)$$

Proof. The theorem was proved independently by A. Tarski [1930] and J. Herbrand [1930]. Below we give an outlined version of the proof to be found in J. Łoś [1955a]. Prove that A1, A2, and MP yields $\alpha \to \alpha$. Then define

$$Y_\alpha = \{\beta : \alpha \to \beta \in C(X)\}$$

and verify that $C(X, \alpha) \subseteq Y_\alpha$. This gives the "if" part of Dt. The "only if" is obvious. ■

The equivalence DT is known sa *Deduction Theorem*. The following is worthwhile noticing. Let C be a standard logic for with DT holds true. Then (cf. R. Suszko [1961]) C is derivational and both A1 and A2 are (schemata of) theorems of C.

23.5. LEMMA. Let $\mathbb{X}_K$ be the set of all relatively maximal theories of the logic determined by $\mathbb{B}(K)$. For all α, β and for each $X \in \mathbb{X}_K$ the following conditions hold true:

($\wedge$) $\alpha \wedge \beta \in X$ iff $\alpha, \beta \in X$,

($\vee$) $\alpha \vee \beta \in X$ iff $\alpha \in X$ or $\beta \in X$,

($\rightarrow$) $\alpha \rightarrow \beta \in X$ iff $\alpha \notin X$ or $\beta \in X$,

($\neg$) $\neg\alpha \in X$ iff $\alpha \notin X$.

PROOF. Each $X \in \mathbb{X}_K$ contains all instances of the schemata A1 – A11 and is closed under MP. Apply axioms for conjunction to get ($\wedge$). In order to get the "if" part of ($\vee$) apply A6 and A7. The argument that establishes the "only if" part is a bit more involved.

Assume that neither α nor β belongs to X, Since X is relatively maximal, hence for some $\gamma \notin X$, γ is provable by means of $\mathbb{B}(K)$ both from $X \cup \{\alpha\}$ and $X \cup \{\beta\}$. Since the Deduction Theorem (cf. 23.4) holds true for the logic $Cl_{\mathbb{B}(K)}$ determined by $\mathbb{B}(K)$, we conclude that both $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$ are in X. Now X contains A8 and thus $(\alpha \vee \beta) \rightarrow \gamma \in X$. Since, by the assumption we made $\gamma \notin X$, thus $\alpha \vee \beta \notin X$ either. The "only if" part of ($\rightarrow$) results from the fact that X is closed under MP. Before we prove the remaining part of condition ($\rightarrow$) let us note that ($\neg$) follows from ($\vee$), the fact that X contains all instances of A11, and the fact that $\alpha \wedge \neg\alpha \rightarrow \beta$ is provable from axioms A1 – A11 by means of MP.

Now observe that if $\beta \in X$, then $\alpha \rightarrow \beta \in X$ in virtute of A1, and if $\alpha \notin X$, then by ($\neg$), $\neg\alpha \in X$. To conclude the proof it suffices to show that all inferences of the form

$$\vdash \neg\alpha \rightarrow (\lambda \rightarrow \beta)$$

are provable from axioms A1 – A11 by means of MP. We shall leave this part of the proof to the reader. ■

An inspection of the proof given above reveals that ($\vee$) holds true for any derivational logic in which A1, A2, and A6 – A8 are valid. Hence we have the following

23.6. LEMMA. Let C be a derivational logic in which all inferences of he form A1 – A2, A6 – A8 are valid. Them for each relatively maximal theory X of C and for all α, β

$$(\vee) \qquad \alpha \vee \beta \in X \text{ iff } \alpha \in X \text{ or } \beta \in X.$$

■

A theory X that has the property defined by ($\vee$) is called *prime*, or *finitely irreducible*.

23.7. PROOF OF 23.3. Apply two-valued truth-tables in order to verify that all inferences in $\mathbb{B}(K)$ are valid. This establishes soundness of $\mathbb{B}(K)$, i.e. the logic $Cl_{\mathbb{B}(K)}$ determined by $\mathbb{B}(K)$ is weaker than K.

In order to show that formalization of K provided by $\mathbb{B}(K)$ is complete, i.e. K is weaker than the logic determined by $\mathbb{B}(K)$, consider the set $\mathbb{X}_K$ of all relatively maximal theories of $Cl_{\mathbb{B}(K)}$. By Lemma 23.5 the characteristic functions of theories in $\mathbb{X}_K$ are easily seen to be classically admissible (cf. 19.2). Apply completeness Lemma 21.2 in order to conclude the proof.■

We shall conclude this section with a few, rather simple observations.

23.8. LEMMA. Let C be any logic for which all inferences in $\mathbb{B}(J)$ are valid. Then for each formula α and for each relatively maximal theory X of C at most one of the formulas α, $\neg\alpha$ is in X.

PROOF. We shall leave it to the reader to show that the formula

$$\alpha \to (\neg\alpha \to \beta)$$

is provable from axioms A1 – A10 be means of Modus Ponens. Once this is established the lemma follows immediately. ■

Observe that in the case of K, Lemma 23.5 and the fact that the Law of Excluded Middle A11 is a theorem of K imply that for each α, and for each relatively maximal theory X of K, at least one of the formulas α, $\neg\alpha$ is in X. This, of course, yields

23.9. COROLLARY. All relatively maximal theories of K are maximal.

PROOF. Assume that X is relatively maximal. Suppose that $\alpha \notin X$. Hence, $\neg\alpha \in X$. Hence, $K(X, \alpha) = L$. ■

24. An inferential base for N and some adequacy theorems. The method of canonical frames

24.1. Since the condition imposed on epistemic valuations for the language $\mathcal{L}$ coincide with "the positive part" of those of the conditions 18.2 that concern the standard connectives, thus for each epistemic valuation ϵ for $\mathcal{L}_\sim$, the restriction $\epsilon \restriction L$ is an epistemic valuation for $\mathcal{L}$. On the other hand the meaning of $\sim$ defined by conditions $(\sim)_+$ and $(\sim)_-$ of 18.2 is that of the "classical" negation; $\sim$ just changes the truth value of he sentence to which it is applied, of course if the sentence has any truth-value at all, i.e. under the valuation considered its truth-value is determined.

The first of the two observations allows us to conclude that N is an extension of J and thus, in particular, all axioms for J are theorems of N, while the second suggest a way to extend an inferential base for J to an inferential base for N.

24.2. Theorem. Denote by $\mathbb{B}(N)$ the inferential base that results by enlarging $\mathbb{B}(J)$ by the following schemata of inferences:

N1. $\vdash\sim(\alpha\wedge\beta)\longleftrightarrow(\sim\alpha\vee\sim\beta)$,

N2. $\vdash\sim(\alpha\vee\beta)\longleftrightarrow(\sim\alpha\wedge\sim\beta)$,

N3. $\vdash\sim(\alpha\to\beta)\longleftrightarrow(\alpha\wedge\sim\beta)$,

N4. $\vdash\sim\neg\alpha\longleftrightarrow\alpha$,

N5. $\vdash\sim\sim\alpha\longleftrightarrow\alpha$.

$\mathbb{B}(n)$ in as adequate inferential base for N.

24.3. Proof. (some comments) As always one has to start with proving that the formalization is sound i.e. $Cn_{\mathbb{B}(N)}\leqslant N$. This part of the proof amounts to routine verifications, and we shall omit it.

Now, in order to establish completeness with the help of Henkin's methods, i.e. with the help of Completeness Lemma 21.2, one has to show that the characteristic functions of relatively maximal theories of $Cn_{\mathbb{B}(N)}$ are admissible valuations for N, i.e. they are functions of the form ϵ_t, where ϵ in an epistemic valuation for $\mathcal{L}_\sim$ defined relatively to a frame $(T,\leqslant)$, and $t\in T$. (of course, ϵ_t is defined by $\epsilon_t(\alpha)=1$ iff $\epsilon(t,\alpha)=1$).

Denote by $\mathbb{X}_N$ the set of all relatively maximal theories of $Cn_{\mathbb{B}(N)}$. Let $X\in\mathbb{X}_N$ and let $\mathcal{X}_X$ be the characteristic function of X. It is not clear how to find this particular frame $(T_X,\leqslant_X)$ with respect to which $\mathcal{X}_X$ is to be proved to be an epistemic valuation for $\mathcal{L}_\sim$. Thus Henkin's method does not work in an automatic way, rather involves some guessing.

I am afraid that in the case of N the question has no easy solution and, in fact, the proof of completeness we are looking for cannot be based on Completeness Lemma but rather on some modification of it.

When the Completeness Lemma is applied directly to a consequence C and a set of truth-valuations H allegedly adequate for C, we have to show that $H(\mathbb{X}_C)\subseteq H$, and $Cn_{H(\mathbb{X}_C)}\leqslant C$, where $H(\mathbb{X}_C)$ is the set of characteristic functions of relatively maximal theories of C. Suppose that, given any $H_0\subseteq H$, we succeedin proving that $Cn_{H_0}\leqslant C$. Of course, this suffices to establish completeness of C relative to H. In general, $H(\mathbb{X}_C)$ is the most obvious candidate to play the role of H_0, but ant always. This is not just the case of N.

24.4. Proof continued. In order to show that completeness of $\mathbb{B}(N)$ we shall use the method referred to as the method of canonical frames.

Consider the frame $(\mathbb{X}_N,\subseteq)$, where, as we have already defined, $\mathbb{X}_N$ is the set of relatively maximal theories of $Cn_{\mathbb{B}(N)}$. Just this frame will be referred to as *canonical*.

Now, we shall not be interested in all epistemic valuations in $(\mathbb{X}_N, \subseteq)$, but in a specific function that will be denoted by v_N and defined by

$$v_N(X, \alpha) = \begin{cases} 1, & \text{if } \alpha \in X, \\ 0, & \text{if } \sim \alpha \in X, \end{cases}$$

for all $X \in \mathbb{X}_N$. We shall verify that v_N is an epistemic valuation for $\mathcal{L}_\sim$. Suppose that this is already done and assume that for some α and some X_0, $\alpha \notin Cl_{\mathbb{B}(N)}(X_0)$. Then for some relatively maximal theory $X \in \mathbb{X}_N$, $X_0 \subseteq X$ and $\alpha \notin X$. Clearly, for all $\beta \in X$, $v_N(X, \beta) = 1$ but $v_N(X, \alpha)$ is either undetermined or equal 0. Hence $\alpha \notin N(X_0)$. Thus $N \leqslant Cl_{\mathbb{B}(N)}$ which we need to conclude the proof.

(Observe that the set of truth-valuations $v_{N,X'}$, $X \in \mathbb{X}_N$, defined by $v_{N,X}(\alpha) = 1$ iff $v_N(X, \alpha) = 1$ plays just the role of the set H_0 from our comments in 24.3, H being interpreted as the set of all admissible valuations for N. It is a matter of proof, which we shall not present, to show that the truth valuations $v_{N,X}$ need not be characteristic functions of relatively maximal theories of $Cl_{\mathbb{B}(N)}$.)

It is rather an easy task to verify that v_N is an epistemic valuation indeed, Of course, all $X \in \mathbb{X}_N$ contain all formulas of the form A1 – A1- as well as all formulas of the form N1 – N5, and they are closed under MP.

Clearly, v_N satisfies conditions (0) and (1) of 18.2.

Apply A3, A4, and MP (i.e. apply the fact that $X \in \mathbb{X}_N$ contain all substitution instances of A3, A4 and they are closed under MP) in order to show that v_N satisfies $(\wedge)_+$.

The proof that v_N satisfies $(\wedge)_-$ requires making use of N1 and Lemma 23.6.

Apply again 23.6 along with A3, A4 in order to establish $(\vee)_+$.

Use N2 and Axioms for conjunction in order to get $(\vee)_-$.

Since theories in $\mathbb{X}$ are closed under MP, v_N satisfies $(\rightarrow)_+$.

Apply Axioms for conjunction and N3 to show that v_N satisfies $(\rightarrow)_-$.

In order to see that v_N satisfies $(\neg)_+$ make use of Lemma 23.8 and N4.

The fact that v_N satisfies $(\neg)_-$ is provable with the help of N4 and N5.

Conditions $(\sim)_+$ and $(\sim)_-$ follow from N5. ∎

24.5. Proof of 17.4. (An outline). Verify that $\mathbb{B}(J)$ is sound, i.e. $J =_{\text{df}} Cl_{\mathbb{B}(J)} \leqslant Cn_{\text{EFrame}}$. Consider to epistemic frame $(\mathbb{X}_J, \subseteq)$. Define the function $v_J : \mathbb{X}_J \times L \rightarrow \{0, 1\}$ by

$$v_J(X, \alpha) = \begin{cases} 1, \text{if } \alpha \in X, \\ 0, \text{ otherwise}, \end{cases}$$

and verify that v_J is an epistemic valuation for $\mathcal{L}$ defined relatively to $(\mathbb{X}_J, \subseteq)$. Assume that for some X_0 and some α, $\alpha \notin J(X_0)$ and apply v_J to show that $\alpha \notin Cn_{\mathrm{EFrame}}$. ■

Observe that the truth valuations $v_{J,X}$, $X \in \mathbb{X}_J$, corresponding to v_J are just the characteristic functions of relatively maximal theories of J. Hence, in this particular case, Henkin's methods works without modification, though again the use of it involves constructing of the canonical frame in a suitable manner.

Incidentally, observe that the assumption we made that epistemic valuations are particular functions has no formal motivation. We might as well assume that epistemic valuations are just functions, i.e. drop "partial" in Definition 17.2. However, under the assumption we made each epistemic valuation for $\mathcal{L}_\sim$ when restricted to $\mathcal{L}$ becomes an epistemic valuation for $\mathcal{L}$. This fact we shall exploit in the proof of the next theorem.

Let the logics C, C' be defined in the languages $\mathcal{S}$, $\mathcal{S}'$ respectively, $\mathcal{S}'$ being an extension of $\mathcal{S}$ (i.e. it results from $\mathcal{S}$ by adding some new connectives). If $C \leqslant C' \restriction \mathcal{S}$, i.e. $C(X) \subseteq C'(X) \cap S$, for all $X \subseteq S$, the logic C' is referred to as na *extension of* C. It is said to be a conservative extension iff $C = C' \restriction \mathcal{S}$.

24.6. Theorem. N is a conservative extension of J.

Proof. Of course, N is an extension of J; $\mathbb{B}(N)$ results by enlarging $\mathbb{B}(J)$. In order to prove that N is a conservative extension assume that for some α and some $X \subseteq L$, $\alpha \notin J(X)$. Let ϵ be an epistemic valuation for $\mathcal{L}$ defined relatively to $(T, \leqslant)$ that falsifies $X \vdash \alpha$. By Lemma 18.3 a $\epsilon \restriction Var(L_\sim)$ can be extended to a valuation ϵ_N for $\mathcal{L}_\sim$. Now $\epsilon_N \restriction L$ is an epistemic valuation for $\mathcal{L}$ and moreover, by Lemma 17.3, $\epsilon_N \restriction L$ verifies exactly the same formulas as ϵ. Hence $\epsilon_N \restriction L$ falsifies $X \vdash \alpha$. Thus ϵ_N falsifies $X \vdash \alpha$ too, which yields $\alpha \notin N(X)$ concluding the proof. ■

24.7. It is of some interest that $J_{\min}$ can be determined by epistemic frames of a certain modified, more general kind. Call a triple $(T, \leqslant, U)$ an *epistemic frame with non-normal points* iff $(T, \leqslant)$ is an epistemic frame, and $U \subseteq T$. The elements of U are referred to as non-*normal points*. Now, modify Definition 17.2 of epistemic valuations by restricting all quantifiers for all t' that the definition involves to T/U, i.e. replace them by 'for all $t' \in T/U$'. Call the valuations defined in this way *normal epistemic valuations*. $J_{\min}$ is just the truth preserving logic complete with respect to all normal epistemic valuations (or equivalently, with respect to all epistemic frames with normal points).

For adequacy proof cf. K. Segerberg [1968].

25. Canonical frames for modal logics

25.1. The technique of canonical models we have presented in the previous section was originally invented for solving the problem of completeness of some modal logics with respect to suitably selected set of relational frames. As a matter of fact, the epistemic frames are relational frames of a specific kind though, of course, the way in which the intuitionistic logic and some logics related to it ($J_{\min}$ and N among others) are interpreted via epistemic frames differs from that in which modal logics are interpreted in relational frames including those that are epistemic.

Since the modal logics are going to deal with will be of the form $\bar{M}$, M being a modal system based on K (cf. 14.2), they all are derivational extensions of K. Now, a great deal of results that are valid for K, are valid for the derivational extensions of that logic. In particular, of a great use for us will be the following

25.2. Lemma. Let C be a derivational extension of K. Then

(i) All relatively maximal theories of C are maximal, and

(ii) For all α, β and all relatively maximal theories X of C conditions ($\wedge$), ($\vee$), ($\rightarrow$), and ($\neg$) of 23.5 are satisfied.

Proof. Repeat the argument by means of which the corresponding conditions have been established for K.

25.3 The frame $\mathcal{F}_L^N = (\mathbb{X}_L, N_L)$ will be said to be the *canonical neighborhood frame* for the logic $\vec{M}_L$ iff

(i) $\mathbb{X}_L$ is the set of all relatively maximal (and thus maximal) theories of M_L,

(ii) The neighborhood function N_L is defined as follows. For each α we put

$$\mathbb{X}_L^\alpha = \{X \in \mathbb{X}_L : \alpha \in X\}$$

and then, for each $X \in \mathbb{X}_L$, we define

$$N_L(X) = \{\mathbb{X} : \text{ for some } \alpha, \quad \mathbb{X} = \mathbb{X}_L^\alpha \cap \mathbb{X}_L^{\Box\alpha}\}$$

25.4. The frame $\mathcal{F}_L^R = (\mathbb{X}_L, R_L)$ will be said to be the *canonical relational frame* for $\vec{M}_L$ iff

(i) $\mathbb{X}_L$ is the set of all relatively maximal (and thus maximal) theories of $\vec{M}_L$,

(ii) For all $x, y \in \mathbb{X}_L$, XR_LY iff, for all α, $\alpha \in Y$ whenever $\Box\alpha \in X$.

25.5. Before we discuss adequacy of the two definitions with respect to the purpose the canonical frames are expected to serve, let us define for each logic M_L the following classes of frames

$$\mathrm{NFrame}(\vec{M_L}) = \{\mathcal{F} \in \mathrm{NFrame} : \vec{M_L} \leqslant Cn_{\mathcal{F}}\}$$

$$\mathrm{RFrame}(\vec{M_L}) = \{\mathcal{F} \in \mathrm{RFrame} : \vec{M_L} \leqslant Cn_{\mathcal{F}}\}$$

Given any class $\mathbb{F}$ of frames of the same kind, define $\zeta(\mathbb{F})$ to be the set of all sentences α verified by all neighborhood/relational valuations defined relative to frames in $\mathbb{F}$. The set $\zeta(\mathbb{F})$ is often referred to as the *content* of $\mathbb{F}$, and the elements of $\zeta(\mathbb{F})$ as *tautologies* of $\mathbb{F}$. Of course, $\zeta(\mathbb{F})$ is merely another notation for $Cn_{\mathbb{F}}(\emptyset)$. Define

$$\mathrm{NFrame}(M_L) = \{\mathcal{F} \in \mathrm{NFrame} : M_L \subseteq \zeta(\mathcal{F})\},$$

$$\mathrm{RFrame}(M_L) = \{\mathcal{F} \in \mathrm{RFrame} : M_L \subseteq \zeta(\mathcal{F})\}.$$

Of course (the proof is straightforward) the following holds true:

25.6. LEMMA. For each modal system M_L based on classical logic

a. $\mathrm{NFrame}(\vec{M_L})$ =$\mathrm{NFrame}(M_L)$,

b. $\mathrm{RFrame}(\vec{M_L})$ =$\mathrm{RFrame}(M_L)$. ■

Incidentally, let us observe that by 20.5, clause b. of 25.6 is a corollary to a.

Now we are in a position to examine adequacy of Definitions 25.3 and 25.4.

25.7. FUNDAMENTAL LEMMA FOR NEIGHBORHOOD CANONICAL FRAMES (Segerberg [1971]). Let M_L be a classical modal system based on K and let $H(\mathbb{X}_L)$ be a set of all characteristic functions of relatively maximal theories of $\vec{M_L}$. Then

$$H(\mathbb{X}_L) \subseteq H(\mathcal{F}_L^N),$$

$\mathcal{F}_L^N$ being the neighborhood canonical frame for M_L.

PROOF. Let η^L be a neighborhood valuation in $\mathcal{F}_L^N$ such that for each propositional variable p, $\eta^L(X, p) = 1$ iff $p \in X$. Apply induction to prove that for each α, $\eta^L(X, \alpha) = 1$ iff $\alpha \in X$. By Lemma 25.2 the part of the inductive argument that concerns $\wedge$, $\vee$, $\rightarrow$, and $\neg$ is straightforward. Let us consider formulas of the form $\Box\alpha$.

Assume that

(1) $$\eta^L(X, \Box\alpha) = 1,$$

This implies

(2) $$\{X' \in \mathbb{X}_L : \eta^L(X', \alpha) = 1\} \in N_L(X)$$

which by the assumption of the inductive argument, yields, that for some $\beta \in \mathbb{X}^\alpha$, $\mathbb{X}^\alpha = \mathbb{X}^\beta$ and

$$\Box\beta \in X. \tag{3}$$

By the hypothesis of the induction, $\mathbb{X}_\alpha = \mathbb{X}_\beta$ implies that $\eta^L(X,\alpha) = \eta^L(X,\beta)$ and hence, we have $\eta^L(X, \alpha \longleftrightarrow \beta) = 1$, i.e. $\alpha \longleftrightarrow \beta \in X$.

Apply the assumption that M_L is classical in order to get $\Box\alpha \longleftrightarrow \Box\beta \in X$, which by (3) yields

$$\Box\alpha \in X \tag{4}$$

The argument to the effect that (4) implies (1) is straightforward. ■

The reader will easily prove the following.

25.8. Fundamental Lemma for Relational Canonical Frames (Segerberg [1971]). Let M_L be a normal modal system based on K and let $H(\mathbb{X}_L)$ be a set of all characteristic functions of relatively maximal theories of $\vec{M}_L$. Then

$$H(\mathbb{X}_L) \subseteq H(\mathcal{F}_L^R),$$

$\mathcal{F}_L^R$ being the relational canonical frame for M_L. ■

26. Are all classical modal systems natural? System $K4.3W$

26.1. From the two lemmas we have stated above and Lemma 25.6 it follows immediately that

a. If M_L is classical and $\mathcal{F}_L^N \in$NFrame(M_L) then $\vec{M}_L = Cn_{\mathcal{F}_L^N}$.

b. If M_L is normal and $\mathcal{F}_L^R \in$RFrame(M_L) then $\vec{M}_L = Cn_{\mathcal{F}_L^N}$.

K. Segerberg [1971] suggested to call a classical (normal) system M_L *natural* when $M_L = \zeta(\mathcal{F}_L^N)$ ($M_L = \zeta(\mathcal{F}_L^R)$, resp.) and asked the question whether all modal systems are natural.

26.2. The question was answered in negative by R. I. Goldblatt [1976]. The system he examined was $K4.3W$, thus $M_{K4.3W}$ under our convention (cf. 9.4). It results from M_K by adding as new axioms all formulas of the form

K1. $\Box\alpha \to \Box\Box\alpha$,

K2. $(\Diamond\alpha \wedge \Diamond\beta) \to (\Diamond(\alpha\wedge\beta) \vee \Diamond(\alpha \wedge \Diamond\beta) \vee \Diamond(\beta \wedge \Diamond\alpha))$,

K3. $\Box(\Box\alpha \to \alpha) \to \Box\alpha$.

Let us examine the system closer. But before we do that, some auxiliary notions will be needed.

26.3. Let $\mathcal{F} = (T, R)$ be a relation frame, and let $t \in T$. Denote by T_t the least subset T' of T such that $t \in T'$ and T' is closed under R (i.e. if $t_1 \in T'$ and $t_1 R t_2$, then $t_2 \in T'$). The frame $\mathcal{F}_L = (T_t, R \restriction T_t)$ is called the *subframe of $\mathcal{F}$ generated by t.*

If $\mathcal{F} = (T, N)$ is a neighborhood frame, the notion of the *subframe* $\mathcal{F}_t$ *generated* by t is defined as follows. We define T_t, $t \in T$, to be the least subset $T' \subseteq T$ such that $t \in T$, and for all $t' \in T'$, $\bigcup N(t') \subseteq T'$. Then we put $\mathcal{F}_t = (T_t, N \restriction T_t)$.

If for some point t, $\mathcal{F} = \mathcal{F}_t$, the frame will be called *principal* and t will be called a *principal* point of $\mathcal{F}$.

Clearly, the following holds true.

26.4. Lemma. Let $\mathcal{F}$ be a relational (neighborhood) frame and let T be the set of points of reference of T. Then,

(i) $\mathcal{F}$ and $\{\mathcal{F}_t : t \in T\}$ are equivalent, i.e. $Cn_{\mathcal{F}} = Cn_{\{\mathcal{F}_t : t \in T\}}$. Moreover

(ii) If $\mathcal{F}$ is principal and t is a principal point of $\mathcal{F}$ then $\mathcal{F}$ and $\mathcal{F}_t$ are equivalent. ■

Denote by NFrame$^*(M_L)$ the class of all principal neighborhood frames of M_L, and denote by RFrame$^*(M_L)$ the set of all principal relational frames of M_L. As an immediate corollary to Lemma 26.4 we have

26.6. Theorem. For each modal system M_L the semantics NFrame$^*(ML)$ is equivalent to NFrame(M_L), and the semantics RFrame$^*(ML)$ is equivalent to RFrame(M_L). ■

Now, we are in a position to turn back to the question of our concern. The following two theorems were established by K. Segerberg [1970].

26.6. Theorem. The class RFrame$(M_{K4.3W})$ of all principal frames of $M_{K4.3W}$ consist of all relational frames (T, R) such that

(i) T is finite,

(ii) R is a linear ordering on T. (i.e. R is antireflexive, antisymetric, and transitive).

26.7. Theorem. $M_{K4.3W} = \zeta(\text{RFrame}(M_{K4.3W}))$.

What Theorem 26.7 amounts to is that $\vec{M}_{K4.3W}$ is weakly adequate with the respect to all finite and linear relational frames (cf. 16.3b).

Suppose that $M_{K4.3W}$ is natural, i.e. $\vec{M}_{K4.3W}$ is weakly adequate with the respect to the canonical frame. Then, by 26.1b, $\vec{M}_{K4.3W}$ is adequate with respect to the relational canonical frame, and thus it is adequate with the respect to the whole class RFrame$(M_{K4.3W})$. by 26.5 this implies that $\vec{M}_{K4.3W}$ is adequate with respect to RFrame$(\vec{M}_{K4.3W})$. And this is exactly what R. I. Goldblatt, [1976] shows not to be true.

26.8. THEOREM. $\vec{M}_{K\,K4.3W}$ is not adequate with the respect to the class RFrame($M_{K4.3W}$) of all frames for $M_{K4.3W}$.

PROOF. Verify that each finite subset X' of the set $X = \{\Diamond^k p : k \in \omega\}$ is satisfied by a relational valuation in a frame of $M_{K4.3W}$, and hence it is consistent with respect to $\vec{M}_{K4.3W}$. Since $\vec{M}_{K4.3W}$ is standard, this implies the consistency of X with respect to $\vec{M}_{K4.3W}$.

In turn, verify that there is no principal frame for $M_{K4.3W}$ (all of them, 26.6, are finite and linear) in which X is satisfied. Thus, by Lemma 26.4, there is no frame for $M_{K4.3W}$ in which X is satisfied. This implies that X is inconsistent with the respect or the consequence determined by RFrame($M_{K4.3W}$), and hence this consequence and $\vec{M}_{K4.3W}$ do not coincide. ∎

27. The problem of completeness

27.1. It turns out (cf. 26.8) that modal systems need be strongly complete with respect to the relational frame semantics. Are they always weakly complete? The question has been answered negative, but before we dwell on that, let us define the classes of frames for some of the modal systems. We have to start with some auxiliary definitions (cf. e.g. Segerberg [1976]).

a. Let (T, N) be a neighborhood frame. A reference point $t \in T$ will be said to be *normal* iff $N(t)$ is a filter (i.e. $N9t) \neq \emptyset$ and for all $x, y \subseteq T$, $x, y \in N(T)$ iff $x \cap y \in N(t)$). If $N(t) = \emptyset$ we say that t is *singular*. A reference point that is either normal or singular is called *regular*. The frame itself is called *normal, singular* or *regular* according to whether its reference points are all normal, all singular, or all regular. Thus both normal and singular frames are regular.

b. Let (T, R) be a normal relational frame. We shall say that (T, R) is *reflexive, symmetric, transitive* or *universal* (according to whether R is reflexive, symmetric, transitive or universal).

27.2. As it is known (cf. e.g. Segerberg [1971])

a. NFrame(M_E) = the class NFrame of all neighborhood frames.

b. NFrame(M_C) = the class of all regular neighborhood frames.

c. NFrame(M_K) = the class of all normal neighborhood frames.

d. RFrame(M_K) = the class RFrame of all normal relational frames.

e. RFrame(M_T) = the class of all normal reflexive frames.

f. RFrame(M_{S4}) = the class of all normal transitive frames.

g. RFrame(M_B) = the class of all normal symmetric frames.

h. RFrame(M_{S5}) = the class of all normal universal frames.

(Just for an illustration let us prove a. Of course, we have to prove only that NFrame $\subseteq$ NFrame(M_E), or equivalently that $M_E \subseteq \zeta$(NFrame).

To begin with observe that if $\alpha \in Sb(K(\emptyset))$ then $\alpha \in \zeta$(NFrame) because all neighborhood valuations are classically admissible. Now $\alpha \in M_E$ iff α is provable from $Sb(K(\emptyset))$ by means of Modus Ponens MP and Replacement Rule RE. But ζ(NFrame) is closed under MP and RE, hence $M_E \subseteq \zeta$(NFrame). ■

Since to each relational frame there corresponds a semantically equivalent neighborhood frame each of clauses f., g., f., has its neighborhood counterpart.

27.3. In view of 25.8 and 25.9 it is enough to verify that for each of the logics considered above the canonical frame (neighborhood for all of them, relational for normal) belongs to the class of frames characteristic of a given logic to obtain

a. The logics $\vec{M}_E$, $\vec{M}_C$, $\vec{M}_K$, $\vec{M}_T$, $\vec{M}_{S4}$, $\vec{M}_B$, $\vec{M}_{S5}$ are complete with the respect to neighborhood frames, and thus NFrame(M) is an adequate semantics for each of the logic M listed above.

b. Those of the logics listed above that are normal, are complete with the respect to relational frames, and thus RFrame(M) is an adequate semantics for each such logic M.

27.4. As we already know there are some modal logic that are not complete with the respect to relational frames. Of this kind is $M_{K4.3W}$, the system we have discussed in the previous section. But the system $M_{K4.3W}$ itself is determined by relational frames. This gives rise to the following question: are all normal modal systems (not logic but just systems!) complete with respect to relational frames?

K. Fine [1974] and S. K. Thomason [1974] presented independently some examples of normal systems incomplete with respect to relational frames. Their results concerned relational frames, but they opened way to solve similar problem for neighborhood frames. M. S. Gerson [1975] showed that neither K. Fine's system nor that defined by S. K. Thomason are complete with respect to neighborhood frames.

From that time many further examples of systems incomplete with respect to both relational and neighborhood frames were given. A very simple one, incomplete with respect to relational frames, has been invented by J. K. van Benthem [1978]. It is the least normal modal system that contains all formulas of the form

$$\Box(\Box\alpha \to \Diamond\alpha) \to \Box(\Box(\Box\alpha \to \alpha) \to \alpha,$$

$\Diamond$ being defined as usual by $\Diamond\alpha =_{\text{df}} \neg\Box\neg\alpha$.

27.5. M. S. Gerson [1975a], [1976] and D. Gabbay [1975] gave examples of normal modal systems that are complete with respect to neighborhood frames but they are not complete with respect to relational frames. This proves that for some neighborhood semantics there does not exists an equivalent relational semantics.

27.6. Given a normal modal system M define

(1) $$\delta_{\mathrm{RFrame}}(M) = \mathrm{card}\{M' : \mathrm{RFrame}(M') = \mathrm{RFrame}(M)\},$$

(2) $$\delta_{\mathrm{NFrame}}(M) = \mathrm{card}\{M' : \mathrm{NFrame}(M') = \mathrm{NFrame}(M)\},$$

(M' running over normal modal systems).

$\delta_{\mathrm{RFrame}}(M)$ and $\delta_{\mathrm{NFrame}}(M)$ are called *degrees of incompleteness* of the system M with respect to relational frames and neighborhood frames, respectively. Of course, M is complete with respect to relational (neighborhood) frames if $\delta_{\mathrm{RFrame}}(M) = 1$ ($\delta_{\mathrm{NFrame}}(M) = 1$, resp.)

Just substitute $\vec{M}$ for M and $\vec{M}'$ for M' in (1) and (2) to get the definitions of $\delta_{\mathrm{RFrame}}(\vec{M})$ and $\delta_{\mathrm{NFrame}}(\vec{M})$: the degree of incompleteness of the logic $\vec{M}$ with respect to relational frames, and that with respect to neighborhood frames.

27.7. Quite surprising result was established by W. J. Blok, [1978]. He proved that for each normal modal system M, either

$$\delta_{\mathrm{RFrame}}(M) = 1$$

or

$$\delta_{\mathrm{RFrame}}(M) = 2^{\aleph_0}.$$

Is the same true about NFrame(M)? Thus far this question remains open. W. Dziobak [1978], adopting some ideas of Blok's paper just mentioned, succeeded in showing that if either

$$\Box\alpha \to \alpha \in M$$

or

$$\Box\alpha \to \Diamond\alpha, \qquad \Box^n\alpha \to \Box^{n+1}\alpha \in M,$$

for all α and for some $n \geqslant 0$, and M is normal, then $\delta_{\mathrm{RFrame}}(M) = 2^{\aleph_0}$.

Chapter 6

Finitary Consequences — Some Semantic Equivalences of the Property

28. Some preparatory results

28.1. Let $\mathbb{X}$ be a family of sets of formulas. We say that $\mathbb{X}$ is *inductive* iff for each upward directed family $\mathbb{Y} \subseteq \mathbb{X}$ the union $\bigcup \mathbb{Y} \in \mathbb{X}$. (A family of sets $\mathbb{Y}$ is said to be *upward directed* iff for all $X, Y \in \mathbb{Y}$ there is $Z \in \mathbb{Y}$ such that $X \cup Y \subseteq Z$).

28.2. Let all X_1, $i \in I \neq \emptyset$ be sets of formulas, and let ∇ be a filter on I. Define

$$\alpha \in \bigcap_{\nabla}\{X_i : i \in I\} \text{ iff } \{i \in I : \alpha \in X_i\} \in \nabla$$

The set $\bigcap_{\nabla}\{X_i : i \in I\}$ (or just $\bigcap_{\nabla} X_i$ under the abbreviated notation) will be referred to as the *product of X_i reduced modulo* ∇; a *reduced product*, if ∇ is fixed. If ∇ is an ultrafilter, $\bigcap_{\nabla} X_i$ will be called an *ultraproduct* of X_i.

Observe that $\{I\}$ is a filter on I, and $\bigcap_{\{I\}} X_i = \bigcap X_i$, hence the notion of a reduced product provides a generalization of that of an intersection.

A family $\mathbb{X}$ of sets of formulas will be said to be *closed under reduced products (ultraproducts)* iff for each $\{X_i : i \in I\} \subseteq \mathbb{X}$, and each (ultra)filter ∇ on I, $\bigcap_{\nabla} X_i \in \mathbb{X}$.

28.3. Lemma. $\bigcap_{\nabla}\{X_i : i \in I\} = \bigcup\{\bigcap\{X_i : i \in F\} : F \in \nabla\}$, for all filters ∇ on I.

PROOF. ($\subseteq$). Define $I(\alpha) = \{i \in I : \alpha \in X_i\}$. If $\alpha \in \bigcap_\nabla X_i$ then, by the definition, $I(\alpha) \in \nabla$ and obviously $\alpha \in \bigcap\{X_i : i \in I(\alpha)\}$. Hence $\alpha \in \bigcup\{\bigcap\{X_i : i \in F : F \in \nabla\}$.

($\subseteq$). Suppose that $\alpha \in \bigcap\{X_i : i \in F\}$, for some $F \in \nabla$. Then each X_i, $i \in F$, contains α and hence $F \subseteq I(\alpha)$. Thus $I(\alpha) \in \nabla$, for ∇ is a filter. This yields $\alpha \in \bigcap_\nabla X_i$, concluding the proof. ■

28.4. LEMMA. Let $\mathbb{X}_0$ be a family of sets of formulas. Define $\mathbb{X} = \{\bigcap \mathbb{Y} : \mathbb{Y} \subseteq \mathbb{X}_0\}$. Then

a. If $\mathbb{X}_0$ is closed under reduced products, so is $\mathbb{X}$.

b. If $\mathbb{X}_0$ is closed under ultraproducts, so is $\mathbb{X}$.

PROOF. Assume that $\mathbb{X}_0$ is closed under reduced products (ultraproducts) and consider any family $\{X_i : i \in I\} \subseteq \mathbb{X}_0$. Let ∇ be a filter (ultrafilter) on I. For each $i \in I$ define $\{X_{i,j} : j \in J_i\}$ to be a subfamily of $\mathbb{X}_0$ such that,

$$\bigcap\{X_{i,j,} : j \in J_i\} = X_i. \tag{1}$$

Put $J = \{(i, j) : i \in I, j \in J_i\}$ and define the set ∇_0 as follows. For each $F \in \nabla$ put

$$F_0 = \{(i, j) : i \in F, j \in J_i\}, \tag{2}$$

and then define

$$\nabla_0 = \{F_0 : F \in \nabla\}. \tag{3}$$

Verify that ∇_0 is a filter (ultrafilter) on J and thus

$$\bigcap\nolimits_{\nabla_0}\{X_{i,j} : (i, j) \in J\} \in \mathbb{X}_0. \tag{4}$$

Apply Lemma 28.3 in order to get

$$\bigcap\nolimits_{\nabla_0}\{X_{i,j} : (i, j) \in J\} = \bigcup\{\bigcap\{X_{i,j} : (i, j) \in F_0\} : F_0 \in \nabla_0\}. \tag{5}$$

But

$$\bigcap\{X_{i,j} : (i, j) \in F_0\} = \bigcap\{X_i : i \in F\}, \tag{6}$$

which yields

$$\bigcup\{\bigcap\{X_{i,j} : (i, j) \in F_0\} : F_0 \in \nabla_0\} = \bigcup\{\bigcap\{X_i : i \in F\} : F \in \nabla\}. \tag{7}$$

Apply one more Lemma 28.3, this time to get

$$\bigcap\nolimits_\nabla\{X_i : i \in I\} \in \mathbb{X}_0 \subseteq \mathbb{X} \tag{8}$$

thus concluding the proof. ■

29. Conditions for a consequence to be finitary

29.1. Theorem. Let C be a consequence. The following conditions are equivalent

(i) C is finitary,

(ii) Th_C is inductive,

(iii) Th_C is closed under ultraproducts,

(iv) There is a closure base $\mathbb{X}$ for C closed under ultraproducts,

(v) Th_C is closed under reduced products,

(vi) There is a closure base $\mathbb{X}$ for C closed under reduced products.

Proof. (i)$\to$(ii). Assume (i) and consider any non-empty upward directed subset $\mathbb{X} \subseteq Th_C$. If $\alpha \in C(\bigcup \mathbb{X})$ then, by (i), there exists a finite subset $X_0 \subseteq \bigcup \mathbb{X}$ such that $\alpha \in C(X_0)$. Hence there exists a finite subset $\mathbb{X}_0 \subseteq \mathbb{X}$ such that $\alpha \in C(\bigcup \mathbb{X}_0)$. But $\mathbb{X}$ is upward directed, and thus for some $Y \in \mathbb{X}$, $\bigcup \mathbb{X}_0 \subseteq Y$. This yields $\alpha \in C(Y) = Y \subseteq \bigcup \mathbb{X}$. It turns out that $C(\bigcup \mathbb{X}) \subseteq \bigcup \mathbb{X}$, i.e. Th_C is inductive.

(ii) $\to$ (iii) Assume (ii) and consider any family $\mathbb{X} = \{X_i : i \in I\}$ of theories of C. Let ∇ be an ultrafilter on I. Define

$$\mathbb{X}^* = \{\bigcap\{X_i : i \in F\} : F \in \nabla\}$$

Of course, $\bigcap \mathbb{X}^* \in Th_C$. Moreover, if $X = \bigcap\{X_i : i \in F\}$, and $Y = \bigcap\{X_i : i \in G\}$, for some $F, G \in \nabla$, then $X \cup Y \subseteq \bigcap\{X_i : i \in F \cap G\}$. Since $F \cap G \in \nabla$, we conclude that family $\mathbb{X}^*$ is upward directed. Now, note that by Lemma 28.3, $\bigcap_\nabla \mathbb{X} = \bigcup \mathbb{X}^*$ and, by (ii), $\bigcup \mathbb{X}^* \in Th_C$.

(iii) $\longleftrightarrow$ (iv). The part "if" obvious. The part "only if" by Lemma 28.4.

(iii) $\to$ (v). Consider any family $\mathbb{X} = \{X_i : i \in I\}$ of theories of C and any proper filter ∇ on I. Define F_∇ to be the set of all ultrafilters ∇' such that $\nabla \subseteq \nabla'$. As known, $\nabla = \bigcap F_\nabla$

Put $I(\alpha) = \{i \in I : \alpha \in X_i\}$ and observe that the following conditions are equivalent

a. $\alpha \in \bigcap_\nabla \mathbb{X}$

b. $I(\alpha) \in \nabla$

c. $I(\alpha) \in \nabla'$, for all $\nabla' \in F_\nabla$,

d. $\alpha \in \bigcap_\nabla$, for all $\nabla' \in F_\nabla$

Thus $\bigcap_\nabla \mathbb{X} = \bigcap\{\bigcap_{tr'} \mathbb{X} : \nabla' \in F_\nabla\}$. By (iii) $\bigcap_{\nabla'} \mathbb{X} \in Th_C$, for all $\nabla' \in F_\nabla$. Since Th_C is a closure system, we conclude that $\bigcap_\nabla \mathbb{X} \in Th_C$

If ∇ is not proper then $\bigcap_{\nabla} \mathbb{X} = S$, S being the set of all formulas of the language of C and, of course, $S \in Th_C$. Hence for all filter ∇, $\bigcap_{\nabla} \mathbb{X} \in Th_C$

(v) $\longleftrightarrow$ (vi). The part "if" obvious, "only if" by Lemma 28.4.

(v) $\rightarrow$ (i). Let $\alpha \notin C(X')$, for any finite $X' \subseteq X$. Assume (v). We have to show that $\alpha \notin C(X)$.

Let I be the family of all finite subsets of X. For each $i \in I$ define $\mathbb{X}_i = \{j \in I : i \subseteq j\}$ and put

$$\nabla = \{F \subseteq I : \text{ for some } i \in I,\ \mathbb{X}_i \subseteq F\}.$$

Since $\mathbb{X}_i \cap \mathbb{X}_j = \mathbb{X}_{i \cup j}$, ∇ is a filter.

Consider the family $\mathbb{X} = \{C(i) : i \in I\}$. By 28.3 we have,

$$X \subseteq \bigcap_{\nabla} \mathbb{X}, \quad \alpha \notin \bigcap_{\nabla} \mathbb{X}.$$

On the other hand, by (v), $\bigcap_{\nabla} \mathbb{X} \in Th_C$, and hence $\alpha \notin C(X)$, which concludes the proof. ■

29.2. Given a set $\{v_i : i \in I\}$ of truth valuations for a language $\mathcal{S}$, call a valuation v the *direct product of the valuations* v_i, $v = \sqcap_i v_i$ (or more pedantically $v = \sqcap \{v_i : i \in I\}$) iff for each α

$$v(\alpha) = 1 \text{ iff for all } i \in I,\ v_i(\alpha) = 1.$$

Now, let ∇ be a filter on I. We shall say that a valuation v is the *product of* v_i, $i \in I$, *reduced modulo* ∇ (a *reduced product*, if ∇ is fixed), in symbols $v = \sqcap_t rv_i$ (or $v = \sqcap_{\nabla} \{v_i : i \in I\}$) iff for each α

$$v(\alpha) = 1 \text{ iff } \{i \in I,\ v_i(\alpha) = 1\} \in \nabla$$

If ∇ is an ultrafilter, v is called the *ultraproduct* of v_i.

A set of truth-valuations H is said to be closed under reduced products (ultraproducts) iff for each $\{v_i : i \in I\} \subseteq H$, and for each filter (ultrafilter) ∇ on I, $\sqcap_{\nabla} v_i \in H$.

All these definitions are obvious as well as the following

19.3. LEMMA. Let each v_i, $i \in I$, be a characteristic function for the corresponding X_i, $i \in I$. Then

$\sqcap_i v_i$ is the characteristic function for $\bigcap_i X_i$, and for each filter ∇ on I,

$\sqcap_{\nabla} v_i$ is the characteristic function for $\bigcap_{\nabla} X_i$. ■

The lemma we have just allows us to translate Theorem 29.1 onto the language of truth-valuation. In particular we have

29.4 COROLLARY For each consequence C, C is finitary iff there exists a set of truth-valuations H for $\underline{S}$ such that

(i) $C = Cn_H$, and

(ii) H is closed under ultraproducts. ■

29.5 NOTE The notion of a reduced product of truth-valuations was examined by H.van Fraassen [1973] who, in particular, proved 29.4. The equivalence of conditions (i) and (ii) of 29.1 is due to J. Schmidt [1952]. The idea to link Schmidt's theorem with ultraproducts via the notion of ultraproducts of sets of formulas, as well as the details of the proof of 29.1, Lemma 28.4 including, are due to M. Maduch.

30. All Ł$_\eta$, $n \in \omega$, are standard: an example of application of theorem 29.1.

30.1. Given any valuation λ in Łukasiewicz truth-table algebra define the binary truth valuation λ^+ corresponding to λ in an expected way, i.e. by $\lambda^+(\alpha) = 1$ iff $\lambda(\alpha) = 1$. Denote by $H(\mathcal{L}_\eta)$ the set of truth valuations λ^+, λ being a valuation in $\mathcal{L}_\eta$. Of course, for each $\eta = 2, \ldots, \omega$

$$Ł_\eta = Cn_H(\mathcal{L}_\eta)$$

We shall prove that

30.2. LEMMA. For each finite $n \geqslant 2$, the set of truth valuations $H(\mathcal{L}_\eta)$ is closed under ultraproducts.

PROOF. Given any family $\{\lambda_i,\ i \in I\}$ of valuations $\mathcal{L}_n$, n finite, and any ultrafilter ∇ on I, define a function $\lambda :\ L \rightarrow \mathcal{L}_n$ by

(1) $$\lambda(\alpha) = x \text{ iff } \{i \in I,\ \lambda_i(\alpha) = x\} \in \nabla$$

In order to verify that λ is well defined, observed that for each α there is exactly one $x \in \mathcal{L}_n$ for which the right hand side of (1) is valid.

Indeed, put $I_x(\alpha) = \{i \in I :\ \lambda_i(\alpha) = x\}$. We have

(2) $$I_x(\alpha) \cap I_y(\alpha) = 0,$$

for all $x \neq y$. This implies that at most one $I_X(\alpha)$ belongs to ∇. At the same time, though perhaps in a less straightforward way, this implies that at least one $I_x(\alpha)$ is in ∇. In order to see the latter define

(3) $$\bar{I}_x(\alpha) = \bigcup\{I_y(\alpha) :\ y \in \mathcal{L}_n\ \ \{x\}\}$$

Since is an ultrafilter, for each x either $I_x(\alpha) \in \nabla$ or $\bar{I}_x(\alpha) \in \nabla$. Now suppose that $I_x(\alpha) \notin \nabla$, for any $x \in \mathcal{L}_n$. Then for all $x \in \mathcal{L}_n$, $\bar{I}_x(\alpha) \in \nabla$, and hence, also

(4) $$\bigcap\{\bar{I}_x(\alpha) :\ x \in \mathcal{L}_x\} \in \nabla$$

But this is impossible since this intersection is the empty set. The contradiction at which we arrived yields $I_x(\alpha) \in \nabla$ for some x.

In order to complete the proof we have to verify that λ is a homomorphism from $\ll$ into $\mathcal{L}_n$, i.e. it is compatible with the operation of $\mathcal{L}_n$. Consider, for instance, $\to$. Suppose that $\lambda(\alpha \to be) = x$, i.e. $\{i \in I : \lambda_i(\alpha \to \beta) = x\} \in \nabla$. We have to show that

$$x = min(1,\ 1 - \lambda(\alpha) + \lambda(\beta))$$

(cf. 9.1 (Ł $\to$)).

Suppose that $\lambda(\alpha) = x_1, \lambda(\beta) = x_2$. Hence $\{i : \lambda_i(\alpha) = x_1\}, \{i : \lambda_i(\beta) = x_2\} \in \nabla$, which yields $\{i : \lambda_i(\alpha) = x_1 \text{ and } i : \lambda_i(\beta) = x_2\} \in \nabla$. But

$$\{i : \lambda_i(\alpha) = x_1 \text{ and } i : \lambda_i(\beta) = x_2\} \subseteq \{i : 1 - \lambda_i(\alpha) + \lambda_i(\beta) = 1 - x_1 + x_2\},$$

and hence the latter set is in ∇. This obviously implies that still larger set $\{i : min(1, 1 - \lambda_i(\alpha) + \lambda_i(\beta)) = min(1, 1 ^{a} x_1 + x_2)\} \in \nabla$ $\{i : min(1, 1 - x_1 + x_2) = x\} \in \nabla$ as desired.

The cases of $\wedge$, $\vee$, *lnot* are left to the reader. ■

As an immediate corollary to the lemma we have proved we have

30.3. Theorem. All Łukasiewicz logics $Ł_n$, n finite, are standard. ■

Verify that the operation $\to_n$ in $\mathcal{L}_n$ that correspond to the connective $\to_n$ (cf. 13.1) has the following property

$$x \to_n y = min(1,\ n - n(x + y))$$

Use this to show that

30.4. Lemma. For each $Ł_n$, n finite,

$$\alpha \in Ł_{n+1}(X,\ \beta) \text{ iff } \beta \to_n \alpha \in Łn + 1(X).$$

■

The lemma, and Theorem 30.3 yield

30.5. Theorem. All $Ł_n$, b finite, are derivational.

30.6. Corollary. For each finite n, $(Ł_n(\emptyset), MP)$ is an inferential base for $Ł_n$.

Chapter 7

Logical Matrices–Main Notions and Completeness Theorems

31. Matrices and matrix semantics

31.1. *A logical matrix* for a propositional language $\mathcal{S}$ is a couple $M = (\mathring{A}, D)$ where $\mathring{A}$ is an algebra similar to $\mathcal{S}$ and $D \subseteq A$, A being the set of elements of algebra $\mathring{A}$. The elements in D are referred to as the *designated elements* of M. If M is a logical matrix, $\bar{M}$ will denote the set of its designated elements, i.e. we put $\bar{M} = D$, if $M = (\mathring{A}, D)$.

Observe that we rule out neither that the set of designated elements $D = \emptyset$ nor that $D = A$. The matrices of the form $(\mathring{A}, \emptyset)$ and $(\mathring{A}, A)$ will be referred to as *trivial*. One element trivial matrices of the form $(\mathring{A}, A)$ (i.e. trivial matrices of this form such that $\mathring{A}$ is a one element algebra) will be denoted by τ ($\tau_{\mathcal{S}}$ if it desirable to indicate the language $\mathcal{S}$ explicitly), while one element trivial matrices of the form $(\mathring{A}, \emptyset)$ will be denoted as τ^0 (or $\tau^0_{\mathcal{S}}$, respectively). Of course, all matrices of the form τ for the same language $\mathcal{S}$ are isomorphic, the same remark applies to matrices of the form τ^0. The class of all isomorphic copies of a τ, i.e. the class of all matrices of the form τ for a fixed language $\mathcal{S}$, will be denoted by $I(\tau)$, similarly $I(\tau^0)$ denotes the class of isomorphic copies of τ^0.

31.2. Let $M = (\mathring{A}, D)$ be a logical matrix for $\mathcal{S}$. Then, homomorphisms h from $\mathcal{S}$ into $\mathring{A}$, $h \in Hom(\mathcal{S},\ \mathring{A})$ will be referred to as *valuations* in M. Similarly as in the case of truth–valuations, given an inference $H \vdash \alpha$ and a valuation (possibly partial) h we shall say that h *satisfies* (*or verifies*) $X \vdash \alpha$, if either $h(\alpha) \in \bar{M}$ or $h(\beta) \notin \bar{M}$ for some $\beta \in X$. If $h(\alpha) \notin \bar{M}$, and $h(\beta) \in \bar{M}$ for all $\beta \in X$ we shall say that h *falsifies* $X \vdash \alpha$. The valuation h will be said to *satisfy* (*verify*) X, if $h(\alpha) \in \bar{M}$, for all

$\alpha \in X$, and it will be said to *falsify* X when $h(\alpha) \notin \bar{M}$, for some $\alpha \in X$. Accordingly, h *verifies* (*falsifies*) α if $h(\alpha) \in \bar{M}(h(\alpha) \notin \bar{M})$. Occasionally, if $h(\alpha) \in \bar{M}(h(\alpha) \notin \bar{M})$ we shall refer to α as *true* (false) under h. If for some valuation h all formulas in X are true under h, the set X will be called *satisfiable*.

Under the truth condition set out, to each valuation h in M there corresponds a truth–valuation h^+ defined by

$$h^+\alpha = 1 \text{ iff } h\ \alpha \in \bar{M}.$$

The set of all truth valuations defined relatively to M will be denoted by $H(M)$.

31.3. The symbol Cn_M will denote the strongest consequence preserving truth under $H(M)$, i.e. it will serve as abbreviation for $Cn_{H(M)}$. Of course,

$\alpha \in Cn_M(X)$ iff for all valuations h in M, $h\ \alpha \in \bar{M}$, whenever $hX \subseteq \bar{M}$.

Classes of similar matrices (i.e. for the same language $\mathcal{S}$) will be referred to as (*matrix*) *semantics*. Given such a semantics $\mathbb{K}$, we define $H(\mathbb{K}) = \bigcup\{H(M) : M \in \mathbb{K}\}$, and we abbreviate $Cn_{H(\mathbb{K})}$ *as* $Cn_{\mathbb{K}}$. One verifies easily that:

a. For each $\mathbb{K}$, $H(\mathbb{K})$ is a logical space and thus $Cn_{\mathbb{K}}$ is structural,

b. $Cn_{\mathbb{K}} = \inf\{Cn_{\mathbb{K}} : M \in \mathbb{K}\}$.

31.4. The following convention will be observed. Whenever we shall use the symbol $Cn_{\mathbb{K}}$ (or Cn_M) we shall assume that the consequence it denotes is defined in a denumerable language $\mathcal{S}$ corresponding to the matrices in $\mathbb{K}$ (to M) in the sense that all matrices in $\mathbb{K}$ (the matrix M) are matrices for $\mathcal{S}$. All languages we deal with are assumed to be denumerable, if otherwise is not stated explicitly. This assumption becomes particularly important now, because consequences defined by the same class $\mathbb{K}$ of matrices but in different languages, say $\mathcal{S}_1$, $\mathcal{S}_2$, of different cardinality may have different properties. For instance $Cn_{\mathbb{K},\mathcal{S}_1}$ may be finitary while $Cn_{\mathbb{K},\mathcal{S}_2}$ not. Of course this cannot happen if $\mathcal{S}_1$, $\mathcal{S}_2$ are of the same cardinality and thus isomorphic.

31.5. Note. The idea of logical matrices goes back to Ch. Pierce [1885] and E. Schröder[1891] who were the first to apply truth tables (Werttaffein in German) in dealing with logical problems. Of course, both Pierce and Schröder restricted their attention to the classical logic only. In a rigorous and general way the notion of a logical matrix was defined by J. Łukasiewicz and A. Tarski [1930]. The definition given in this Section is (up to some inessential details) the same as that of Łukasiewicz and Tarski.

The foundations of the theory of logical matrices were set in J. Łoś [1949], and in a series of papers by J. Kalicki [1950], [1950a], [1950b], [1952]. Of considerable importance are papers by M. Wajsberg [1935], S. Jaśkowski [1936], and A. Tarski [1938]. A systematic investigations into the theory brings a survey paper by R. Suszko [1957] which, in matrices (cf. 32.2), and more generally, Lindenbaum methods in application to logical matrices.

32. First two completeness theorems

32.1. Given any logic C, denote by $Matr(C)$ the class of all matrices M, such that $C \leqslant Cn_M$. They will be referred to as *matrices of* C. Let $\mathcal{S}$ be the language of C. Observe that

a. For each $X \subseteq S$, $(\mathcal{S}, C(X))$ is a matrix for C, moreover

b. $C(\emptyset) = Cn_{(\mathcal{S},C(\emptyset))}$.

Indeed, valuations in matrices of the form $(\mathcal{S}, X)$, are endomorphisms of $\mathcal{S}$, i.e. in our terminology substitutions. Now, assume that $\alpha \in C(X)$ and consider any matrix of the form $(\mathcal{S}, C(Y))$. If for some valuation (substitution) e, $eX \subseteq C(Y)$, then by structurality of C (C is assumed to be a logic !) and condition T2 of definition 2.2 of consequence operation we have $eC(X) \subseteq C(eX) \subseteq C(C(Y)) = C(Y)$. Hence $e\ \alpha \in C(Y)$ which yields $\alpha \in Cn_{(\mathcal{S},C(Y))}$ and proves a.

From a. it follows that $C(\emptyset) \subseteq C_{(S,C(\emptyset))}$. To have the converse assume that $\alpha \notin C(\emptyset)$, and verify that under the identity substitution the assumption yields $\alpha \notin Cn_{(\mathcal{S},C(\emptyset))}$.

32.2. Proposition 32.1b is known as Lindenbaum theorem. In honor of him the matrices of the form $(\mathcal{S}, C(X))$ are called *Lindenbaum matrices*. Now, we define

$$\ll_C = \{(S, C(X)) : X \subseteq S\},$$

and we shall call $\ll_C$ *Lindenbaum bundle* (of matrices). As we shall see, Lindenbaum matrices are of enormous importance in matrix investigations.

As an almost immediate corollary to 32.1a we have

32.3. THE FIRST COMPLETENESS THEOREM. Each structural consequence C is complete with respect to $Matr(C)$, i.e. $Cn_{Matr(C)} \leqslant C$ (and thus it is adequate as well, i.e. $Cn_{Matr(C)} = C$).

PROOF. Let $\alpha \notin C(X)$. Apply the identity substitution to verify that $\alpha \notin Cn_{(\mathcal{S},C(X))}$. By 32.1a $(\mathcal{S}, C(X)) \in Matr(C)$. ■

Observe that the proof of 32.3 establishes not only 32.3 but also

32.4. THE SECOND COMPLETENESS THEOREM (R. Wójcicki [1970]) Each structural consequence C is complete (and hence adequate) with respect to Lindenbaum bundle $\ll_C$. ■

As usual, two semantics (of any kind whatsoever) are called equivalent if they determine the same consequence. Thus by 32.3 and 32.4 we have

32.5. COROLLARY. For each structural consequence C, the semantics $Matr(C)$ and $\ll_C$ are equivalent.

Since $C \leqslant C'$ iff $Th'_C \subseteq Th_C$ (cf. 4.2) we have

32.6. COROLLARY. Let C be structural. For each $C' \in [C)_0$, there exists $\mathbb{K} \subseteq \ll_C$ such that $C' = Cn_{\mathbb{K}}$.

33. Simple matrices and two more completeness theorems

33.1. We shall apply symbols of the form $\equiv_\Theta$ to denote equivalences, congruences in particular. Instead of $a =_\Theta b$ we shall rather write $a = b(\Theta)$ (a coincides with b modulo Θ).

Now, given an algebra $\mathring{A}$ and a congruence $\equiv_\Theta$ on $\mathring{A}$,

a. $|a|_\Theta$ denotes the equivalence class $\{b: \; a \equiv b(\Theta)\}$ of a,

b. $\mathring{A}/\Theta$, denotes the quotient of $\mathring{A}$ by Θ,

c. For each $A \subseteq \mathring{A}$, we put $A/\Theta = \{|a|_\Theta: \; a \in A\}$.

33.2. Let $M = (\mathring{A}, D)$ be a logical matrix. A congruence $\equiv_\Theta$ on $\mathring{A}$ is said to be a *congruence on* M (a *matrix congruence*) iff for each $a \in D$, $|a|_\Theta \subseteq D$.

If $\equiv_\Theta$ is a congruence on M, we define

$$(q) \qquad M/\Theta \;=\; (\mathring{A}/\Theta, D/\Theta),$$

and we shall refer to M/Θ as the *quotient of* M by Θ.

Observe that definition (q) makes sense when $\equiv_\Theta$ is a congruence on $\mathring{A}$ we shall merely, not necessarily on M. If $\equiv_\Theta$ is a congruence on $\mathring{A}$ we shall refer to M/Θ as an *algebraic quotient* of M. It need not be a *matrix quotient*; the latter being meant to be the quotient by a matrix congruence.

The reader will find it very easy to verify that

33.3 LEMMA. For each matrix M and for each matrix congruence $\equiv_\Theta$, M and M/Θ are equivalent. ■

A matrix M is said to be simple, if there is no congruence on M but identity. Of considerable importance is the following

33.4 LEMMA. (i) For each matrix M the relation $\equiv_{\bar{M}}$ defined on the set of elements of (the algebra of) M by $a \equiv b(\bar{M})$ iff $a \equiv b(\Theta)$ for some congruence Θ on M, is a congruence on M, and moreover,

(ii) $\equiv_{\bar{M}}$ is the greatest congruence on M, and hence $M/\bar{M}$ is a simple matrix.

PROOF. The proof goes by easy verifications. Make use of the fact that congruences on any algebra $\mathring{A}$ form a complete lattice, and moreover for each set $\{\equiv_t : t \in T\}$ of congruences, the lowest upper bound of that set is the congruence $\equiv_\Theta$ satisfying the condition: $a \equiv b(\Theta)$ iff there is a finite, possibly empty, sequence $a_1, \ldots, a_n$ of elements of $\mathring{A}$, and there are $t_1, \ldots, t_n \in T$ such that $a \equiv_{t_1} a_1, a_1 \equiv_{t_2} a_2, \ldots, a_n \equiv_{t_{n+1}} b$ (cf. e.g. P. Cohn [1965]).

Now, (ii) is an immediate corollary to (i). ■

Given any semantics $\mathbb{K}$, by $\mathbb{K}^*$ we shall denote the class $\{M/\bar{M} : M \in \mathbb{K}\}$. Still, instead of $(Matr(C))^*$ we shall prefer to use the symbol $Matr^*(C)$. Finally, we put: $M = M/\bar{M}$.

By 32.3, 32.4 and 33.4 we have the following two theorems.

33.5. THE THIRD COMPLETENESS THEOREM. Each structural consequence C is complete (and thus adequate as well) with respect to $Matr^*(C)$. ■

33.6. THE FOURTH COMPLETENESS THEOREM. Each structural consequence C is complete (and thus adequate as well) with respect to $\ll^*_C$. ■

Of course, the following counterpart of 32.6 holds true.

33.7. COROLLARY. Let C be a structural. For each $C' \in [C)_0$ there exists $\mathbb{K} \subseteq \ll^*_C$ such that $C' = Cn_{\mathbb{K}}$.

34. Łoś–Suszko's theorem

34.1. One can prove rather easily that, given any class $\mathbb{F}$ of frames, all $\mathcal{F}$ in $\mathbb{F}$ being of the same kind, one can combine frames in $\mathbb{F}$ into a single frame $\mathcal{F}_{\mathbb{F}}$ equivalent to $\mathbb{F}$. Is the same true about matrix semantics? The answer is 'no'. There are structural consequences that are determined by no single matrix. The consequences of the form Cn_M are of special kind then, and they will be referred to as *matrix consequences.* Łoś and Suszko's theorem states some necessary and sufficient conditions for a consequence to be a matrix one. But before we present the theorem, let us produce an example of a consequence which is not a matrix one.

34.2. Let $\mathcal{B}$ be the Boolean algebra of the two elements 0 and 1. Since 1 can be defined as a nullary operation in $\mathcal{B}$ we may treat $\mathcal{B}$ and the matrix $(\mathcal{B}, 1)$ as identical, still for some reason that will be clear in a moment, we shall prefer to keep the designated element explicit.

Of course, the logic $Cn_{(\mathcal{B},1)}$ determined by $(\mathcal{B}, 1)$ is the familiar classical two–valued K. But there is one more two valued logic, namely the one determined by $(\mathcal{B}, 0)$. We shall denote the latter by dK and call it *dual* with respect to K. Of course, unless the interpretation of 0 and 1 is unchanged, dK preserves falsity, not truth, and of course, both K and dK are matrix consequences.

34.3. Define $\tilde{K} = \inf(K, dK)$. Thus (cf.31.3b) $\tilde{K}$ is determined by the class $(\mathcal{B}, 1)$, $(\mathcal{B}, 0)$ of our two matrices, and thus it is the strongest logic in the standard language that preserves both truth and falsity. The rules of $\tilde{K}$ are the rules that are both rules of K and dK, and the same is true about theorems witch, incidently, amounts to that $\tilde{K}(\emptyset) = \emptyset$. Indeed the theorems of dK are just all inconsistent sentences of K, and vice versa.

34.4. The argument to the effect that $\tilde{K}$ is not a matrix consequence is rather simple. Suppose that for some matrix M, $\tilde{K} = Cn_M$. Let p and q be two distinct propositional variables. We have

$$\tilde{K}(p \vee \neg p, q \wedge \neg q) \;=\; L \tag{1}$$

At the same time, for each variable r different both from p and q, we have $r \notin \tilde{K}(p \vee \neg p)$ and $r \notin \tilde{K}(q \wedge \neg q)$. Hence for some valuations h_1, h_2 in M, $h_1(p \vee \neg p) \in \bar{M}$, $h_2(q \wedge \neg q) \in \bar{M}$ but neither $h_1 r \in \bar{M}$ nor $h_2 r \in \bar{M}$.

Select any valuation h such that $hp = h_1 p$, $hq = h_2 q$ and, say, $hr = h_1 r$ (we may put $hr = h_2 r$ as well). Of course, $h(p \vee \neg p, q \wedge \neg q) \subseteq \bar{M}$ but $hr \notin \bar{M}$, and thus

$$r \notin Cn_M(p \vee \neg p, q \wedge \neg q), \tag{2}$$

which combined with (1), contradicts the assumption that $\tilde{K} = Cn_M$, concluding the proof.

34.5. The argument we have presented suggests a necessary condition for a logic C to be a matrix logic

Call C *separable* iff given any two sets of formulas X, Y of language of C such that $Var(X) \cap Var(Y) = \emptyset$ (i.e. X and Y have no variable in common, cf.1.4) and given any variable $r \notin Var(X \cup Y)$ the following condition is satisfied

(s) If $r \in C(X \cup Y)$ then either $r \in C(X)$ or $r \in C(Y)$.

(Of course, if $Var(X \cup Y)$ involves all variables, (s) is satisfied vacuously).

34.6. The separability can take the following stronger form. A consequence C will be said to be *absolutely separable* iff for each family $\mathbb{X}$ of sets of formulas such that for any two X, $Y \in \mathbb{X}$, if $X \neq Y$ then $Var(X) \cap Var(Y) = \emptyset$, and for each propositional variable $r \notin Var(\bigcup \mathbb{X})$,

(as) If $r \in C(\bigcup \mathbb{X})$ then $r \in C(X)$, for some $X \in \mathbb{X}$.

34.7. Of course,

a. Each absolutely separable consequence is separable.
We have also

b. There are separable consequences that are not absolutely separable.

(Let C be a consequence defined, in the language that involves $\Box$ as the only connective, by the condition

$$C(X) = \begin{cases} X, \text{if for some substitution } e,\ eX \text{ is finite,} \\ \text{the set of all formulas, otherwise.} \end{cases}$$

Verify that C is separable but not absolutely separable.)

34.8. LEMMA. If C is standard then C is separable iff C is absolutely separable.

PROOF. We have 34.7a. Now assume that C is separable. If C is standard then the assumption that $r \in C(\bigcup \mathbb{X})$ implies that $r \in C(X_1 \cup \ldots \cup X_n)$ for some $X_i \in \mathbb{X}$, and hence (s) implies (as). ■

34.9. LEMMA. In order for a C to be a matrix consequence it is necessary that C be absolutely separable.

PROOF. The proof is a modification of the argument presented in 34.4. Suppose that C is not absolutely separable but still $C = Cn_M$ for some matrix M. Let r and $\mathbb{X}$ satisfy the assumptions that precede (as), and let $r \in C(\bigcup \mathbb{X})$, though $r \in C(X)$, for no $X \in \mathbb{X}$. Then, just as in 34.4, one may find a valuation h that verifies $\bigcup \mathbb{X}$ but not r. Hence $r \notin Cn_M(\bigcup \mathbb{X})$, and thus $C \neq Cn_M$ contrary to the assumption. ■

34.10. A consequence C is said to be *uniform* iff for all X, Y, α,
if

(i) $Var(X, \alpha) \cap Var(Y) = \emptyset$
(ii) $C(Y) \neq S$, S being the set of all formulas,
(iii) $\alpha \in C(X \cup Y)$, then
(iv) $\alpha \in C(X)$.

34.11. Verify that,

a. If C is uniform it is separable.

b. There are separable consequences that are not uniform.

c. There are uniform consequences that are not absolutely separable (C defined in 37.4b is of this kind).

34.12. A consequence that is both uniform and absolutely separable will be called *absolutely uniform*.

From 34.8 and 34.11a it follows

34.13. LEMMA. If C is standard then C is uniform iff C is absolutely uniform.

34.14. ŁOŚ–SUSZKO THEOREM. Let C be a logic. Then C is a matrix consequence iff C is absolutely uniform.

(J. Łoś and R. Suszko [1958]) demanded only uniformity of C. Under this condition, however, the theorem is, in general, false. The present version of the theorem was given by Wójcicki [1969], see also [1970].

PROOF. One verifies easily that if C is a matrix consequence then C is absolutely uniform. The proof of the converse is much more involved. Still, its idea is rather simple and we shall present it leaving details to the reader.

The first step consists in enlarging the set of all variables of the language $\mathcal{S}$ of C, so that it becomes possible to "separate" in the enlarged language $\mathcal{S}^+$ all non–trivial theories of C. What we mean here is that it becomes possible to assign to each $X \in Th_C$ a substitution e_X of $\mathcal{S}^+$ such that any two different non–trivial theories X, $Y \in Th_C$, $Var(e_X(X)) \cap Var(e_Y(Y)) = \emptyset$.

The consequence C is defined on $\mathcal{S}$. But we already know (cf.14.1) how to define the natural extension C^+ of C onto $\mathcal{S}^+$ (we need just to close the set $\vdash_C$ of all inferences of C under the substitutions in $\mathcal{S}^+$ and to obtain in that way an inferential base for C^+).

Verify that for each non–trivial $X \in Th_C$, $C^+(e_X X)$ is non–trivial again. Verify also that the assumption that C is absolutely separable implies that C^+ is absolutely separable. Hence

$$C^+(\bigcup\{e_X X : X \in Th_C, \quad X \neq S) \neq S^+$$

Denote this theory by $\triangle$ and verify that

$$C = Cn_{(\mathcal{S}^+,\triangle)}.$$

■

By 34.13 and 34.14 we have

34.15. COROLLARY. Let C be a standard logic. Then C is a matrix consequence iff C is uniform.

35. A few comments on Łoś–Suszko theorem

35.1. Of logics we have defined thus far only well determined Łukasiewicz logics $\zeta(\mathcal{L}_\eta)$, Johansson's minimal logic J_{min}, and, of course, $\tilde{K}$ are not matrix ones. They are not uniform.

It may be of some interest that some logics are "hereditarily" matrix in the sense that they themselves, and all their strengthenings are matrix logics. Intuitionistic logic J is of this kind. Let us outline a proof of this claim.

35.2. Lemma. Let C be a logic in $\mathcal{L}$. If for each X such that $C(X) = L$ there is a substitution e such that $eX \subseteq C(\emptyset)$ then C is absolutely uniform and thus a matrix consequence.

Proof. Straightforward. ■

35.3. The logic J has the following property. If, for any substitution e,

(1) $$ep \;\in\; J(\emptyset) \;\cup\; \{\beta : J(\beta) = L\}, \text{ for all } p,$$

then

(2) $$e\,\alpha \;\in\; J(\emptyset) \;\cup\; \{\beta : J(\beta) = L\}, \text{ for all } \alpha.$$

This property is quite common, not universal though. For instance Łukasiewicz logics of neither kind share it. What will be of importance for our argument is that this property is shared by K.

In virtue of Glivienko's theorem (cf. Glivienko [1929]) J has the following property. For all α

(3) $$J(\alpha) = L \text{ iff } K(\alpha) = L$$

(or equivalently $\neg\alpha \in J(\emptyset)$ iff $\neg\alpha \in K(\emptyset)$).

Now, (3) implies that for each X

(4) $$J(X) = L \text{ iff } K(X) = L.$$

The "if", part of (4) is obvious. In order to prove the converse that $K(X) = L$ iff $\neg(p \to p) \in K(X)$ which yields $\neg(p \to p) \in K(\beta_1 \wedge \ldots \wedge \beta_n)$ for some $\beta_1, \ldots, \beta_n \in X$. But then $\neg(\beta_1 \wedge \ldots \wedge \beta_n) \in K(\emptyset) \cap J(\emptyset)$ and thus $J(X) = L$.

Let $J(X) = L$. Consider any substitution e that has both the property (1) and sends X into $K(\emptyset)$. Such a substitution can be defined in the way we have described in 8.6. Then, $eX \subseteq J(\emptyset)$ for (1) implies (2), and we have (4). In view of Lemma 35.2 this proves that J is absolutely uniform and thus matrix consequence.

Suppose that $J \leqslant J^+$, J^+ being structural and non–trivial, then, from (4) it follows that

(5) $$J^+(X) \;=\; L \text{ iff } K(X) \;=\; L.$$

The part of the proof which now follows presupposes some results that will be obtain later.

Let $\mathbb{K}$ be a matrix semantics adequate for J^+, i.e. $J^+ = Cn_{\mathbb{K}}$. Form the direct product $\sqcap\,\mathbb{K}$ of all elements of $\mathbb{K}$ and apply (5) and Lemma 41.5 to obtain $J^+ = Cn_{\mathbb{K}} = Cn_{\sqcap\mathbb{K}}$. ■

35.4. Very interesting but technically difficult are investigations into the cardinality of matrices adequate for a given logic. That a matrix consequence need not have a denumerable matrix adequate for it, was showed for the first time by R. Suszko who proved that all matrices adequate for SCI are not denumerable. (SCI stands for Sentential Calculus with Identity, that is a logic that results from K by adding identity $=$ as a new connective. $p = q$ reads the proposition p is identical with q. For more details, cf. R. Suszko [1973] and [1973a]).

A. Wroński [1974] succeeded to prove that J has no denumerable matrix. His papers opened a series of results of similar kind. In particular, A. Wroński and E. Graczyńska [1974] examined some of intermediary logics (i.e. axiomatic strengthenings if J). Some fragments of J were examined again by A. Wroński and then by W. Dziobiak. The latter author extended his investigations onto relevant logics making use of some relations between J and C_B, C_R, C_T, C_E, C_{EM} established by R. Meyer [1973]. Again it has turned out that the fragments examined have no denumerable matrices adequate for them. It should be said that uniformity of C_E, C_R and some other relevant logics was established by L. L. Maksimova [1976].

36. Ramified matrices and ramified logics

36.1. From the philosophical standpoint, logics that are not uniform can be viewed as logics that preserve more than one logical value. For instance $\tilde{K}$ preserves both truth and falsity, and well determined Łukasiewicz logics $\overrightarrow{\zeta(\mathcal{L}_\eta)}$ preserve "degrees of truth". This remark suggests to generalize the notion of a logical matrix in the following manner.

Let $\mathring{A}$ be an algebra similar to a propositional language $\mathcal{S}$ and let $\{D_t : t \in T\}$ be a family of subsets of A. Then, the structure

(1) $$M = (\mathring{A}, \{D_t : t \in T\})$$

will be referred to as a *generalized logical matrix* (cf. R. Wójcicki [1970]) or as a *ramified matrix*. The cardinality of T will be called the *degree of ramification* of M.

The consequence operation Cn_M determined by M as well as the set $\zeta(M)$ of all sentences valid in M are defined in an expected manner:

(2) $$\alpha \in Cn_M(X) \text{ iff for all } t \in T,\ \alpha \in Cn_{(\mathring{A}, D_t)}(X)$$

and

(3) $$\alpha \in \zeta(M) \text{ iff } \alpha \in \bigcap\{\zeta(\mathring{A}, D_t) : t \in T\}.$$

Thus, each ramified matrix of the form (1) is semantically equivalent to the boundle of matrices

(2) $$\mathbb{B}_M = \{(\mathring{A}, D_t) : t \in T\},$$

and thus, the notion of a ramified matrix is perfectly dispensable. All the same it is convenient in some applications. In particular, referential matrices we are going to discuss in Chapters XI, XII will be defined as ramified matrices of certain particular kind.

If ξ is the least of all cardinals for which there exists a ξ–ramified matrix M for a logic C (a matrix the degree of ramification of which is ξ), the logic C will be said to be *ξ–ramified*. Thus, matrix consequences are just 1–ramified. The degree of ramification of $\tilde{K}$ is, of course, 2.J_{min} is 2–ramified, also. The latter result was established by J. Hawranek, cf. J. Hawranek and J. Zygmunt [1981].

Chapter 8

Matrix Vrs Algebraic Semantics

37. Implicative logics

37.1. The notion of an implicative logic due to H. Rasiowa. The definition given below differs only inessentially from that one can find in H. Rasiowa [1974].

A logic C is said to be *implicative* iff in terms of the connectives of the language of C one may define a binary connective $\to$ such that:

(i) $\alpha \to \alpha \in C(\emptyset)$, for all formulas α

(ii) The following are rules of C :

(MP) $p{,}p \to q/q$, (Modus Ponens)

(PR) $p/p \to p$, (Prefixing)

(TR) $p \to q,\ q \to r/p \to r$, (Transitivity)

$(\mathrm{RP})_\alpha$ $p \to q,\ q \to p/\alpha(p/r) \to \alpha(q/r)$. (Replacement)

(The last one of the rules is given in the form of a schema. For each α, we obtain another sequential rule. Now, $\alpha\ (q/p)$ is a substitution instance of α that results by replacing all occurrences of p by q, cf.1.4).

37.2. Theorem. Let C be an implicative logic. Then all matrices $M \in Matr^*(C)$ are of the form $(\mathcal{A}, 1_{\mathcal{A}})$, $1_{\mathcal{A}}$ being an element of $\mathcal{A}$ such that for all $a \in \mathcal{A}$

$$1_{\mathcal{A}} \;=\; a \;\to\; a.$$

Proof. Let $M = (\mathcal{A}, D)$. With the help of definition 37.1, verify that the relation $\equiv_I$ defined on $\mathcal{A}$ by $a \equiv b(I)$ iff $a \to b$, $b \to a \in D$ is a congruence on M. Assume, that for some congruence $\equiv_\Theta$ on M, $a \equiv b(\Theta)$. Since $a \to a \in D$, as it is implied by condition (i) of 37.1, then $a \to b$,

$b \to a \in D$, and hence $a \equiv b(I)$. But this means that $\equiv_I$ is the same as $\equiv_M$.

Now, suppose that $a, b \in D$. Then, in order for PR to be valid in M we must have $a \to b$, $b \to a \in D$. And this gives $a \equiv_M b$, and implies that D/M is a singleton. Of course, it is of the form $a \to a$, for any $a \in \mathcal{A}$ since, by condition (i) of 37.1, all $a \to a$ are designated elements of matrices of C. ■

37.3. If C is an implicative logic and $(\mathcal{A}, 1_{\mathcal{A}})$ is a simple matrix of C the algebra $\mathcal{A}$ will be referred to as *C–algebra* (or *algebra of C*) (cf. H. Rasiowa [1974] and the class of all such algebras will be denoted by $\mathrm{Alg}^*(C)$. More generally, given any logic (implicative or not), $\mathrm{Alg}^*(C)$ will denote the set of all algebras $\mathcal{A}$ such that for some element d of $\mathcal{A}$, $(\mathcal{A}, d)$ is a simple matrix of C. If, moreover, $(\mathcal{A}, d) \in \ll^*_C$, the algebra $\mathcal{A}$ will be referred to as a *Lindenbaum algebra*.

We shall often take advantage of the fact that $1_{\mathcal{A}}$ is definable in an uniform way in all C–algebras of an implicative C, and we shall identify C–algebras with simple matrices of C. Thus, e.g. we shall write $Cn_{\mathcal{A}}$ instead of $Cn_{(\mathcal{A},1_{\mathcal{A}})}$. By a valuation in $\mathcal{A}$ we shall mean a valuation in $(\mathcal{A}, 1_{\mathcal{A}})$, etc.

Incidentally, we have already applied this convention in the case of Łukasiewicz truth–table algebras (cf. 9.1). Łukasiewicz logics $Ł_\eta$, $\eta = 3, 4, \ldots, \omega$ are implicative with respect to $\to$ (verify that $\to$ satisfies conditions of definition 37.1) and each of Łukasiewicz truth–table algebras $\mathcal{L}_\eta$ is easily seen to be an $Ł_n$–algebra. In connection with this remark let us notice the following. All $\mathcal{L}_n$ are simple algebras, i.e. the only congruence relation they admit is identity. In general, however, in order for an algebra $\mathcal{A}$ to be a C–algebra for an implicative C it is not necessary that $\mathcal{A}$ be simple, though of course it is necessary that the matrix $(\mathcal{A}, 1_{\mathcal{A}})$ be simple. For instance 4–element Boolean algebra is a simple matrix for K but not a simple algebra, though, of course, it is a K–algebra.

Verify the following.

37.4. Lemma. Let C be an implicative logic defined in $\mathcal{S}$. Then an algebra $\mathcal{A}$ is a C–algebra iff

(i) $\mathcal{A}$ is similar to $\mathcal{S}$,

(ii) $a \to a = b \to b$, for all a, $b \in \mathcal{A}$,

(iii) $(\mathcal{A}, a \to a)$ is a matrix of C,

(iv) If $a \to b = b \to a = a \to a$, then $a = b$, for all a, $b \in \mathcal{A}$.

Proof. Repeat the argument applied in the proof of 37.2 in order to show that $\equiv_I$ defined as in that proof, but with respect to $(\mathcal{A}, a \to a)$ coincides with $\equiv_{(\mathcal{A},\ a \to a)}$. Then, observe that by (iv) the latter congruence is identity. Hence $(\mathcal{A}, a \to a)$ is simple. ■

To conclude our brief survey of the most general properties of implicative logics and their algebras let us observe that by 33.5 and 37.2 we immediately obtain.

37.5. COROLLARY. Each implicative logic C is complete (and thus adequate as well) with respect to $\mathrm{Alg}^*(C)$.

38. More on algebraic semantics

38.1 We shall say that a logic C *is l–algebraic* iff there exists a class $\mathbb{A}$ of matrices of C such that:

(i) All matrices in $\mathbb{A}$ are of the form $(\mathcal{A}, d)$, (pedantically, $(\mathcal{A}, \{d\})$ where d is an element of $\mathcal{A}$. The matrices of this form will be called *singular* and the classes of singular matrics will be referred to as an *algebraic semantic*.

(ii) C is complete with respect to $\mathbb{A}$.

The class of logics that have an adequate algebraic semantics is much larger then that of implicative logics.

Preparatory to the theorem we are going to state let us define a few notions.

38.2. a. A logic C will be said to be *implicitly implicative* iff there is an implicative logic C^+ being a conservative extension of C (i.e. the language $\mathcal{S}$ of C is a sublanguage of the language $\mathcal{S}^+$ of C^+ and $C = C^+ \restriction \mathcal{S}$, cf. 24.5).

b. A consequence C is said to be *pseudo–axiomatic* (cf. J. Łoś and R. Suszko [1958] iff $C(\emptyset) \neq \bigcap \; \{C(X) : X \neq \emptyset\}$.

c. Let C be a consequence. Formulas α, β will be said to be *congruent modulo C restricted* to X, $\alpha \equiv \beta_{(C\restriction X)}$, iff for all variables p and all formulas φ,

$$C(X,\ \alpha,\ \varphi\ (\alpha\ /\ p))\ =\ C(X,\ \beta,\ \varphi\ (\beta,\ /\ p)).$$

38.3. Let C be a consequence in $\mathcal{S}$. Then, as one easily verifies,

a. for each $X \subseteq S$, $\equiv_{(C\restriction X)}$ is a congruence on $\mathcal{S}$.
Moreover,

b. $\equiv_{(C\restriction X)}$ is congruence on $\mathcal{S}$ such that for each $\alpha \in C(X)$, $|\alpha|_{(C\restriction X)} \subseteq C(X)$, and hence it is a congruence on matrix $(\mathcal{S}, C(X))$.

c. If $(\mathcal{S}, C(X))$ is non–trivial, so is the quotient $(\mathcal{S}, C(X))/(C \restriction X)$.

Observe also the following

38.4. LEMMA. Let C be a logic. The following two conditions are equivalent:

(i) C is not pseudo–axiomatic, i.e. $C(\emptyset) = \bigcap\{C(X) : X \neq \emptyset\}$.

(ii) C is complete with respect to the set $\ll_C^0$ of all non–trivial Lindenbaum matrices of C.

PROOF. (i) $\to$ (ii). Suppose that $\alpha \notin C(X)$ for some $X \neq \emptyset$. Then, of course, $\alpha \notin Cn_{\ll_C^0}(X)$ since $\alpha \notin Cn_{(\mathcal{S},C(X))}(X)$. The identity valuation in the matrix $(\mathcal{S}, C(X))$ falsifies $X \vdash \alpha$. The case left to deal with is $\alpha \notin C(\emptyset))$. But C is not pseudo–axiomatic, hence $\alpha \notin C(X)$ for some non–empty X which implies that $\alpha \notin Cn_{(\mathcal{S},C(X))}(\emptyset)$ concluding this part of the proof.

(ii) $\to$ (i). Suppose that C is pseudo–axiomatic. Then, for some $\alpha \notin C(X)$, $\alpha \in C(X)$ for all $X \neq \emptyset$. Hence $\alpha \in Cn_{(\mathcal{S},C(X))}(\emptyset)$ for all $X \neq \emptyset$, i.e. $\alpha \in Cn_{\ll_C^0}(\emptyset)$, which contradicts (ii). ■

38.5. THEOREM. Let C be a logic that is not pseudo–axiomatic. Then the following conditions are equivalent:

(i) C is l–algebraic.

(ii) C is implicitly implicative.

(iii) All non–trivial simple matrices of C are singular.

(iv) For all α, the rule

$$p,\ q,\ \alpha\ (p\ /\ r)\ /\ \alpha\ (q\ /\ r)$$

is a rule of C.

(v) For all X, all α, β, $C(X, \alpha) = C(X, \beta)$ implies $\alpha = \beta(C \restriction X)$.

PROOF. The case of C being inconsistent is trivial. Assume then that it is not.

(i) $\to$ (ii). Let $\mathbb{A}$ be an adequate algebraic semantics for C. Given a matrix $M = (\mathcal{A}, d)$ in $\mathbb{A}$ define a matrix M^I for the language $\mathcal{S}$ of C enlarged by a new connective $\to$ (any new binary connective, if $\mathcal{S}$ already involves $\to$) by enlarging the set of operations of M with a new operation $\to$ (any new binary operation, if M already involves $\to$) defined as follows

$$a \ \to\ b \ =\ \begin{cases} b, & \text{if } a \ \neq\ b, \\ d, & \text{if } a\ =\ b. \end{cases}$$

Verify that the consequence Cn_{M^I} is implicational. Note also that, for all $X \subseteq S$, $\alpha \in S$,

$$\alpha\ \in\ Cn_{M^I}(X)\ \text{ iff }\ \alpha\ \in\ Cn_M(X). \tag{2}$$

Indeed, if X, α are as assumed than in order to establish whether $\alpha \in Cn_{M^I}(X)$ or not, we make use only of the 'old' part M of M^I.

Define $\mathbb{A}^I = \{M^I : M \in \mathbb{A}\}$. The consequence $Cn_{\mathbb{A}^I}$ is implicational, since all Cn_{M^I} are implicational (cf. 37.5). It is a conservative extension of $Cn_{\mathbb{A}^I} \restriction \mathcal{S}$, since all Cn_{M^I}, $M^I \in \mathbb{A}^I$ are conservative extensions of $Cn_{M^I} \restriction \mathcal{S}$. But, at the same time

$$(3) \qquad Cn_{\mathbb{A}^I} \restriction \mathcal{S} \;=\; Cn_{\mathbb{A}} \;=\; C,$$

and hence C is implicitly implicational.

(ii) $\to$ (iii). Let $M = (\mathcal{A}, D)$ be a non–trivial simple matrix of C. Select any d D and define a new operation $\to$ on $\mathcal{A}$ by the conditions (1) stated already above. Denote by M^I the matrix that results by extending $\mathcal{A}$ by the new operation $\to$.

Let C^I be the weakest implicative extension of C. Since C is assumed to be implicitly implicative, it is conservative. Verify that $M^I \in Matr(C^I)$. Hence $M^I/\bar{M}^I \in Matr^*(C^I)$. But, of course, $\equiv_{M^I}$ restricted to M is a congruence on M, and M has no congruences but identity. Thus $\equiv_{M^I}$ must be identity as well; the set of elements of M^I is exactly the same as that of M. If so, then

$$Matr^*(C) \;\subseteq\; Matr^*(C^I) \restriction \mathcal{S}$$

which implies (iii).

(iii) $\to$ (iv). Obvious.

(iv) $\to$ (v). Assume that $C(X,\alpha) = C(X,\beta)$. Then $C(X,\alpha, \varphi(\alpha/p)) = C(X,\alpha,\beta,\varphi(\alpha/p))$. Apply the rule defined in (iv) to get $\varphi(\beta/p) \in C(X,\alpha,\beta,\varphi(\alpha/p))$, and thus to get

$$C(X,\alpha,\varphi(\beta \,/\, p)) = C(X,\beta,\varphi(\beta \,/\, p)) \subseteq C(X,\alpha,\varphi(\alpha \,/\, p))$$

By the symmetry of the assumptions with respect to α and β the converse is valid sa well.

(v) $\to$ (i). Consider any non–trivial Lindenbaum matrix $(\mathcal{S}, C(X))$. Under the assumption, if $\alpha, \beta \in C(X)$ then $\alpha \equiv \beta(C \restriction X)$. Since (cf.38.3) $\equiv_{C\restriction X}$ is a congruence on $(\mathcal{S}, C(X))$, the quotient of $(\mathcal{S}, C(X))$ by $\equiv_{(C\restriction X)}$ is a matrix for C, non–trivial since $(\mathcal{S}, C(X))$ is assumed to be non–trivial. Moreover, it is a singular matrix, which by 38.4 and the assumptions of the theorem yields (i), concluding the proof. ■

39. Properly l–algebraic logics

39.1. Let C be a logic in $\mathcal{S}$ and let $\mathcal{A}$ be an algebra similar to $\mathcal{S}$. We shall say that a set $D \subseteq \mathcal{S}$ is a *C–filter* on $\mathcal{A}$, in symbols $D \in F_C(\mathcal{A})$, iff $(\mathcal{A}, D)$ is a C–matrix. The notion of C–filter thus defined is a straightforward generalization of that of a deductive filter, cf. H. Rasiowa [1974].

Let $M = (\mathcal{A}, D)$ be a C–matrix. Given any two elements a, b of M define

$$a \leqslant_C b \text{ iff for each } \nabla \in F_C, b \in \nabla \text{ whenever } a \in \nabla.$$

Of course, $\leqslant_C$ is a quasi–ordering and hence $\equiv_C$ defined on elements of M by

$$a \equiv b(C) \text{ iff } a \leqslant_C b \text{ and } b \leqslant_C a$$

is an equivalence. In general, however, $\equiv_C$ is not a congruence.

39.2. We shall say that C is *properly l–algebraic* iff it is algebraic and for each simple non–trivial C–matrix M there exists exactly one element d of M such that for all elements a of $M a \leqslant_C d$. The element d will be usually denoted as $\mathbf{1}_C$.

Of course, the following holds true:

39.3. THEOREM.

a. Let C be properly l–algebraic and let $M = (\mathcal{A},\ d)$ be a simple matrix for C. Then $d = \mathbf{1}_C$. Moreover

b. If $C(\emptyset) \neq \emptyset$, then $\mathbf{1}_C = h^\alpha$ for each α *in* $C(\emptyset)$ and each valuation h in M.

PROOF. Straightforward.

39.4. K and J are obvious examples of logic that are properly l–algebraic. The class $Alg^*(K)$ is the variety of all Boolean algebras. $Alg^*(J)$ is the variety of all Heyting (or pseudo–Boolean) algebras. For the proof, cf. H. Rasiowa [1974]. As a matter of fact it is not very difficult. It consist in verifying that if $\mathcal{A}$ is a J–algebra (and hence a K–algebra as well) then the ordering on $\mathcal{A}$ defined by $a \leqslant b$ iff $a \to b \in \mathbf{1}_C$ is a lattice ordering and the lattice it determines is distributive. To establish this one has to make use, in an rather obvious manner, of axioms A1–A8 (cf. 23.2) (they have to be satisfied in $\mathcal{A}$, in $(\mathcal{A}, \mathbf{1}_C)$ more accurately, under all valuations). Now if $\mathcal{A} \in Alg^*(J)$ we have to verify that $\to$, $\neg$ are the relative pseudo–complement and complement operations respectively in the lattice determined by $\leqslant$, and if $\mathcal{A} \in Alg^*(K)$, $\neg$ is the complement operation, and $\to$ is defined in the familiar manner.

39.5. The logic $K \upharpoonright \{\wedge, \vee\} = J \upharpoonright \{\wedge, \vee\}$ is an example of a logic that is properly l–algebraic and, at the same time, purely inferential, $K \upharpoonright \{\wedge, \vee\}(\emptyset) = \emptyset$. The class $Alg^*(K \upharpoonright \{\wedge, \vee\})$ consists of all distributive lattices.

The logic $\tilde{K}$ we have defined in section 34 (cf. 34.3) may serve as an example of a logic that is l–algebraic but not properly l–algebraic. Curiously enough, though not surprisingly, the class $Alg^*(\tilde{K}) = Alg^*(K) =$

$Alg^*(dK)$, i.e. it is the variety of all Boolean algebras. But of course, at the same time we have

$$Matr^*(\tilde{K}) \;=\; Matr^*(K) \;\cup\; Matr^*(dK) \;\cup\; I(\tau^0)$$

and

$$Matr^*(K) \;\cap\; Matr^*(dK) \;=\; I(\tau)$$

Why is it obvious? Because $\mathbf{1}_{\,C} \neq \mathbf{1}_{\,dC}$.

The class of logics that are not implicative at all is quite large (of course, we have in mind logics that are known and studied). For instance, relevant logics C_E, C_R, C_{RM}, da Costa paraconsistent logics C_n, Suszko's SCI, various modal logics, Łukasiewicz's well–determined logics $\overrightarrow{\zeta(\mathcal{L}_\eta)}$, all of them belong to this category.

40. A bit of philosophy

40.1. Consider the following interpretation of classical logic K. Let $\mathcal{A}$ be an atomic σ–complete Boolean algebra. Any such an algebra can be viewed as an algebra of events, to each event $a \in \mathcal{A}$ being assigned a real number $x \in [0,1]$ called the probability of a, $P(a)$. The function P satisfies familiar Kolmogorov's axioms.

Given any atom a, call the prime filter ∇_a determined by a an *elementary experiment*. The event a can be viewed as the outcome of the experiment (the answer 'yes' to the question of whether a will take place). The events b in ∇_a are all determined by a, i.e. they take place if a does. Now, each proper filter ∇ in $\mathcal{A}$ can be viewed as an experiment though not necessarily elementary one.

Now, given any space of events $\mathcal{A}$ and any experiment ∇, and given any sentence α in $\mathcal{L}$, we shall say that α is true under a valuation h in $\mathcal{A}$ (a homomorphism $h \in Hom(\mathcal{L}, \mathcal{A})$) and under the experiment ∇ iff $h\alpha \in \nabla$. Of course, another experiment ∇' can falsify α under the same valuation h, i.e. $h\alpha \notin \nabla'$. There is no surprise in that, for the events have been assumed to be probabilistic. A "good" logic is the logic that preserves truth relative to any space of events and relative to any experiment. And this good logic is K again. For each Boolean algebra $\mathcal{A}$ and for each filter ∇, $(\mathcal{A}, \nabla)/\nabla$ is a simple Boolean matrix $(\mathcal{A}/\nabla, 1_K)$. Since $\equiv_\nabla$ is a congruence on $(\mathcal{A}, \nabla)$ the matrix $(\mathcal{A}, \nabla)$ and its quotient are equivalent.

40.2. While Boolean algebras provide an adequate mathematical interpretation of the structure of "classical" events, the interpretation does not provide an adequate account of quantum events. Some of quantum quantities are complementary which amounts to that some events that are classically possible, from the point of view of Quantum Mechanics just do not exist or, some experts would subscribe rather to this standpoint, they do but they are not experimentally accessible.

If from the two standpoints we take the former one, the structure of quantum events becomes that of a Boolean algebra from which some elements were removed, and thus it is not a Boolean algebra any longer. On the ground of some physical considerations it is claimed that the structure of quantum events is that of an orto–lattice, or perhaps an ortolattice that satisfies some additional conditions.

If we take the second standpoint, the structure of quantum events is that of a Boolean algebra with some elements undefined. A rigorous account of this, rather special situation, was given in terms of Partial Boolean Algebras by S. Kochen and E.P. Specker [1965] and [1965a]. Curiously enough, Quantum Logic in the sense of Kochen and Specker is not a logic at all from our point of view. The consequence operation they define is not structural. Roughly speaking non–structurality of Kochen and Specker quantum consequence is caused by the assumption that propositional variables represent only elementary events and thus they do not refer to event (or propositions) to which compound sentences refer.

40.3. An algebra $\mathcal{A} = (\mathcal{A}, \wedge, \vee, \neg, 1)$ is said to be an *ortolattice* iff it is a lattice with 1 being the greatest element and $\neg$ being an unary operation such that:

(i) $\neg a \vee \neg b = (a \wedge b)$,

(ii) $\neg\neg a = a$,

(iii) $a \vee \neg a = 1$,

for all a, $b \in \mathcal{A}$. The class of all ortolattices is usually denoted by OL.

Now Cn_{OL} defined in $\mathcal{S} \upharpoonright \{\wedge, \vee, \neg\}$, each $\mathcal{A} \in OL$ being treated as a matrix with 1 being the designated element is called the *minimal quantum logic*. A logic C is said to be a quantum logic iff

$$Cn_{OL} \leqslant C \leqslant K$$

As it can be proved for each such a logic C there exists $\mathbb{K} \subseteq OL$ such that $C = Cn_{\mathbb{K}}$.

40.1. In view of remarks we made in subsections 40.1 and 40.2 one may doubt whether quantum logics, minimal quantum logic in particular, were define in an adequate manner. More appropriate seems to be the following approach.

Define MOL to be the class of all matrices of the form $(\mathcal{A}. \nabla)$ where $\mathcal{A}$ is an ortolattice and ∇ is a filter on $\mathcal{A}$. Define the *minimal quantum logic* as Cn_{MOL} and call a logic C a *quantum logic* if

$$Cn_{MOL} \leqslant C \leqslant K.$$

Cn_{OL} need not be an implicative logic and thus it need not coincide with Cn_{MOL}.

We leave open the question how the two classes of logics we defined are related to each other. Our intention was to make a certain philosophical point rather then to contribute to any technical problem concerning quantum logics.

40.5. NOTE. As the class OL, being a variety, is closed under S and P, we easily obtain, applying Theorem 44.7, that for each C satisfying (1) there exists $\mathbb{K} \subseteq OL$ such that $C = Cn_{\mathbb{K}}$.

Similarly the class MOL is closed under S and P. Hence, again by Theorem 44.7, for each C satisfying (2) there is a class $\mathbb{K} \subseteq MOL$ strongly adequate for C.

I own these two remarks to J. Czelakowski.

Chapter 9

The Class $Matr(C)$

41. Some operations on matrices

41.1. Matrices are relational structures (models) of a special kind and all operations on relational structures are applicable to matrices. In particular

a. Let $M = (\mathring{A},\ D)$, $N = (\mathcal{B},\ E)$. If $\mathcal{B}$ is a subalgebra of $\mathring{A}$ and $E = D \cap \mathcal{B}$, N is said to be a *submatrix* of M.

b. Let $M_t = (\mathring{A}_t,\ D_t)$, $t \in T$ be similar–matrices. The *direct product* $\sqcap_t M_t(\sqcap\{M_t : t \in T\}$, pedantically) of the matrices M_t is the matrix $(\mathring{A},\ D)$ such that $\mathring{A}$ is the direct product $\sqcap_t \mathring{A}_t$ of the algebras $\mathring{A}_t$, and D is the Cartesian product $\sqcap_t D_t$ of the sets D_t.

c. The notion of homomorphism of matrices should be applied with some care, since it can be defined in at least two different ways. The one which will be of particular importance for us is that which is sometimes (cf. e.g. A.I. Malcev [1970]) referred to as *strong homomorphism*, we shall call it also *matrix homomorphism*. Let $M = (\mathring{A},\ D)$, $N = (\mathcal{B},\ E)$ be matrices. A homomorphism h from $\mathring{A}$ into $\mathcal{B}$ is said to be a *homomorphism from M into N* iff $hD \subseteq E$. Now h is said to be a strong (or matrix) homomorphism if moreover $\overleftarrow{hE} = D$.

41.2. Observe that congruences on matrices correspond to matrix homomorphisms in the familiar way, i.e. if $\equiv_\Theta$ is congruence on M than the mapping $a \to |a|_\Theta$ from M onto M/Θ is a matrix homomorphism called the *canonical homomorphism* from M onto the quotient M/Θ. On the other hand, if h is a matrix homomorphism from M into N then $\equiv_h$ defined by $a \equiv b(h)$ iff $h(a) \equiv h(b)$ is a congruence on M. It is referred to as the *kernal* of h. Of course $M/h = h(M)$, i.e. the quotient of M by $\equiv_h$ is isomorphic with the image of M by h. $h(M)$ will be referred to as a homomorphic copy of M. As usual, one–to–one homomorphism are referred to as *embedings* and one–to–one and onto as *isomorphisms*.

41.3. The following notation will be applied. Given any class of similar matrices $\mathbb{K}$,

$S(\mathbb{K})$ – is the class of all isomorphic copies of all submatrices of matrices in $\mathbb{K}$.

$P(\mathbb{K})$ – is the class of all direct products of all subfamilies of $\mathbb{K}$, the empty family including. We define $\sqcap\emptyset$ to be the trivial one element matrix τ similar to matrices in $\mathbb{K}$.

$H_S(\mathbb{K})$ – is the class of all homomorphic copies, under matrix homomorphisms, of matrices in $\mathbb{K}$.

$\overleftarrow{H}_S(\mathbb{K})$ – is the class of all such matrices M that for some matrix homomorphism h, $h(M)$ is in $\mathbb{K}$, i.e. homomorphic counterimages, under matrix homomorphisms, of matrices in $\mathbb{K}$.

41.4. Theorem. For each logic C, the class $Matr(C)$ is closed under the operations S, P, H_S, $\overleftarrow{H}_S$.

Proof. The proof requires only easy and obvious verifications. In the case of the operation P of forming direct products the following lemma is helpful. ■

41.5. Lemma. Let M_i, $i \in I$ be a logical matrices for $\mathcal{S}$ and let $M = \sqcap_i M_i$. Then, for each set of formulas X

$$Cn_M(X) = \begin{cases} \bigcap Cn_{M_i}(X), & \text{if } X \text{ is satisfiable in all } M_i, \\ S, & \text{otherwise.} \end{cases}$$

Proof. Assume that all M_i are of the form $(\mathring{A}_i, D_i)$. For each valuation h in $\sqcap_i M_i$ and for each $i \in I$, define the mapping $h_i : \mathcal{S} \to \mathring{A}_i$, by

$$h_i\alpha \;=\; (h\;\alpha)_i, \tag{1}$$

$(h\;\alpha)_i$ being the projection of $h\alpha$ onto the i–th coordinate, and verify that h_i is a homomorphism from $\mathcal{S}$ into $\mathring{A}_i$, thus a valuation in M_i. The valuation h will be referred to as the *direct product of the valuations* h_i.

Observe that for each α

$$h\;\alpha \;\in\; D \;\text{ iff }\; h_i\;\alpha \;\in\; D_i, \;\text{ for all }\; i \;\in\; I, \tag{2}$$

which, of course, implies

$$\bigcap_i Cn_{M_i}(X) \;\subseteq\; Cn_M(X) \tag{3}$$

The converse of (3) need not hold true however. For suppose that $\alpha \in Cn_{M_i}(X)$ and at the same time for some j, X is not satisfiable in M_j, i.e. there is no valuation h_j in M_j such that $h_j(X) \subseteq D_j$. Then, clearly, X is not satisfiable in M either, and we have $Cn_M(X) = S$. But this is precisely when (3) cannot be reversed. ■

Of course, we have also the following

41.6. LEMMA. For each semantics $\mathbb{K}$,$\mathbb{K}$,$H_S(\mathbb{K})$ and $vekH_S(\mathbb{K})$ are equivalent. ■

42. Reduced products of matrices

42.1. Let $M = (\mathring{A}, D)$ be the direct product of matrices $M_i = (\mathring{A}_i, D_i)$, $i \in I$, and let ∇ be a filter on I. Then, which is a standard, very well know procedure, we define on $\mathring{A}$ the congruence $\equiv_\nabla$ by

$$a \;\equiv\; b(\nabla) \;\text{ iff }\; \{i:\; a_i \;=\; b_i\} \;\in\; \nabla$$

The matrix $(M/\nabla, D/\nabla)$ is denoted by $\sqcap_\nabla M_i$ ($\sqcap_t r\{M_i : i \in I\}$, pedantically) and is called the *product* of M_i *reduced* modulo ∇ (a *reduced product*). If ∇ is an ultrafilter, $\sqcap_i M_i$ is referred to as an *ultraproduct*.

Observe that, in general, the congruence $\equiv_\nabla$ is not a congruence on M but merely on $\mathring{A}$, and hence $\sqcap_\nabla M_i$ need not be equivalent to $\sqcap_i M_i$ (cf. 42.4).

42.2. The following notation will be applied. Let $\mathbb{K}$ be a class of similar matrices, then

$P_R(\mathbb{K})$ – is the class of all isomorphic copies of reduced products of non–empty subfamilies of $\mathbb{K}$

$P_U(\mathbb{K})$ – is the class of all isomorphic copies of ultraproducts of non–empty subfamilies of $\mathbb{K}$.

Now, in some parte of further discussion we shall need the notion of σ–reduced products, i.e. products reduced modulo a σ–filter. A filter is said to be a σ–filter iff it is closed under countable intersections. Thus, we need one symbol more:

$P_{\sigma-R}(\mathbb{K})$ – is the class of all isomorphic copies of σ–reduced products of subfamilies of $\mathbb{K}$.

42.3. Let $M_i = (\mathring{A}_i,\ D_i)$, $i \in I$ be matrices for $\mathcal{S}$ and let each h_i, $i \in I$ be a valuation in M_i. Given any filter ∇ on I define the mapping $\sqcap_\nabla h_i$ from $\mathcal{S}$ into $\sqcap_\nabla M_i$ by

$$(\sqcap_\nabla h_i)(\;\alpha\;) \;=\; |(\sqcap h_i)(\;\alpha\;)|_\nabla$$

and verify that

a. h is a valuation in $\sqcap_\nabla M_i$ iff it is of the form $\sqcap_\nabla h_i$. ($\sqcap_\nabla h_i$ will be referred to as the *product of* h_i *reduced modulo* ∇).

b. Let $M = (A,\ D) = \sqcap_\nabla M_i$, $h = \sqcap_\nabla h_i$. Then, for each α, $h\alpha \in D$ iff $\{i : h_i\alpha = 1\} \in \nabla$, or equivalently

c. $h^+\alpha = 1$ iff $\{i : h_i^+\alpha = 1\} \in \nabla$ (h^+ and h_i^+ being the truth–valuations determined by h and h_i, respectively).

42.4. LEMMA. Let M_i, $i \in I$ be similar matrices and let ∇ be a filter on I. Put $M = \sqcap_\nabla M_i$. Then, for each finite X,

$$Cn_M(X) \;=\; \bigcap\{\bigcup\{Cn_{M_i}(X):\ i\ \in\ F\}:\ F \in \nabla\}.$$

PROOF. ($\subseteq$). Assume that $\alpha \notin Cn_M(X)$, X being finite. Hence, for some valuation h in M, $hX \subseteq \bar{M}$, and $h\alpha \notin \bar{M}$. Let $h = \sqcap_\nabla h_i$ (cf. 42.3). Then both $\{i : h_i(X) \subseteq \bar{M}_i\} \in \nabla$, and $\{i : h_i(\alpha) \notin \bar{M}_i\} \in \nabla$, which implies that the intersection, denote it by F, is in ∇, too. But for each $i \in F, h_iX \subseteq \bar{M}_i$, $h_i\alpha \notin \bar{M}_i$, hence for this particular $F \in \nabla$ we have $\alpha \notin \bigcup\{Cn_{M_i}(X) : i \in F\}$, which concludes this part of the proof.

($\subseteq$). Assume that for some $F \in \nabla$, $\alpha \in Cn_{M_i}(X)$, for no $i \in F$. Select any valuations h_i, $i \in I$ such that, for all $i \in F$, $h_i(X) \subseteq \bar{M}_i$, $h_i\alpha \notin \bar{M}_i$, and let $h = \sqcap_\nabla h_i$. Then of course, $h(X) \subseteq \bar{M}$, $h\alpha \notin \bar{M}$. ∎

By Theorem 41.4 and Lemma 42.4 we obtain

42.5. THEOREM. For each standard C, the class $Matr(C)$ is closed under $S, P, H_S, \overleftarrow{H}_S, P_R, P_U$. ∎

From 42.5 it follows immediately that for each standard C, $Matr(C)$ is *quasivariety*, i.e. a class of structures definable by quasi–identies. We shall return to this point later.

43. Czelakowski's theorems

43.1. THEOREM (J. Czelakowski [1979]). Let C be a standard consequence, and let $C = Cn_{\mathbb{K}}$, for some matrix semantics $\mathbb{K}$. Then

$$Matr(C) \;=\; \overleftarrow{H}_S H_S S P_R(\mathbb{K}).$$

The proof of this theorem will be given in the next section.

43.2. In fact, 43.1 is a corollary to a more general result established in J. Czelakowski [1980]. In the complete form his result is the following.

Let C be a structural consequence defined on a language $\mathcal{S}$ on whose cardinality no restrictions are imposed. Let m be a regular infinite cardinal such that:

(i) $m \leqslant card(\mathcal{S})^+$ (i.e. the least cardinal greater than that of $\mathcal{S}$).

(ii) If $\alpha \in C(X)$ than there is $X' \subseteq X$ of the cardinality less than m, such that $\alpha \in C(X')$.

If this assumptions are satisfied then for each class $\mathbb{K}$ of matrices such that $C = Cn_{\mathbb{K}}$,

$$Matr(C) \ = \ \overleftarrow{H}_S H_S SP_{m-R}(\mathbb{K}),$$

where $P_{m--R}(\mathbb{K})$ is the class of all isomorphic copies of products of non–empty subfamilies of $\mathbb{K}$ reduced modulo filters closed under intersections of cardinality $\leqslant m$.

From this theorem it follows not only 43.1 but also

43.3. Theorem. (J. Czelakowski [1979]). Let C be a structural consequence in $\mathcal{S}$ ($\mathcal{S}$ being, as usual, assumed to be denumerable). And let for some matrix semantics $\mathbb{K}$, $C = Cn_{\mathbb{K}}$. Then

$$Matr(C) \ = \ \overleftarrow{H}_S H_S SP_{\sigma-R}(\mathbb{K})$$

We shall (cf. the next section) comment on the proof of this theorem rather than provide it in all details.

44. The proof of Czelakowski's theorems

44.1. Lemma (Czelakowski [1980]). Let $\mathbb{K}$ be a set of matrices for $\mathcal{S}$, and let $C = Cn_{\mathbb{K}}$. Then

$$\ll^*_C \ \subseteq \ H_S SP(\mathbb{K})$$

Proof. Given any theory $X = C(X) \neq S$, select matrices $M_i \in \mathbb{K}$, $i \in I$ and select valuations h_i, $i \in I$ for $\mathcal{S}$ in M_i so that

$$X \ = \ \bigcap\{\overleftarrow{h}_i(M_i) \ : \ i \ \in \ I\}.$$

The assumptions of the theorem guarantee that there are such M_i and h_i. Of course, $i \neq j$ need not imply $M_i \neq M_j$, and $M_i = M_j$ need not imply $h_i = h_j$.

Let M_X be the direct product of all M_i, $i \in I$, and let h_X be the direct product of h_i, $i \in I$, hence, $(h_X\alpha)_i = h_i\alpha$, for all α (cf. the proof of 41.5). Observe that $h_X(\mathcal{S})$ is a subalgebra of M_X and denote by N_X the matrix M_X restricted to $h_X(\mathcal{S})$, i.e.

$$N_X \ = \ (h_X(\mathcal{S}), \bar{M}_X \ \cap \ h_X(\mathcal{S})).$$

Of course, $N_X \in SP(\mathbb{K})$ for each non–trivial theory X of C.

Let $L_X = (\mathcal{S},\ C(X))$. Verify that for all α_1, α_2, β if $h_X(\alpha_1) = h_X(\alpha_2)$, then $\alpha_1 \equiv \beta(L_X)$ iff $\alpha_2 \equiv \beta(L_X)$. Then define

$$h_X\alpha \ = \ h_X\beta(\Theta_X) \ \text{ iff } \ \alpha \ \equiv \ \beta(L_X),$$

and verify that $\equiv_{\Theta_X}$ is a congruence on N_X.

In turn, define $f : N_X/\Theta_X \to L^*_X$ by

$$f(|h_X\alpha|_{\Theta_X}) \;=\; |\alpha|_{L_X},$$

and verify that f is isomorphism and hence

$$N_X/\Theta_X \;=\; L^*_X.$$

Since $N_X/\Theta_X \in H_SSP(\mathbb{K})$, then also $L^*_X \;\in\; H_SSP(\mathbb{K})$.

If $C(X) = S$, L_X is a one element trivial matrix, and hence, it is isomorphic with the direct product of the empty set (cf. 41.3).

44.2. Lemma. Let $M \in Matr(C)$ and let the cardinality of M (i.e. of the set of elements of M) be $\leqslant \aleph_0$. Then $M \in \overleftarrow{H}(\ll_C)$.

Proof. Let $\mathring{A}$ be the algebra of M and let $\mathcal{S}$ be the language of C. Since $\mathcal{S}$ is free in the class of all similar algebras and the cardinality of $\mathring{A}$ is not greater than that of $\mathcal{S}$, there is a homomorphism h from $\mathcal{S}$ onto $\mathring{A}$. Let $X_h = \overleftarrow{h}(\bar{M})$. Since $M \in Matr(C)$, $\overleftarrow{h}(\bar{M}) \in Th_C$, and thus, $(\mathcal{S}, X_h) \in \ll_C$.

Let $\equiv_{Th_h}$ be the kernal of h. Then $(\mathcal{S},\ X_h)/\Theta_h \cong \underline{M}$ and since $\equiv_{Th_h}$ is a subcongruence of $\equiv_{(\mathcal{S},X_h)}$, there is a matrix homomorphism g such that $g(M) = (\mathcal{S}, X_h)^*$. Hence $M \in \overleftarrow{H}(\ll^*_C)$. ■

The two lemmas that follow are purely model–theoretic, they hold true not only for matrices but for algebraic structures of any kind. We shall leave them without proof.

44.3. Lemma. Let $S_\omega(M)$ be the set of all submatrices generated by finite subsets of the set of elements of M. Then $M \in SP_U(S_\omega(M))$ and in particular $M \in SP_R(S_\omega(M))$.

44.4. Lemma. For each class of similar matrices $\mathbb{K}$, the class $\overleftarrow{H}_SH_SSP_R(\mathbb{K})$ is closed under all operations in terms of which it has been defined.

44.5. We are now in a position to prove Theorem 43.1. To begin with observe that by 42.5, $\mathbb{K} \subseteq Matr(C)$ implies that

$$\overleftarrow{H}_SH_SSP_R(K) \;\subseteq\; Matr(C).$$

Now, let $M \in Matr(C)$. All matrices in $S_\omega(M)$, i.e. all matrices generated by finite subsets of the set of elements of M are of cardinality $\leqslant \aleph_0$, which yields $S_\omega(M) \subseteq \overleftarrow{H}_S(\ll^*)$. Since $\ll^*_C \subseteq H_SSP(\mathbb{K})$, by the assumption of the theorem and by 44.2 we obtain $S_\omega(M) \subseteq \overleftarrow{H}_SH_SSP(\mathbb{K}) \subseteq \overleftarrow{H}_SH_SSP_R(\mathbb{K})$. But $M \in SP_RS_\omega(M)$ (cf. 44.3) which yields $M \in SP_R\overleftarrow{H}_SH_SSP_R(\mathbb{K})$. Apply 30.5 in order to get $M \in \overleftarrow{H}\neg_SH_SSP_R(\mathbb{K})$. ■

44.6. The proof of Czelakowski's theorem in its complete form, cf. 43.2, does not involves any essentially new kind of argument. In all places when reduced products are applied, one has to use m–reduced product (σ–reduced products in particular, when the language is kept denumerable), m–being the cardinal satisfying conditions (i) and (ii) of 43.2. Of course, it means that the lemmas on which the proof of 43.1 is based and which concern reduced products should be replaced by more general counterparts.

Observe that Lemma 44.4 implies the following

44.7. Corollary. (P. Wojtylak [1979]). Let $C = Cn_{\mathbb{K}}$. Then for each $C' \in [C)_0$ C' $Cn_{\mathbb{K}'}$ for some $\mathbb{K} \subseteq SP(\mathbb{K})$.

Proof. Since $\ll_C^* \in H_S SP(\mathbb{K})$ then $\ll_C$ is semantically equivalent to some $K_0 \subseteq SP(\mathbb{K})$. Now $\ll_{C'} \subseteq \ll_C$ and by Completeness Theorem 32.4, cf. also 32.6, the Corollary follows. ■

Chapter 10

The Class $Matr^*(C)$ – More On Standard Logics

45. Some more conditions for a logic to be standard

45.1. Theorem. The following conditions are equivalent

(i) C is standard,

(ii) $Matr(C)$ is closed under ultraproducts,

(iii) There is a matrix semantics $\mathbb{K}$ adequate for C closed under ultraproducts.

Proof. (i) → (ii). Select any family M_i, $i \in I$, of C–matrices. Let ∇ be an ultrafilter on I. Put $M = \sqcap_\nabla M_i$. Assume that $\alpha \in C(X)$ for some finite X. We have to show that $\alpha \in Cn_M(X)$. Select any valuation h in M. Let $h = \sqcap_\nabla h_i$ (cf. 42.3a), and suppose that $h(X) \subseteq \bar{M}$ but $h\alpha \notin \bar{M}$. Since X is finite this implies. $\{i : h_i(X) \subseteq \bar{M}_i\} \in \nabla$ and $\{i : h\alpha \notin \bar{M}_i\} \in \nabla$. Hence, the intersection of the two sets is in ∇ and thus it is not empty. This implies that for some i, $h_i(X) \subseteq \bar{M}_i$ and $h_i\alpha \notin \bar{M}_i$ contradicting the assumptions and concluding the argument

(ii) → (iii). Obvious.

(iii) → (i). If $\mathbb{K}$ is closed under ultraproducts, so is the class of all valuations in $\mathbb{K}$ and hence (cf. 42.3c) the class of truth–valuations $H(\mathbb{K})$ as well. Apply (29.4) in order to conclude the proof. ■

45.2. Theorem. Let $C = Cn_{\mathbb{K}}$ for some finite class $\mathbb{K}$ of finite matrices. Then C is standard.

Proof. Given any finite class $\mathbb{M}$ of finite algebraic structures, the class $I(\mathbb{M})$ of all isomorphic copies of members of $\mathbb{M}$ is closed under ultraproducts. Hence under the assumptions of the theorem, $I(\mathbb{K})$ is closed under ultraproducts, and since the semantics $\mathbb{K}$ and $I(\mathbb{K})$ are equivalent then, by 45.1, the theorem follows. ■

(One can show that $I(\mathbb{K})$ is closed under ultraproducts by the following argument. Let $\mathbb{K} = \{M_1, \ldots, M_k\}$ and let all N_i, $i \in I$, be in $I(\mathbb{K})$. Assume that ∇ is an ultrafilter on I. Define $I_K = \{i \in I : N_i = M_K\}$. Verify that for some k_0, $1 \leqslant k_0 \leqslant k$, $I_{k_0} \in \nabla$ and define $\nabla_{k_0} = \{F \cap I_{k_0} : F \in \nabla\}$. Now ∇_{k_0} is easily seen to be an ultrafilter on I_{k_0}. Verify that $N = \sqcap_\nabla \{N_i : i \in I\}$ is isomorphic to $N_0 = \sqcap_{\nabla k_0} \{N_i : i \in I_{k_0}\}$ and then verify that $N_0 =$ is isomorphic to M_{k_0}.) ■

In J. Łoś and R. Suszko [1958] it was proved (by a topological argument) that for each finite matrix M, Cn_M is finite. Now, of course, 45.2 is a rather straightforward corollary to Łoś–Suszko's theorem. By 31.3 $Cn_\mathbb{K} = \inf\{Cn_M : M \in \mathbb{K}\}$. Apply 5.4 to see that if all Cn_M are finitary and $\mathbb{K}$ is finite, $Cn_\mathbb{K}$ must be finitary.

46. Some corollaries to theorem 43.1

46.1. Let $\mathbb{K}_\mathcal{S}$ be the class of all matrices for $\mathcal{S}$. Define $\mathcal{S}_{PCI}$ to be the first order propositional language corresponding to $\mathbb{K}$ with identity symbol $=$ (i.e. structures in $\mathbb{K}$ are models for the language $\mathcal{S}_{PCI}$) such that:

(1) The propositional variables of $\mathcal{S}$ are the individual variables of $\mathcal{S}_{PCI}$.

(2) The connectives of $\mathcal{S}$ are the symbols for operations of $\mathcal{S}_{PCI}$.

(3) D is the unary predicate symbol of I_{PCI}, to be interpreted in each $M \in \mathbb{K}$ as $\bar{M}$.

Observe that, under the assumptions made, the set S of all formulas of the language $\mathcal{S}$ coincides with the set of all terms of $\mathcal{S}_{PCI}$. Note also that $\mathbb{K}_\mathcal{S}$ is the class of all models for $\mathcal{S}_{PCI}$.

46.2. By an *identity* in $\mathcal{S}_{PCI}$ (for all notions defined in this and the next sections, cf. Malcev [1966] we shall mean the universal closure of an atomic formula of the language, i.e. any formula of the form

$$\forall p_1 \ldots \forall p_n(\alpha = \beta) \tag{1}$$

or of the form

$$\forall p_1 \ldots \forall p_n \, D(\alpha) \tag{2}$$

α, β being terms of $\mathcal{S}_{PCI}$ (formulas of $\mathcal{S}$), and $p_1, \ldots, p_n$ being all variables appearing in α and β.

Now, by a *quasi–identity* we shall mean any sentence of $\mathcal{S}_{PCI}$ of the form

(3) $$\forall\, p_1\ \ldots\ \forall\, p_n((\sigma_1\ \wedge\ \ldots\ \wedge\ \sigma_n)\ \rightarrow\ \sigma),$$

where $\sigma_1, \ldots, \sigma_n$, σ are atomic formulas of $\mathcal{S}_{PCI}$, i.e. formulas of the form $\alpha = \beta$, or $D(\alpha)$, and $p_1, \ldots, p_n$ are all variables they involve.

46.3. Given any set χ of sentence of $\mathcal{S}_{PCI}$ denote by $\mathbb{K}(\chi)$ the class of models for χ.

a. A class of matrices $\mathbb{K}$ for $\mathcal{S}$ is said to be a *variety* iff $\mathbb{K} = \mathbb{K}(\chi)$ for a set of identities χ.

b. $\mathbb{K}$ is said to be *quasi–variety* iff $\mathbb{K} = \mathbb{K}(\chi)$ for some set χ of quasi–identities.

As we have already mentioned, the notion we have defined in 46.2 and the present section are due to Malcev, more exactly they are adaptations of the notions defined by Malcev for algebraic structures of any kind to logical matrices. Of course, all of them are generalizations of corresponding notions applicable to algebras.

Denote by $q(\mathbb{K})$ the least quasi–variety that includes $\mathbb{K}$. Let us state, without proof, in the form restricted to matrices the following well known theorem of model–theory (universal algebra).

46.4. Lemma. Let $\mathbb{K}$ be a class of similar matrices. The following conditions are equivalent

(i) $\mathbb{K} = q(\mathbb{K})$, i.e. $\mathbb{K}$ is a quasi–variety.

(ii) $SP_R(\mathbb{K}) \subseteq \mathbb{K} \cup I(\tau)$.

(iii) $SPP_u(\mathbb{K}) \subseteq \mathbb{K}$.

(The equivalence of (i) and (ii) was established by Malcev [1966], for the proof of the equivalence (ii) and (iii) cf. e.g. Grätzer and Lakser [1973].)

Observe that 46.4 applied to 43.1 yields.

46.5. Corollary. For each standard logic C, the class $Matr(C)$ is a quasi–variety. ■

Of course we have also,

46.6. Corollary. If C is standard and $C = Cn_{\mathbb{K}}$ then $Matr(C) = \overleftarrow{H}_S H_S q(\mathbb{K})$.

Proof. The least quasi–variety that includes $\mathbb{K}$ is $SP_R(\mathbb{K})$. Apply 43.1. ■

47. $Matr^*(C)$ for equivalential logics

47.1. In general, the class $Matr^*(C)$ need be closed neither under the operation S of forming submatrices nor under the operation P_R of forming reduced products. Suppose it is, and suppose that $C = Cn_{\mathbb{K}^*}$, all $M \in \mathbb{K}^*$ being simple. Assume also that C is standard. We have

(1) $$Matr(C) = \overleftarrow{H}_S H_S S P_R(\mathbb{K}^*),$$

but under the assumption we made all matrices in $SP_R(\mathbb{K}^*)$ are simple and hence (1) yields

(2) $$Matr(C) = \overleftarrow{H}_S S P_R(\mathbb{K}^*)$$

(homomorphisms applied to simple matrices are embeddings). Clearly, (2) implies

(3) $$Matr^*(C) = SP_R(\mathbb{K}^*)$$

The usefulness of the observation we made depends on whether the class of logics C such that $Matr^*(C)$ is closed under S and P_R contains sufficiently many logics of considerable significance. It does. In fact, it covers nearly all known and studied logics, cf. 47.7.

47.2. Let E be a set of formulas in two variables p, q (i.e. $Var(E) = \{p, q\}$) of a language $\mathcal{S}$. Given any α, β, define

(1) $$E(\alpha, \beta) = eE,$$

for any substitution e such that $ep = \alpha$, $eq = \beta$. Similarly, given a matrix M for $\mathcal{S}$ and any two elements a, b of M define

(2) $$E(a, b) = hE,$$

for any valuation h in M such that $hp = a$, $hq = b$.

47.3. A logic C is said to be *equivalential* (cf. T. Prucnal and A. Wroński [1974]) iff there exists a set $E(p, q)$ in two variables p, q such that for all α, β, γ the following conditions are satisfied:

(E1) $E(\alpha, \alpha) \subseteq C(\emptyset)$,

(E2) $E(\beta, \alpha) \subseteq C(E(\alpha, \beta))$,

(E3) $E(\alpha, \gamma) \subseteq C(E(\alpha, \beta) \cup E(\beta, \gamma))$,

(E4) $\beta \subseteq C(E(\alpha, \beta), \alpha)$,

(E5) For each n–ary connective § of language of C and for all $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$,

$$E(\S(\alpha_1, \ldots, \alpha_n),\ \S(\beta_1, \ \ldots, \beta_n)) \ \subseteq \ C(E(\alpha_1,\ \beta_1) \cup \ \ldots \ \cup E(\alpha_n,\ \beta_n)).$$

If, moreover, E is finite C will be said to be *finitely equivalential.*

If C is equivalential, we shall denote by E_C the largest of all sets E that satisfy conditions (E1)–(E5). Any union of sets that satisfy (E1)–(E5) is again a set that satisfies those conditions. Hence E_C is the union of all such sets.

Observe that conditions (E1)–(E5) amount to that certain sequential rules are valid for C. For instance (E2) is equivalent to that all rules of the form

$$E(p,\ q)/\gamma,$$

where $\gamma \in E(q, p)$ are valid for C. Since the rules of C are rules of all strengthenings of C we arrive at

47.4. Lemma.

a. Let C be an equivalential logic. Then for each $C' \geqslant C$, C' is also equivalential. Moreover

b. If C is finitely equivalential so are all $C' \geqslant C$. ■

47.5. Let M be a logical matrix for $\mathcal{S}$. The congruence $\equiv_M$ will be said to be *polynomial* (*finitely polynomial*) iff there is a set (a finite set resp.) of formulas E in two variables p, q such that:

$$(1) \qquad a \ \equiv \ b(M) \ \text{ iff } \ E(a,\ b) \ \subseteq \ \bar{M},$$

for all elements a, b of M.

Now, if such a set exists then, clearly, among all E that satisfy (1) there exists the largest one (the union of all E in two variables p, q, for which (1) holds true). We shall denote it by E_M.

47.6. Lemma.

a. $\equiv_M$ is (finitely) polynomial iff Cn_M is (finitely) equivalential. Moreover

b. If $\equiv_M$ is polynomial $E_M \equiv E_{Cn_M}$.

Proof. The proof requires some obvious verifications only. ■

Now, we are in a position to turn back to the problem raised in 47.1.

47.7. Lemma. (J. Czelakowski [1980]). If C is standard and finitely equivalential then $Matr^*(C)$ is closed under S and P_R.

Proof. (S). Let $M \in Matr^*(C)$. Then for all a, $b \in M$,

$$(1) \qquad a \ \equiv \ b(M) \ \text{ iff } \ a \ = \ b$$

and, on the other hand, by 47.5

$$(2) \qquad a \equiv b(M) \text{ iff } E_C(a,\ b) \subseteq \bar{M}.$$

Assume that N is a submatrix of M. Then $N \quad Matr(C)$, and hence, by 47.4,

$$(3) \qquad a \equiv b(N) \text{ iff } E_C(a,\ b) \subseteq \bar{N}$$

which combined with (1) and (2) yields

$$(4) \qquad a \equiv b(N) \text{ iff } a = b,$$

i.e. N is simple.

(P_R) Let M_i, $i \in I$, be all in $Matr^*(C)$ then

$$(5) \qquad a \equiv b(M_i) \text{ iff } a = b,$$

$$(6) \qquad a \equiv b(M_i) \text{ iff } E_C(a,\ b) \subseteq \bar{M}_i.$$

Let ∇ be a filter on I, and let $M \ = \sqcap_\nabla M_i$. By Lemma 42.4, and the assumptions of the theorem, $M \in Matr(C)$. Thus for any $\bar{a}, \bar{b} \in \sqcap_i M_i$,

$$(7) \qquad |\bar{a}|_\nabla = |\bar{b}|_\nabla\ (M) \text{ iff } E_C(|\bar{a}|_\nabla,\ |\bar{b}|_\nabla) \subseteq \bar{M}.$$

Now, C is finitely equivalential, hence for some finite subset $E \subseteq E_C$ we have

$$(8) \qquad E_C(|\bar{a}|_\nabla,\ |\bar{b}|_\nabla) \subseteq \bar{M} \text{ iff } E(|\bar{a}|_\nabla,\ |\bar{b}|_\nabla) \subseteq \bar{M}.$$

and, of course also,

$$(9) \qquad E_C(a,\ b) \subseteq \bar{M}_i \text{ iff } E(a,\ b) \subseteq M_i.$$

In turn,

$$(10) \qquad E(|\bar{a}|_\nabla,\ |\bar{b}|_\nabla) \subseteq M \text{ iff } \{i:\ E(a_i,\ b_i) \subseteq M_i\} \in \nabla,$$

which by (5), (6), (7), (9) implies

$$(11) \qquad |\bar{a}|_\nabla \equiv |\bar{b}|_\nabla\ (M) \text{ iff } |\bar{a}|_\nabla = |\bar{b}|_\nabla.$$

■

47.8. Theorem (J. Czelakowski [1980]). Let C be standard and finitely equivalential, then for each $\mathbb{K}$, such that $C = Cn_{\mathbb{K}}$,

$$Matr^*(C) \;=\; SP_R(\mathbb{K}^*) \;=\; SPP_u(\mathbb{K}^*).$$

Proof. cf. 47.1. Apply Lemma 47.7 and 46.4. ■

The vast majority of all logics which are subject to investigation are finitely equivalential. In particular, all implicative logics are in that category. Thus the scope of Theorem 47.8 is very large indeed.

Observe, that if $\mathbb{K}$ is a finite class of finite matrices then $P_u(\mathbb{K}) = I(\mathbb{K})$ (cf. 45.2). Hence, by 46.3, 46.5, and 47.8 we obtain

47.9. Theorem. Let $C = Cn_{\mathbb{K}}$, $\mathbb{K}$ being a finite set of finite matrices then

a. $Matr(C) = \overleftarrow{H}_S H_S SP(\mathbb{K})$.
   Moreover, if $\mathbb{K} = \mathbb{K}^*$, i.e. all matrices in $\mathbb{K}$ are simple and C is finitely equivalential then
b. $Matr^*(C) = SP(\mathbb{K})$. ■

The logics that are determined by finite sets of finite matrices will be referred to as *strongly finite*. We shall study them closer in Chapter XII.

48. Subdirectly irreducible matrices

48.1. The notion of a subdirectly irreducible matrix plays in the investigations concerning logical consequences a role of importance comparable to that played by subdirectly irreducible algebras in the theory of abstract algebras.

Although the notion of a subdirectly irreducible matrix can be defined for all matrices, it is convenient to restrict it to simple matrices only. Let $M = (\mathring{A}, D)$ and all $M_i = (\mathring{A}_i, D_i)$, $i \subseteq I$ be simple C–matrices.

a. We say that M is a *subdirect product* of the matrices M_i, $i \in I$ iff there exists an embedding f of M onto $\sqcap_i M_i$ such that for each $i \in I$, $\{a_i : a \in f(A)\} = A_i$, $\mathring{A}_i$ being the algebra of M_i. As usual, a_i is the projection of the element a onto the coordinate.
b. Let $\mathbb{K}$ be a class of matrices, all matrices in $\mathbb{K}$ being simple. We say that M is *subdirectly irreducible in* $\mathbb{K}$ iff for each family M_i, $i \in I$ of matrices from $\mathbb{K}$, if M is a subdirect product of M_i and f is the embedding of M into $\sqcap_i M_i$ satisfying the condition stated in the definition of a subdirect product then for some i, $\pi_i(f(M)) \cong M_i$, i.e. the projection of $f(M)$ onto the i–th coordinate is isomorphic with M_i.

48.2. A C–filter D on $\mathring{A}$ will be said to be *proper* iff $D \neq A$. D will be said to be *irreducible* iff

$$D \neq \bigcap\{D^+ \in F_C(\mathring{A}) : D \subseteq D^+ \text{ and } D \neq D^+\},$$

48.3. LEMMA. Let C be standard and let D be a C–filter on $\mathring{A}$. Then D is the intersection of all irreducible C–filters on $\mathring{A}$ that include D.

PROOF. If D is not proper then D is irreducible itself. If D is proper then select any $a \in D$ and consider any chain $\mathbb{D}$ of proper C–filters D' such that $D \subseteq D'$ and $a \notin D'$. Observe, that $\bigcup\mathbb{D}$ is C–filter. Indeed, let $\alpha \in C(X)$, for some finite X, and let for some homomorphism from the language of C into $\mathring{A}$, $hX \subseteq \bigcup\mathbb{D}$. Then, since X is finite, and $\mathbb{D}$ is a chain, $hX \subseteq D'$ for some $D' \in \mathbb{D}$. But then $h\alpha \subseteq \bigcup\mathbb{D}$. Apply Zorn's lemma to conclude that for each $a \notin D$ there exists a maximal C–filter D^+_a in the class of all C-ªfilter D' such that $D \subseteq D'$ and $a \notin D'$. Of course, D^+ is irreducible, and of course $D = \bigcap\{D^+_a : a \in D\}$. ■

48.4. LEMMA. Let $M = (\mathring{A}, D)$ be a simple C–matrix. Then the following conditions are equivalent:

(i) M is subdirectly irreducible in $Matr^*(C)$.

(ii) D is C–irreducible.

PROOF. (i)→(ii). Assume that D is not C-ªirreducible. Then $D = \bigcap\{D_i : i \in I\}$, where D_i, $i \in I$ are all C–filters on $\mathring{A}$ such that $D \subseteq D_i$ and $D \neq D_i$. Put $M_i = (\mathring{A}, D_i)$, for all $i \in I$, and define $f : M \to \sqcap M_i$ by

$$(f) \qquad (f(a))_i = a,$$

for all elements a of M. Verify that f is an embedding. In turn, define $f' : \sqcap_i M_i \to \sqcap_i M_i/\bar{M}_i$ by

$$(f'(\bar{a}))_i = |\bar{a}_i|_{\bar{M}_i},$$

and verify that f' is a matrix homomorphism. Thus

$$f' \diamond f : M \to \sqcap_i M_i/\bar{M}_i$$

is a matrix homomorphism, too. But M is simple and hence $f' \diamond f$ is one–to–one.

Now, observe that for each i and each element a of $M(f'(f(a)))_i = |a|_{\bar{M}_i}$. But D is a proper subset of D_i and hence for some element $a \in D, (f'(f(a)))_i \notin M_i/\bar{M}_i$. Consequently for no $i \in I$, $\pi_i \diamond f' \diamond f$ (π_i being, as usual, the projection on the coordinate i) is a isomorphism between M and $M_i/\bar{M}_i$. All M_i, and thus all $M_i/\bar{M}_i$ as well are in $Matr^*(C)$, which implies that M is not subdirectly irreducible.

(ii) $\to$ (i). Assume that M is not subdirectly irreducible in $Matr^*(C)$, and let $M_i \in Matr(C)$, and let $f : M \to \sqcap_i M_i$ satisfy conditions (f). Put

$$D_i^+ \;=\; \{a \;\in\; A : \;(f(a))_i \;\in\; \bar{M}_i\}, \;\; i \;\in\; I.$$

Since $\{a_i : \;\; a \in f(A)\} = \{a_i a \in \sqcap_i A_i\}$, $\mathring{A}_i$ being the algebra of M_i, the matrix $(\mathring{A}, D_i^+) \cong \pi_i(\sqcap_i M_i) = M_i$ and hence, $D_i^+ \in F_C(\mathring{A})$. Since f is an embedding, we have $D = \bigcap_i D_i^+$. But M is a simple matrix and, moreover, for no i, $\pi_i \diamond f$ is an isomorphism. Hence $D \neq D_i^+$ for any i, which implies that D is not irreducible. ■

48.5. Subdirect representation theorem. Let C be standard. Then each simple C–matrix is isomorphic to a subdirect product of a family of matrices subdirectly irreducible in $Matr^*(C)$.

Proof. Let $M = (\mathring{A}, D)$ be a simple C–matrix. Since C is assumed to be standard than by Lemma 48.3, $D = \bigcap\{D_i : i \in I\}$ where all D_i, $i \in I$ are irreducible filters such that $D \subseteq D_i$. Put $M_i = (\mathring{A}, D_i), i \in I$ and define $f : M \to \sqcap_i M_i$ by

$$(f(a))_i \;=\; a,$$

for all $a \in A$. Verify that f is an embedding and that $\pi_i(f(A)) = A$. Now, define $f' : \sqcap_i M_i \to \sqcap_i M_i/\bar{M}_i$ by

$$(f'(\bar{a}))_i \;=\; |\bar{a}_i|_{M_i}$$

and verify that f' is a matrix homomorphism "onto". Since M is simple then $f' \diamond f$ is an embedding of M into $M_i/\bar{M}_i$. One verifies easily that the fact that D_i is irreducible in $\mathring{A}$ implies that $D_i/\bar{M}_i$ is irreducible in $\mathring{A}/\bar{M}_i$. Apply Lemma 48.4 to conclude the proof. ■

Chapter 11

Referential Matrices Vrs Frames

52. K–standard referential matrices

52.1. A logic C will be said to be a *referential extension* of K iff it is selfextensional and includes K as its fragment, i.e. $C \restriction \mathcal{L} = K$. For instance, all well–determined classical modal logics based on K are of this kind (cf. 52.3).

A referential matrix $(\mathcal{R})$ for a referential extension C of K will be said to be K*–standard* iff all truth–valuations in $H(\mathcal{R})$ are classically admissible.

Verify that if $(\mathcal{R})$ is K–standard then

$$(DT) \qquad \beta \to \alpha \in Cn_{\mathcal{R}}(X) \text{ iff } \alpha \in Cn_{\mathcal{R}}(X, \beta),$$

i.e. $Cn_{(\mathcal{R})}$ satisfies Deduction Theorem.

52.2. Theorem. Let a standard logic C be a referential extension of K. The following conditions are equivalent

(i) The set $\mathbb{X}_C$ of all relatively maximal theories of C is a closure base for C.

(ii) C is complete with respect to K–standard referential matrices.

Proof. (i) → (ii). Assume (i). By Lemma 51.2, C is derivational and by Lemma 25.2, the canonical matrix $(\mathcal{R}_{\mathbb{X}C})$ of C is K–standard, which implies (ii).

(ii) → (i). Let $\mathbb{R}$ be a set of referential matrices such that $C = Cn_{\mathbb{R}}$. Form $\between \mathbb{R}$, and consider all theories of C of form $\overleftarrow{h}_t(1)$, $t \in T_{\between\mathbb{R}}$. For each such a theory X and for each α, either α or $\neg\alpha \in X$. Hence X is relatively maximal, and since $H(\between\ \mathbb{R})$ determines C, so does the set of relatively maximal theories. ■

From 52.2 it follows immediately that each referential well–determined logic based on K is complete with respect to K–standard referential matrices. The following theorem tells us more on how well–determined logics are related to referential semantics.

52.3. THEOREM. Let $\vec{M}$ be a well determined logic based on K. The following conditions are equivalent.

(i) $\vec{M}$ is a classical modal logic,

(ii) $\vec{M}$ is complete with respect to referential matrices,

(iii) $\vec{M}$ is complete with respect to K–standard referential matrices.

PROOF. We have to prove only the equivalence of (i) and (ii). Assume (i). In order to show that $\vec{M}$ is referential we shall show that it is selfextensional. The argument to this effect is inductive. Assume that $\vec{M}(\alpha) = \vec{M}(\beta)$, and observe that

$$\vec{M}(\gamma(\alpha/p) = \vec{M}(\gamma(\beta/p)) \tag{1}$$

is satisfied, if γ is a propositional variable. We have to prove that the set X_γ of all γ's that satisfy (1) is closed under the connectives of $\mathcal{L}_\Box$. Since K is an implicative logic and M inherits all rules of K, certainly X_γ is closed under standard connectives. Suppose that for some φ, $\varphi \in X_\gamma$. We have to prove that $\Box_\varphi \in X_\gamma$. Since the Deduction Theorem is valid for $\vec{M}$ (cf. 52.1) the assumption that $\varphi \in X_\gamma$ implies that

$$\varphi(\alpha/p) \longleftrightarrow \varphi(\beta/p) \in \ \vec{M}(\emptyset),$$

and this, be RE implies

$$\Box\varphi(\alpha/p) \longleftrightarrow \ \Box\varphi(\beta/p) \ \in \ \vec{M}(\emptyset),$$

or equivalently,

$$\vec{M}(\Box\varphi(\alpha/p)) \ = \ \vec{M}(\Box\varphi(\beta/p)).$$

In order to complete our remarks on relations between classical modal logics and referential matrices observe that 50.6 yields.

52.4. COROLLARY. Let C be a well–determined classical logic based on K. Then the canonical referential matrix $(\mathcal{R}_{RMTh_C})$ is adequate for C. ■

53. Referential matrices vs neighborhood frames

53.1. THEOREM. Each neighborhood frame $\mathcal{F} = (T, N)$ is semantically equivalent to the referential matrix $(\mathcal{R}_{\mathcal{F}})$ defined as follows

(i) $T_{\mathcal{R}_{\mathcal{F}}} = T$,

(ii) $\mathcal{R}_{\mathcal{F}}$ is full,

(iii) $\mathcal{R}_{\mathcal{F}}$ is K–standard,

(iv) For each element $r \in \mathcal{R}$ and for each $t \in T$,

$$\Box r(t) = 1 \text{ iff } \{t' : r(t') = 1\} \in N(t).$$

PROOF. Assume that η is a neighborhood valuation determined by $\mathcal{F}$. Since $\mathcal{R}_{\mathcal{F}}$ is full there is a valuation h in $(\mathcal{R}_{\mathcal{F}})$ such that

$$(hp)_t = \eta(t, p), \tag{1}$$

for all variables p. But $\mathcal{R}_{\mathcal{F}}$ is K–standard and η_t is classically admissible for all $t \in T$, and this allows us to conclude that the set of all formulas α for which $(h\alpha)_t = \eta(t, \alpha)$ is closed under standard connectives. We have to show that it is closed under $\Box$.

Assume that $\eta(t, \Box\alpha) = 1$. Hence $\{t' : \eta(t', \alpha) = 1\} \in N(t)$. This, under the assumption of the recursive argument, yields $\{t' : (h\alpha)_{t'} = 1\} \in N(t)$. Assume that $(h\Box\alpha)_t = 1$ and just reverse the argument to get $\eta(t, \Box\alpha) = 1$.

What we have established is $H(\mathcal{F}) \subseteq H(\mathcal{R}_{\mathcal{F}})$. But, of course, to each valuation h in $(\mathcal{R})$ there is a neighborhood valuation η such that (1) is satisfied, and hence (1) establishes one–to–one correspondence between valuations defined with respect to $\mathcal{F}$ and those in $\mathcal{R}_{\mathcal{F}}$ and we have $H(\mathcal{F}) = H(\mathcal{R}_{\mathcal{F}})$. ■

Observe that conditions (ii) and (iii) imply that $\mathcal{R}_{\mathcal{F}}$ is a complete atomic Boolean algebra ($\mathcal{R}_{\mathcal{F}}$ is isomorphic to the algebra of all subsets of T), and hence the theorem is "referential" counterpart of M. Gerson theorem [1974] which says that a Boolean frame (= a modal algebra) is isomorphic to a neighborhood frame iff it is atomic and complete.

53.2. Let $\mathcal{R}$ be a referential algebra such that for all r_1, r_2 and for each $t \in T$

$$\{t' \in T : r_1(t') = 1\} = \{t' \in T : r_2(t') = 1\}$$

iff

$$\Box r_1(t) = \Box r_2(t).$$

If this condition is satisfied we shall call $\mathcal{R}$ a *neighborhood* (*referential*) *algebra* and $(\mathcal{R})$ a *neighborhood* (*referential*) *matrix*.

Given any such algebra $\mathcal{R}$ we define $N_{\mathcal{R}}$ to be a mapping from $T_{\mathcal{R}}$ into the power set of the power set of $T_{\mathcal{R}}$ such that

$$(N_{\mathcal{R}}) \qquad \{t' \in T : r_1(t') = 1\} \in N_{\mathcal{R}}(t) \text{ iff } \Box r(t) = 1$$

and call it the *neighborhood function* of $\mathcal{R}$.

With the help of this definition we can extend the notions defined in 26.1a for neighborhood frames onto referential matrices. Thus, for instance, $t \in T_{\mathcal{R}}$ will be said to be *normal* iff $N_{\mathcal{R}}(t)$ is a filter. The notions of a *singular/regular* point of $\mathcal{R}$, a *normal/singular/regular referential algebra* (*matrix*) are to be defined by simulating the corresponding definitions for frames. Of course, all these notion are applicable only to neighborhood matrices.

53.3. In order to pursue the analogy between referential matrices and frames and on the other hand, in order to bring to light some essential differences, let us introduce the following notation.

a. Given any logic C, we shall denote by $RMatr(C)$ the class of all referential matrices for C, see also 50.1.

b. If moreover the language of C is an extension of $\mathcal{L}$, we shall denote by $R_K Matr(C)$ the class of K–standard referential matrices for C.

c. If M is a modal system based on K we define $RMatr(M)$ and $R_K Matr(M)$ to be the class of all referential matrices and, correspondingly, the class of all K–standard referential matrices $\mathcal{R}$ such that $M \subseteq \xi(\mathcal{R})$.

53.4. The reader will easily verify that for each modal system M based on K

$$(+) \qquad R_K Matr(M) = R_K Matr(\vec{M}),$$

which corresponds to 25.6b. We also have

a. $R_K Matr(M_E)$ =the class of all referential K–standard matrices for $\mathcal{L}_{\Box}$.

b. $R_K Matr(M_C)$ =the class of all regular referential K–standard matrices for $\mathcal{L}_{\Box}$.

c. $R_K Matr(M_K)$ =the class of all normal referential K–standard matrices for $\mathcal{L}_{\Box}$.

The results correspond to 27.2a, b, c. Of course, in view of 52.3 the identities a, b, and c imply corresponding completeness theorems.

54. Referential matrices vrs relational frames

54.1. The relation between normal relational frames and referential matrices is rather obvious. We already know that each relational frame (T, R) is equivalent to the neighborhood one (T, N_R); for the definition of N_R, cf. 20.5. Thus, in view of 53.1, a normal relational frame is equivalent to a referential matrix.

Let us then examine how referential matrices are related to relational frames in general sense, i.e. not necessarily normal. Hence again, as it was established by K. Segerberg [1971], for each relational frame there exists an equivalent neighborhood frame. We shall adopt Segerberg's argument for our purpose.

To begin with, let us define a referential matrix $(\mathcal{R})$ to be *augmented* iff it is full, regular, and for each normal $t \in T$

$$\bigcap N_{\mathcal{R}}(t) \ \in \ N_{\mathcal{R}}(t).$$

54.2. THEOREM. A relational frame $\mathcal{F} = (T, R, Q)$ is semantically equivalent to each referential matrix $(\mathcal{R})$ that satisfies the following conditions

(i) $T_{\mathcal{R}} = T$,

(ii) $\mathcal{R}$ is augmented,

(iii) $\mathcal{R}$ is K–standard,

(iv) Q is the set of singular elements of $\mathcal{R}$,

(v) If t is normal in $\mathcal{R}$ then tRt' iff $t' \in \bigcap N_{\mathcal{R}}(t)$.

PROOF. Assume that η is a neighborhood valuation determined by $\mathcal{F}$, and observe that for each such valuation there exists a valuation h in $\mathcal{R}_{\mathcal{F}}$ ($\mathcal{R}_{\mathcal{F}}$ is augmented and thus full) such that

(1) $$\eta(t, p) = hp)_t,$$

for all variables p and all $t \in T$. And vice versa, for each valuation h in $\mathcal{R}_{\mathcal{F}}$ there exists neighborhood valuation η such that (1) is satisfied. As in the proof of 53.1, we shall show that (1) yields

(2) $$\eta(t, \alpha) = (h\alpha)_t.$$

Of course, as it follows from (iii) and the properties of neighborhood valuations, the set of those α's for which (2) holds true is closed under standard connectives. Let us assume that (2) is valid for α, and all t. We have to prove that

(3) $$\eta(t, \Box\alpha) \ = \ (h\Box\alpha)_t,$$

We have

(4) $\eta(t, \Box\alpha) = 1$ iff $t \notin Q$ and for all t', if tRt' then $\eta(t', \alpha) = 1$

and, taking into account (iv),

(5) $(h\Box\alpha)_t = 1$ iff $t \notin Q$ and $\{t' : (h\alpha)_{t'} = 1\} = N_{\mathcal{R}}(t)$.

Let us compare the two conditions. If $t \in Q$ then we have immediately $(h\Box\alpha)_t \neq 1$ and $\eta(t, \Box\alpha) \neq 1$, and thus the case we have to consider is $t \notin Q$. Assume $\eta(t, \Box\alpha) = 1$. If so, then by (v) $\{t' : tRt'\} = \bigcap N_{\mathcal{R}}(t)$. But for each $t' \in \{t' : tRt'\}$ we have $\eta_{t'}\ \alpha = h_t, \alpha = 1$, and hence the set $\{t' : (h\alpha)_{t'} = 1\}$ is a superset of $\bigcap N_{\mathcal{R}}(t)$. $\mathcal{R}$ is regular and thus $\{t' : (h\alpha)_{t'} = 1\} \in N_{\mathcal{R}}(t)$, and we again have $h_t\Box\alpha = 1$.

Assume $(h\Box\alpha)_t = 1$. (The case $t \in Q$ is obvious, then assume that $t \notin Q$). Consider any t' such that tRt'. By (v) this implies $t' = \bigcap N_{\mathcal{R}}(t)$ and consequently $t' \in \{t' : (h\alpha)_{t'} = 1\}$. Since $(h\alpha)_{t'} = \eta(t', \alpha)$ we obtain $\eta(t', \alpha) = 1$. This holds true for all t' such that tRt', hence $\eta(t, \Box\alpha) = 1$, and the proof of a. is concluded. ■

From theorem 54.2 it follows immediately that for each relational frame, a normal relational frame in particular, there is a referential matrix semantically equivalent to it. This, of course, makes it possible to extend completeness results established for relational frames to referential matrices.

55. Comparing the relative strength of different semantics

55.1. Denote by $\mathbb{M}_N$ the class of well–determined normal modal logics based on K. We already know that:

a. Each system $M \in \mathbb{M}_N$ that is complete with respect to the relational frames is complete with respects to the neighborhood frames. But not vice versa (cf. 20.5 and 26.5) in symbols, $\mathbb{M}_N : RFrame \rightharpoonup NFrame$.

b. Each system $M \in \mathbb{M}_N$ that is complete with respect to neighborhood frames is complete with respect to K–standard referential matrices but not vice versa (cf. 53.1, 52.3, and 26.4). In symbols, $\mathbb{M}_N : NFrame \rightharpoonup R_K Matr$.

c. All system $M \in \mathbb{M}_N$ are complete with respect to K–standard referential matrices, and hence K–standard referential matrix semantics is (with respect to $\mathbb{M}_N$) equivalent to referential matrix semantics and furthermore matrix semantics (cf. 52.3). In symbols, $\mathbb{M}_N : R_K Matr \rightleftharpoons RMatr \rightleftharpoons Matr$.

Combining a., b., c. in a single diagram we obtain

$$\mathbb{M}_N \ : \ RFrame \ \rightharpoonup \ NFrame \ \rightharpoonup \ R_K Matr \ \rightleftharpoons \ RMatr \ \rightleftharpoons \ Matr$$

55.2. a. Each logic $\vec{M} \in \mathbb{M}_N$ that is complete with respect to $RFrame$ is complete with respect to $NFrame$. In symbols, $\mathbb{M}_N : RFrame \rightharpoonup NFrame$

b. By 26.7 and 52.3 we have $\mathbb{M}_N : NFrame \rightharpoonup R_K Matr$

c. And from 52.3 it follows that $\mathbb{M}_N : R_K Matr \rightleftharpoons RMatr \rightleftharpoons Matr$.

Gathering a., b., c. in a single diagram we obtain

$$\mathbb{M}_N : \ RFrame \ \rightharpoonup \ NFrame \ \rightharpoonup \ R_K Matr \ \rightleftharpoons \ RMatr \ \rightleftharpoons \ Matr$$

It goes without saying that if we consider another class of logics, the picture could be different. For instance let M be the class of all modal systems based on K. Then

$$\mathbb{M} : \ RFrame \ \rightharpoonup \ NFrame \ \rightharpoonup \ R_K Matr \ \rightharpoonup \ RMatr \ \rightharpoonup \ Matr$$

55.3. Now, what is the corresponding picture for J and its structural strengthenings i.e. for $[J)_0$.

To begin with, observe that for each epistemic frame $\mathcal{F} = (T, \leqslant)$ there exists a referential matrix $\mathcal{R}_{\mathcal{F}}$ semantically equivalent to $\mathcal{F}$. We form $\mathcal{R}_{\mathcal{F}}$ as follows.

(i) We take $\{0,1\}^T$ to be the set of elements of $\mathcal{R}_{\mathcal{F}}$. And then

(ii) we define the operations $\wedge$, $\vee$, $\rightarrow$, $\neg$ on $\{0,1\}^T$ in an expected way ($f, g \in \{0,1\}^T$):

(1) $(f \wedge g)_t = 1$ iff $f_t = 1$, and $g_t = 1$,

(2) $(f \vee g)_t = 1$ iff either $f_t = 1$ or $g_t = 1$ or $f_t = g_t = 1$.

(3) $(f \rightarrow g)_t$ iff for all $t' \geqslant t$, $g_t = 1$ whenever $f_{t'} = 1$,

(4) $(\neg f)_t = 1$ iff $f_{t'} = 1$ for no $t' \geqslant t$.

Since, as we easily verify $\mathbb{H}(\mathcal{F}) = \mathbb{H}(\mathcal{R}_{\mathcal{F}})$, we obtain

55.4. Theorem. Each epistemic frame $\mathcal{F}$ is semantically equivalent to the referential matrix $\mathcal{R}_{\mathcal{F}}$ defined by conditions (i), (ii) of 55.3. ■

55.5. Denote by PBA the class of all pseudoboolean algebras, and recall that $EFrame$ is the class if all epistemic frames.

From 55.4 it follows immediately that referential semantics in application to $[J)_0$ is at least as strong as epistemic. But, it follows from Lemma 51.2 that all intermediate "logics" $J^+(\emptyset)$, $J^+ \in [J)_0$ are complete with respect to referential matrices. Indeed, for each such J^+, $J^+(\emptyset) = \overrightarrow{(J^+(\emptyset))}(\emptyset)$ and of course $\overrightarrow{J^+(\emptyset)}$ satisfies condition (ii) of Lemma 51.1 and thus is referential. On the other hand, as it has been established by Shehtman [1977] only some of intermediate logics are complete with respect to epistemic frames. This given us $J)_0$: $EFrame \rightharpoonup RMatr$.

To examine how referential matrices and PBA are related observe that each referential J–matrix $(\mathcal{R})$ is equivalent to the bundle $(\mathcal{R}, D_t)$, $t \in T$. The latter is equivalent to the class of simple matrices $(\mathcal{R}, D_t)|D_t$, $t \in T$, and simple matrices for J are all of the form $(\mathring{A}, \mathbf{1}_J)$ where $\mathring{A}$ is a PBA. Hence $\mathcal{R}|D_t$, $t \in T$ are PBA. To combine them into a single pseudo–Boolean algebra equivalent to $(\mathcal{R})$ we have just to form the direct product $\sqcap\{\mathcal{R}|D_t : t \in T\}$. In 35.3 we have already argued that in the case of PBA for each $\mathbb{K} \subseteq PBA$, $\mathbb{K}$ and the direct product $\sqcap\mathbb{K}$ are equivalent. Since, as we have established in the previous paragraph, all $J^+(\emptyset)$, $J^+ \in [J)_0$ are complete with respect to $RMatr$, we arrive at $[J)_0 : RMatr \rightleftharpoons PBA$. Obviously, we have also $[J)_0 : PBA \rightleftharpoons Matr$.

Combining what we have established together we obtain

a. $[J)_0 : EFrame \rightharpoonup RMatr \rightleftharpoons PBA \rightleftharpoons Matr$
   We also have
b. $[J)_0 : EFrame \rightharpoonup RMatr \rightharpoonup PBA \rightleftharpoons Matr$.

We need to show only that $\rightleftharpoons$ between $RMatr$ and PBA cannot be reversed i.e. PBA–semantics is properly stronger than referential. For this purpose, consider J_{+R}, where R is any standard rule that preserves $J(\emptyset)$ but is not a rule of J. Some examples of such rules were given in 9.3. Suppose that J_{+R} is referential. Then, by Lemma 51.1, RDT holds for J_{+R}. But J_{+R} is standard and $J(\emptyset) = J_{+R}(\emptyset)$ and if RDT were valid for J_{+R} indeed, we would obtain $J = J_{+R}$ which cannot be true. Hence J_{+R} is not referential.

On the other hand, all strengthenings of J, J_{+R} in particular, are implicative (the strengthenings of an implicative logics are implicative) and hence J_{+R} is complete with respect to PBA, which concludes our argument.

Chapter 12

Referential Matrices Some General Results

49. Referential algebras

49.1. Let T be a non–empty set of points of reference (whatever they may be: time instances, space coordinates, states of affair, possible worlds). If all elements of an algebra $\mathcal{R}$ are functions of the form $r : T \to \{0,1\}$, i.e. $R \subseteq \{0,1\}^T$, the algebra $\mathcal{R}$ will be called a *referential algebra on* T. Now, if such an algebra $\mathcal{R}$ is given, we define for each $t \in T$

(1) $$D_t \;=\; \{r \;\in\; R : \; r(t) \;=\; 1\},$$

and we put

(2) $$\bar{\mathcal{R}} \;=\; \{D_t : \; t \;\in\; T\}.$$

The couple

(3) $$(\mathcal{R},\; \bar{\mathcal{R}})$$

is a ramified matrix. It will be referred to as a *referential matrix on* T.

49.2. Observe, that each referential algebra $\mathcal{R}$ determines uniquely the referential matrix $(\mathcal{R},\; \bar{\mathcal{R}})$. Therefore, in what follows, we shall denote the matrix $(\mathcal{R}, \bar{\mathcal{R}})$ as $(\mathcal{R})$ or, if it desirable to point out the set of reference points explicitly, as $(\mathcal{R})_T$. Moreover, if $\mathbb{R}$ is a class of referential algebras, we put $(\mathbb{R}) = \{(\mathcal{R}) : \mathcal{R} \in \mathbb{R}\}$. The classes of similar referential matrices will be referred to as *referential semantics*.

If $R = \{0,1\}^T$, i.e. R includes all functions $r : T \to \{0,1\}$, the algebra $\mathcal{R}$ and the corresponding matrix $(\mathcal{R})$ will be called *full*.

Let $(\mathcal{R})_T$ be a matrix for $\mathcal{S}$. Given any valuation h of $\mathcal{S}$ in $(\mathcal{R})_T$ put

$$h_t\alpha \;=\; (h\;\alpha)_t,$$

for all $t \in T$. The mapping h_t is a truth–valuation, i.e. $h_t : \mathcal{S} \to \{0,1\}$. It will be referred to as the *projection of* h *onto the coordinate* t.

49.3. Since for each formula γ, $h_t\gamma = 1$ iff $h\gamma \in D_t$, we have the following equivalence:

(r) $\alpha \in Cn_{(\mathcal{R})_T}(X)$ iff for all valuations h in $(\mathcal{R})_T$, and for all $t \in T$, $h_t\alpha = 1$ whenever $h_t\beta = 1$, for all $\beta_t \in X$.

In view of (r), we can rule out the sets D_t from technical considerations concerning logics of the form $Cn_{(\mathcal{R})}$. Of some importance also seems to be the fact that (r) tells us what kind of logics are ones of the form $Cn_{(\mathcal{R})}$: the consequence $Cn_{(\mathcal{R})}$ is the strongest logic that preserves truth at each reference point $t \in T$.

Before we provide a syntactical characteristic of referential logics (cf. next section), let us examine referential matrices in more detail.

49.4. Observe, that given any class $(\mathbb{R}) = \{(\mathcal{R}_i) : i \in I\}$ of referential matrices of the same similarity type, one may easily form a single referential algebra $\mathcal{R}$ such that $(\mathbb{R})$ and $(\mathcal{R})$ are semantically equivalent, i.e.

$$Cn_{(\mathbb{R})} \; = \; Cn_{(\mathcal{R})}.$$

The algebra $\mathcal{R}$ can be formed as follows. Let T_i be the set of reference points of $\mathcal{R}_i$. We may assume that T_i are pairwise disjoint; if necessary replace some $\mathcal{R}_i$ by isomorphic copies in order to assure that the assumption made holds true. Now define

$$T \; = \; \bigcup\{T_i : \; i \; \in \; I\} \tag{1}$$

$$\mathcal{R} \; = \; \{r \; \in \; \{0,1\}^T : r \restriction T_i \; \in \; R_i, \text{ for all } i \; \in \; I\}. \tag{2}$$

Since for each $t \in T$ there exists exactly one $i \in I$ such that $t \in T_i$, for each n–ary operation § of the algebras $\mathcal{R}_i$, we may define the corresponding operation § on $\mathcal{R}$ as follows

$$(\S_{\mathcal{R}}(r_1, \ldots, r_n))_t = (\S_{\mathcal{R}_i}(r_1 \restriction T_i, \ldots, r_n \restriction T_i))_t, \; t \; \in \; T_i. \tag{3}$$

Denote the operation that applied to $\mathbb{R}$ produces $\mathcal{R}$, by $\between$, i.e. we put

$$\mathcal{R} \; = \; \between \; \mathbb{R}.$$

We shall call it the operation of *pasting* referential algebras.

To have $\between$ defined for all classes of similar referential algebras put $\between\, \emptyset = \tau$. One easily verifies that $(\between\, \mathbb{R})$ has the property we wanted it to have.

49.5. Lemma. For each class $\mathbb{R}$ of similar referential matrices

$$Cn_{(\mathbb{R})} = Cn_{(\between\mathbb{R})}.$$

■

49.6. If $\mathcal{R} = \between \{\mathcal{R}_i : I \in I\}$, the algebras $\mathcal{R}_i$ will be referred to as *fragments* of $\mathcal{R}$. In a fully general way the notion of a fragment can be defined as follows

Given any referential algebras $\mathcal{R}_T, \mathcal{R}_S$ we say that $\mathcal{R}_S$ is the *S–fragment* of $\mathcal{R}_T$ iff $\mathcal{R}_S = \mathcal{R}_T \restriction S$, more specifically, iff

(i) $\mathcal{R}_T$ and $\mathcal{R}_S$ are similar,

(ii) $S \subseteq T$,

(iii) $r' \in R_S$ iff $r' = r \restriction S$, for some $r \in R_T$,

(iv) For each n–ary operation § of the two algebras and for all $r_1, \ldots, r_n \in R_T$,

$$\S_{\mathcal{R}}(r_1, \ldots, r_n) \restriction S = \S_{\mathcal{R}}(r_1 \restriction S, \ldots, r_n \restriction S).$$

49.7. Observe that a fragment $\mathcal{R}_S$ of $\mathcal{R}_T$ is not a subalgebra of the latter algebra but rather a quotient of the latter. Indeed, given any $S \subseteq T$ define the relation $\equiv_S$ on $\mathcal{R}$ as follows

$$r_1 = \; r_2(S) \text{ iff } r_1(t) \; = \; r_2(t), \text{for all} t \; \in \; S$$

Verify the following

a. $\equiv_S$ is a congruence on $\mathcal{R}_T$ iff there is an S–fragment $\mathcal{R}_S$ of $\mathcal{R}_T$. Moreover

b. If $\equiv_S$ is a congruence on $\mathcal{R}_T$ then the S–fragment $\mathcal{R}_T \restriction S$ of $\mathcal{R}_T$ and the quotient $\mathcal{R}_T/S$ are isomorphic.

49.8. Notice, that if $\equiv_S$ is a congruence of $\mathcal{R}_T$ it need not imply that $\equiv_S$ is a congruence on matrices $(\mathcal{R}_T, D_t)$. Still for each logic C the class of all referential C–matrices, $RMatr(C)$, is easily seen to be closed under congruence of the form $\equiv_S$. We have

a. For each referential algebra $\mathcal{R}$ and for each fragment $\mathcal{R}'$ of $\mathcal{R}$, $Cn_{(\mathcal{R})} \leqslant Cn_{(\mathcal{R}')}$. Observe also that

b. Assume that $S_i \subseteq T$ for all $i \in I$, $\bigcup_i S_i = T$, and for each S_i, there exists S_i–fragment of the referential algebra $\mathcal{R}_T$. Then $(\mathcal{R}_T)$ and $(\{\mathcal{R}_T \restriction S_i : i \in I\})$ are semantically equivalent.

(In order to show b. form $\mathcal{R}'_T = (\between \{R_T \restriction S_i : i \in I\})$ and verify that $(\mathcal{R}_T)$ and $(\mathcal{R}'_T)$ are semantically equivalent).

50. Selfextensional logics

50.1. The objective of this section is to define in syntactical terms the class of logics that are complete with respect to referential semantics, i.e. the class of logics C such that $C = Cn_{RMatr(C)}$, $RMatr(C)$ being the class of all referential C–matrices. Given any congruence C on $\mathcal{S}$ we define a congruence $\equiv_C$ on $\mathcal{S}$ by

$$\alpha \ \equiv \ \beta(C) \ \text{ iff } \ C(\gamma(\alpha/p)) \ = \ C(\gamma(\beta/p)),$$

for all variables p and formulas γ. One easily verifies (cf. comments to the proof of 33.4) that

50.2. Lemma. For each consequence C defined on $\mathcal{S}$, $\equiv_C$ is the greatest of all congruences $\equiv_\Theta$ defined on $\mathcal{S}$ such that $\alpha \equiv \beta(\Theta)$ implies $C(\alpha) = C(\beta)$.■

50.3. A logic C will be said to be *selfextensional* iff for all α, β, $C(\alpha) = C(\beta)$ implies $\alpha \equiv \beta(C)$.

With this definition we have

50.4. Theorem (R. Wójcicki [1979]). A propositional logic C is complete with respect to referential matrices iff it is selfextensional.

Proof. $(\rightarrow)$ Assume that $C = Cn_{(\mathcal{R})}$, T being the set of reference points of $\mathcal{R}$. Suppose that for some α, β, $C(\alpha) = C(\beta)$. Hence $h_t\alpha = h_t\beta$ for all valuations h in $\mathcal{R}$ and for all $t \in T$. Clearly, this yields

(1) $$h(\gamma(\alpha/p)) = h(\gamma(\beta/p))$$

for all valuations h, all γ and all p. Now (1) implies $\alpha \equiv \beta(C)$, and we conclude that C is selfextensional.

$(\longleftarrow)$ Select T to be a closure base of C. For instance, put $T = Th_C$. Now, for each formula α define α^T to be the function from T into $\{0,1\}$ such that

(2) $$\alpha^T(X) \ = \ 1 \ \text{ iff } \ \alpha \ \in \ X.$$

Put

(3) $$R \ = \ \{\alpha^T : \ \alpha \ \in \ S\}$$

$\mathcal{S}$ being the language of C. In turn, for each n–ary connective § of $\mathcal{S}$ define the operation $\S_{\mathcal{R}}$ by

(4) $$\S_{\mathcal{R}}(\alpha_1^T, \ldots, \alpha_n^T) = (\S(\alpha_1, \ldots, \alpha_n))^T,$$

for all $\alpha_1, \ldots, \alpha_n$. Of course, we have to verify that (4) does not lead to contradiction, i.e. $\S_{\mathcal{R}}$ is well–defined. In order to be so, we put have

(5) $$(\S(\alpha_1, \ldots, \alpha_n))^T \ = \ (\S(\beta_1, \ldots, \beta_n))^T,$$

whenever

$$\alpha_i^T \;=\; \beta_i^T,\ \text{for all}\ i \;=\; 1,\ldots,n. \tag{6}$$

Let us verify that (6) implies (5). Since T is assumed to be a closure base for C, (6) is equivalent to

$$C(\alpha_i) \;=\; C(\beta_i),\ i \;=\; 1,\ldots,n \tag{7}$$

which implies

$$\alpha_i \;\equiv\; \beta_i(C),\ i \;=\; 1,\ldots,n \tag{8}$$

and hence,

$$\S(\alpha_1,\ldots,\alpha_n) \;\equiv\; \S(\beta_1,\ldots,\beta_n)(C). \tag{9}$$

Once more exploit the fact that T is a closure base for C, this time in order to get (5) from (9).

The set R supplied with the operations $\S_{\mathcal{R}}$ corresponding to the connectives of $\mathcal{S}$, forms the referential algebra $\mathcal{R}$. We shall show that $C = Cn_{(\mathcal{R})}$.

Select any formula α and any set of formulas X and assume that

$$\alpha \;\in\; C(X) \tag{10}$$

Let h be a valuation in $(\mathcal{R})$. For each propositional variable p, there exists a formula α_p such that

$$hp \;=\; \alpha_p^T. \tag{11}$$

Define a substitution e_h in $\mathcal{S}$ by

$$e_h p \;=\; \alpha_p. \tag{12}$$

Since C is structural, we have

$$e_h\alpha \;\in\; C(e_h X). \tag{13}$$

Now, observe that for each β,

$$h\beta \;=\; (e_h\beta)^T. \tag{14}$$

Indeed, by (11) and (12) the identity (14) is valid at least when β is a propositional variable. This allows us to apply a recurrsive argument. Assume that (14) is valid for $\beta = \beta_i$, for $i = 1,\ldots,n$ and consider any formula of the form $\S(\beta_1,\ldots,\beta_n)$, $\S$ being an n-ary connective of $\mathcal{S}$. We have $h\S(\beta_1,\ldots,\beta_n) = \S_{\mathcal{R}}(h\beta_1,\ldots,h\beta_n) = \S_{\mathcal{R}}((e_h\beta_1)^T,\ldots,(e_h\beta_n)^T) = (\S(e_h\beta_1,\ldots,e_h\beta_n))^T = (e_h\S(\beta_1,\ldots,\beta_n))^T$ which establishes (14).

Suppose that for some $Y \in T$,

(15) $$(h\beta)_Y = 1 \text{ for all } \beta \in X.$$

By (3) and (14) we obtain $e_h\beta \in Y$ for all $\beta \in X$ which, by (13) and the assumption that T is a closure base for C, yields

(16) $$e_h\alpha \in Y.$$

Apply (3) and (14) once again to obtain

(17) $$(h\alpha)_Y = 1$$

Thus (15) implies (17) which by 50.3 (r) gives

(18) $$\alpha \in Cn_{(\mathcal{R})}(X).$$

Now, suppose that (10) is not true, i.e. $\alpha \notin C(X)$. Then $\alpha \notin Y$, for some $Y \in T$ such that $X \subseteq Y$. Define h to be the valuation in $(\mathcal{R})$ such that

(19) $$hp = p^T,$$

for all variables p. This, by the same argument that leads from (11) to (13), implies that

(20) $$h\beta = \beta^T,$$

for all β, and hence $(h\gamma)_Y = 1$ for all $\gamma \in X$, and $(h\alpha)_Y = 0$. This yields $\alpha \notin Cn_{(\mathcal{R})}(X)$, concluding the proof. ∎ A logic C that is complete with respect to referential matrice, i.e. it is selfextensional, will be occaxionally refer to as *referential*.

50.5. Given any closure base $\mathbb{X}$ for a logic C denote by $(\mathcal{R}_{\mathbb{X}})$ the referential matrix determined by $\mathbb{X}$. The matrices of this form will be referred to as *canonical*. Observe that, in general, there is more than one canonical matrix for C.

The proof of 50.4 is, at the same time, a proof of

50.6. Theorem. Let C be a self–extensional logic and let $\mathbb{X}$ be a closure base for C. Then the canonical referential matrix $(\mathcal{R}_{\mathbb{X}})$ is adequate for C. ∎

51. An useful lemma

51.1. Define C to be *quasi–implicative* iff C satisfies all conditions by means of which an implicational logic has been defined (cf. 37.1) with the exception that Replacement Rule is not demanded to be a rule of C, but merely a permissible rule of C, i.e. a rule that preserves $C(\emptyset)$.

Explicitly: C is *quasi–implicative* (with respect to $\rightarrow$) iff

(i) $\alpha \to \alpha \in C(\emptyset)$,

(ii) MP, PR, TR are rules of C,

(iii) For each α, $(RP)_\alpha$ preserves $C(\emptyset)$.

51.2. LEMMA. Let C be a logical such that $J \restriction \{\wedge, \to\} \leqslant C \restriction \{\wedge, \to\}$. Then the following two conditions are equivalent

(i) C is selfextensional,

(ii) C is quasi–implicative with respect to $\to$, and for all α and for all finite sets of formulas X_f,

$$(RDT) \qquad \beta \in C(X_f, \alpha) \text{ iff } \alpha \ \to \ \beta \ \in \ C(X_f)$$

(we shall refer to RDT as the *Restricted Deduction Theorem*).

PROOF. (i) $\to$ (ii). Assume (i), an suppose that $\beta \in C(X_f, \alpha)$, for some finite X_f. Let γ be a conjunction of the elements of X_f, if $X_f \neq \emptyset$, or else let $\gamma = \alpha \to \alpha$. We have

$$(1) \qquad C(\gamma \wedge \alpha \wedge \beta) \ = \ C(\gamma \wedge \alpha)$$

Make use of selfextensionality of C to get

$$(2) \qquad C((\gamma \wedge \alpha) \to (\gamma \wedge \alpha)) \ = \ C((\gamma \wedge \alpha) \to (\gamma \wedge \alpha \wedge \beta)),$$

with yields

$$(3) \qquad (\gamma \wedge \alpha) \to (\gamma \wedge \alpha \wedge \beta) \ \in C(\emptyset).$$

Now observe that we have

$$(4) \qquad (\gamma \wedge \alpha) \to (\gamma \wedge \alpha \wedge \beta) \ \to \ (\gamma \to (\alpha \to \beta)) \ \in \ J(\emptyset).$$

Since $J \restriction \{\wedge, \to\} \leqslant C \restriction \{\wedge, \to\}$, (3), (4) and MP yield

$$(5) \qquad \gamma \to (\alpha \to \beta) \ \in \ C(\emptyset).$$

or equivalently

$$(6) \qquad \alpha \to \beta \ \in \ C(X_f).$$

Apply MP to get $\beta \in C(X_f, \alpha)$ from (6). Thus RDT holds true for C.

The only thing we have to prove in order to show that C is quasi–implicative is that for all α of the language of C, $(RP)_\alpha$ is a permissible rule of C. Since $J \restriction \{\wedge, \to\} \leqslant C \restriction \{\wedge, \to\}$ all the remaining rules that a quasi–implicative logic is postulated to satisfy (cf. 51.1) are rules of C.

Select any α, and assume that $(\beta_1 \longleftrightarrow \beta_2) \in C(\emptyset)$, for some formulas β_1, β_2. Obviously, we have $C(\beta_1) = C(\beta_2)$. This yields $C(\alpha(\beta_1/p)) = C(\alpha(\beta_2/p))$. But RDT holds true for C and we have $\alpha(\beta_1/p)) \to \alpha(\beta_2/p) \in C(\emptyset)$ as required.

(ii) $\to$ (i). This part of the proof is straightforward. Let $C(\alpha) = C(\beta)$. Then $\alpha \longleftrightarrow \beta \in C(\emptyset)$ by RDT and by $(RP)_\gamma$ we have $\gamma(\alpha/p) \longleftrightarrow \gamma(\beta/p) \in C(\emptyset)$, i.e. $C(\gamma(\alpha/p)) = C(\gamma(\beta/p))$. ■

The assumption of the lemma is too strong. Only some of the theorems of $J \restriction \{\wedge, \to\}$ are needed to carry out the proof. Of course, one can easily trace them down and formulate the Lemma in an adequate manner.

51.3. From the lemma we have proved it follows e.g. that J_{min}, H, J, numerous strengthenings of those logics, K, numerous modal logics are selfextensional. Thus, no doubt, the class of selfextensional logics is very large.

Łukasiewicz's truth preserving logics $Ł_\eta$, the logic with constructive falsity N are, but many others, not selfextensional. Observe, that both $Ł_n$, n–finite, and N are implicative and at the same time they involve implicity the connective $\Rightarrow$ with respect to which RDT (even DT) holds true; $\Rightarrow$ is definable within those logics. The connective $\Rightarrow$ differs from this with respect to which the logics in question are implicative, and hence they do not satisfy the condition (ii) of 51.2.

Chapter 13

Logics Strongly Finite

56. A syntactical test for strong finiteness

56.1. A logic C is said to be *strongly finite* iff $C = Cn_{\mathbb{K}}$ for some finite class of finite matrices. The notion is an inferential counterpart of a tabular logic. We say that C is *tabular* iff $C(\emptyset) = \xi(M)$ for some finite matrix $M \in Matr(C)$. Often the condition for M to be a C–matrix is dropped out, and the definition becomes less restrictive. It is perhaps worth–while noticing that if $C(\emptyset) = \xi(\mathbb{K})$ for some finite class of finite C–matrices, then there is a single finite C–matrix that establishes tabularity of C. E.g. the product $\sqcap\mathbb{K}$ of all matrices of $\mathbb{K}$ is such a matrix. In the case of strongly finite logic quite often $\mathbb{K}$ cannot be replaced by a single matrix.

56.2. We shall need some auxiliary notion and symbols

a. Given any language $\mathcal{S}$ and given any cardinal number ξ, we shall denote by $\mathcal{S}(\xi)$ the language whose all propositional variables are p_ζ, $\zeta \leqslant \xi$, and whose connectives are exactly such as those of $\mathcal{S}$. $S(\xi)$ will denote the set of all formulas of $\mathcal{S}(\xi)$.

b. Now, given the languages $\mathcal{S}$ and $\mathcal{S}(\xi)$ and given any structural consequence C on $\mathcal{S}$ we define a consequence $C^{(\xi)}$ on $\mathcal{S}$ by

$$\alpha \in C^{(\xi)}(X) \text{ iff for all } e \in Hom(\mathcal{S}, \mathcal{S}(\xi)),\ e\alpha \in C(eX),$$

for all $\alpha \in S$, and all $X \subseteq S$.

56.3. The reader should find it easily to verify that

a. If $card(\mathcal{S}) \leqslant \xi$ than $C^{(\xi)} = C$

b. $C^{(1)} \geqslant C^{(2)} \geqslant \ldots \geqslant C$

c. If $\xi \leqslant card(\mathcal{S})$ then $C \leqslant C^{(\xi)}$.

56.4. THEOREM. (R. Wójcicki [1973]). Let C be a logic defined in $\mathcal{S}$. C is strongly finite iff there exists $k \in \omega$ such that for all X and all α the following two conditions are satisfied

(i) $C = C^{(k)}$,

(ii) The set $S(k)|C$ of all equivalence classes $|\alpha|_C$, $\alpha \in S(k)$ is finite. (For the definition of $\equiv_C$ see 50.1).

PROOF. ($\rightarrow$) Assume that $C = Cn_{\mathbb{K}}$, where $\mathbb{K} = \{M_1, \ldots, M_n\}$ all M_i being finite. Let m_i be the cardinality of M_i. Put $m = max(m_1, \ldots, m_n)$. Since C is structural, certainly we have, for all k

(1) Iff $\alpha \in C(X)$ then for each $e \in Hom(\mathcal{S}, \mathcal{S}(k))$, $e\alpha \in C(eX)$

We have to prove the converse. Assume that $\alpha \notin C(X)$. This gives $\alpha \notin Cn_{M_i}(X)$ for some M_i (cf.31.3b). Select a valuation h in M_i such that $hX \subseteq \bar{M}_i, h\alpha \notin M_i$. But, of course, for some $r \leqslant m, h(Var(\mathcal{S})) \subseteq \{a_1, \ldots, a_r\}$, a_j being all elements of M_i. Define a substitution e_h in $\mathcal{S}$ by

(2) $$e_h p = p_i,$$

whenever $hp = a_i$, and select valuation h' in M_i such that

(3) $$h' e_h p = hp.$$

Obviously, $h' e_h X \subseteq \bar{M}_i$, $h' e_h \alpha \notin bar M_i$, and we arrive at $e_h\alpha \notin C(e_h X)$. The argument shows that if we put $k = m$, the converse of (1) will hold true.

In order to complete the proof we have to show that there are finitely many equivalence classes $|\alpha|_C, \alpha \in S(k)$. Define a relation $\equiv_{(K)}$ on formulas of $\mathcal{S}$ by

$$\alpha \equiv \beta(\mathbb{K}) \text{ iff for all } M_i \in \mathbb{K} \text{ and all valuations } h \text{ in } M_i,\ h\,\alpha = h\,\beta.$$

Verify that $\equiv_{(\mathbb{K})}$ is a congruence, and by Lemma 50.2 $\equiv_{(\mathbb{K})} \subseteq \equiv_C$. Since, what is obvious, there are finitely many equivalence classes of $\equiv_{(\mathbb{K})}$ on $S(k)$ the quotient $S(k)|C$ is finite, too.

($\longleftarrow$). Assume (i) and (ii) and consider the quotient matrices

$$(\mathcal{S}(k), C(X) \wedge S(k))/C, \quad X \subseteq S$$

For each $X \subseteq S$, denote the corresponding matrix of those defined above by $M_{X,\emptyset}$. As an immediate corollary to Lemma 56.5, we are going to state below, we obtain $C = Cn_{\{M_{X,\emptyset}: X \subseteq S\}}$. Now, in virtue of (i) and (ii) each $M_{X,\emptyset}$ is finite and so is the set of all $M_{X,\emptyset}$. ■

56.5. LEMMA. Assume (i) and (ii) of 56.4, and for each X, $Y \subseteq S(k)$, define the quotient matrix

$$M_{X,Y} \;=\; (\mathcal{S}(k),\; (C(X \;\cup\; Sb(Y)) \;\cap\; S(k)))|C$$

Then, the following two conditions are equivalent

(i) $e\alpha \in C(X \wedge Sb(Y))$, for all $e \in Hom(\mathcal{S}, \mathcal{S}(k))$

(ii) $\alpha \in Cn_{M_{X,Y}}(X)$.

PROOF. (i) $\to$ (ii). Assume (i), and assume that for some $M_{Z,Y}$ and some valuation h in that matrix, $h(X) \subseteq \bar{M}_{Z,Y}$. Select any substitution $e_h \in Hom(\mathcal{S}, \mathcal{S}(k))$ such that

(1) $$|e_h p|_C \;=\; hp,$$

for all variables p of $\mathcal{S}$. Of course, (1) implies

(2) $$|e_h \beta|_C \;=\; h\beta,$$

for all $\beta \in S$. The assumption $h(X) \subseteq \bar{M}_{Z,Y}$ yields

(3) $$e_h X \;\subseteq\; C(Z \;\cup\; Sb(Y)) \;\cap\; S(k)$$

Now (3) and (i) imply

(3) $$e_h\; \alpha \;\in\; C(Z \;\cup\; Sb(Y)) \;\cap\; S(k),$$

and thus $h\alpha \in \bar{M}_{Z,Y}$, which gives (ii).

(ii) $\to$ (i). Assume that (i) is false. Let $e \in Hom(\mathcal{S}, \mathcal{S}(k))$ such that (4) is not true. Define a valuation h_e in $M_{X,Y}$ by the condition

(5) $$h_e p \;=\; |ep|_C,$$

for each variable p. This yields $h_e\beta = |e\beta|_C$, for every formula $\beta \in S$. Hence $h_e X \subseteq \bar{M}_{X,Y}$, $h_e\alpha \notin \bar{M}_{X,Y}$ and we arrive at not (ii). ■

57. The lattices of strengthenings of a strongly finite consequences

57.1. THEOREM. (R. Wójcicki [1974]). Let C be SF. Then (a) the number of all structural axiomatic strengthenings of C, and hence the number of all invariant theories of C, is finite. Moreover (b) each such strengthening is SF.

PROOF. (b) Let $C_0 = C_{+Sb(X)}$, for some set of formulas X. To begin with, we shall show that C_0 is SF. Since C_0 is structural (cf. 7.7 and 10.4), we have $C_0 \leqslant C_0^{(k)}$. Assume that $\alpha \notin C_0(Y) = C(Y \cup Sb(X))$. Let

$C = Cn_{\mathbb{K}}$, $\mathbb{K}$ is being a finite set of finite matrices. Let $M \in \mathbb{K}$, and let h be a valuation in M such that

$$h(Y \cup Sb(X)) \subseteq \bar{M} \tag{1}$$

and

$$h\,\alpha \notin \bar{M}. \tag{2}$$

For each element a of M such that $\overleftarrow{h}(a) \cup Var(\mathcal{S}) \neq \emptyset$ select p_a such that $hp_a = a$, and define a substitution e by the condition

$$ep = p_a \text{ iff } hp = hp_a. \tag{3}$$

Clearly we have $eep = ep$ and thus, $ee\beta = e\beta$, for all β.

Suppose that $e\alpha \in C(eX \cup Sb(Y))$. This yields $ee\alpha = e\alpha \in C(eeX \cup eSb(Y)) = C(eX \cup eSb(Y))$. Making use of the valuation h, we immediately see that this is impossible. Hence

$$e\alpha \notin C(eX \cup Sb(Y)). \tag{4}$$

Let $\{a_1, \ldots, a_m\}$ be all elements of M. Select any isomorphic substitution i such that $ip_{aj} = p_j$, $j = 1, \ldots, m$. Since i is an isomorphism, we have $iSb(Y) = Sb(Y)$ and, at the same time

$$ie\,\alpha \notin C(ieX \cup Sb(Y) = C_0(ieX)). \tag{5}$$

Now, observe that $i \diamond e \in Hom(\mathcal{S}, \mathcal{S}(k))$ where k is the number under which C satisfies (i) and (ii) of 56.4. Thus succeeded to show (i) of 56.4.

In order to establish (ii) observe that $\equiv_C \subseteq \equiv_{C_0}$, and since $\mathcal{S}(k)|C$ is finite so is $\mathcal{S}(k)|C_0$.

(a) In order to establish (a) observe that by, 56.4 and Lemma 56.5, $C_{+Sb(Y)} = Cn_{\{M_{X,Y}: X \subseteq S\}}$, and $\{M_{X,Y} : X \subseteq S\} \subseteq \{M_{X,\emptyset} : X \subseteq S\}$. Now, since $\{M_{X,\emptyset} : X \subseteq S\}$ is a finite set, the number of strengthenings of C of the form $C_{+Sb(Y)}$ is finite. ■

We say that a logic C' is *finitely based relative* to C, iff for some finite set Q of standard rules $C' = C_{+Q}$ (i.e. C' is the strengthening of C by means of Q, cf. 7.7).

The next step in our investigations into the properties of the set of strengthenings of a SF consequence will consist in proving.

57.2. Theorem. Let C be SF. Then (a) the lattice $([C)_0, \leqslant)$ is both atomic and coatomic. (b) It contains finitely many atoms and finitely many coatoms. Moreover (c) each atom is finitely based relative to C, and each coatom is SF.

PROOF. Assume that k is the least number under which C satisfies (i) and (ii) of 56.4. Define

(1) $\mathcal{C} = \{C_{+X/\alpha} : X \cup \{\alpha\} \subseteq S(k) \text{ and } \alpha \in C(X)\}$

Since $\mathcal{S}(k)|(C \restriction \mathcal{S}(k))$ is finite, and $\alpha \equiv \beta(C)$ implies $C(\alpha) = C(\beta)$, we conclude that each $C' \in \mathcal{C}$ is of the form $C_{+X/\alpha}$, where X is finite and $X \cup \{\alpha\} \subseteq S(k)$. Hence $\mathcal{C}$ is finite and each C' in $\mathcal{C}$ is finitely based relatively to C.

Let $C_1, \dots, C_n$ be all minimal elements of $(\mathcal{C}, \leqslant)$. Since $\mathcal{C}$ is finite, $n \neq 0$. Observe, that each C_i, $i = 1, \dots, n$ is an atom in $([C)_0, \leqslant)$. For, if $C < C^+ \leqslant C_i$, $C^+ \in [C)_0$, then, by (ii) of 56.4, we have $e\alpha \notin C(eX)$ and $e\alpha \in C^+(eX)$ for some X and α. Hence $C^+ \in \mathcal{C}$ which yields $C^+ = C_i$. But C^+ have been an arbitrary element of $([C)_o, \leqslant)$ and thus we have showed not only that C_i are atoms in $([C)_0, \leqslant)$ but also that the lattice is atomic.

Let $\mathbb{X}$ be the set of all consistent and invariant theories of C. By 57.1(a) we know that $\mathbb{X}$ is finite. Let $X_1, \dots, X_n$ be all maximal elements in $(\mathbb{X}, \subseteq)$. Put $C_i = C_{+X_i}$. Each C_i is structural and $C \leqslant C_i$. Hence $C_i \in \{C^+ \in [C)_0 : C^+(\emptyset) = X_i\}, i = 1, \dots, n$. Put $C_i^+ = \sup\{C^+ \in [C)_0 : C^+(\emptyset) = X_i\}$. Since X_i is maximal in $(\mathbb{X}, \subseteq)$, C_i^+ is a coatom in $([C)_0, \leqslant)$. Consider any $C' \in [C)_0$. We have $C'(\emptyset) \subseteq X_i$ for some $i = 1, \dots, n$. Put $C_i' = C_{+X_i}$. Since $C \leqslant C_i'$ and $C_i'(\emptyset) = X_i$, hence $C_i' \leqslant C_i^+$, and hence $([C), \leqslant)$ is coatomic.

Since $C_i^+(\emptyset) \neq S$ then $\alpha \in C_i^+(\emptyset)$ for some α. Hence $e\alpha \notin C_i^+(\emptyset)$ for some $e \in Hom(\mathcal{S}, \mathcal{S}(k))$, where k is the least number such that $Var(\alpha) \subseteq \{p_1, \dots, p_k\}$. Hence, $\alpha \notin C_i^{+(k)}(\emptyset)$, and because $C_i^+ \leqslant C_i^{+(k)}$, we have $C_i^+ = C_i^{+(k)}$. Moreover, C is SF and $C \leqslant C_i^+$ hence $\mathcal{S}(k)|C_i^+$ is finite and by 56.4 we conclude that C_i^+ is SF. ■

(The theorem covers results established partially by W. Dziobiak [1980], and partially by R. Wójcicki [1979]).

58. Hereditary properties

58.1. A property of a logic C will be called *hereditary* iff it is shared by all $C' \in [C)_0$. Thus, in particular C will be said to be *hereditarily finitary, hereditarily SF, hereditarily finitely approximable* (cf. below) iff all their structural strengthenings are finitary, SF, finitely approximable, respectively.

The notion of finite approximability have not been defined thus far. C is said to be *finitely approximable* iff $C = Cn_{\mathbb{K}}$ for some set $\mathbb{K}$ (finite or not) of finite matrices.

Before we start examining hereditary properties mentioned above, we state some lemmas.

58.2. LEMMA. Let C be a structural consequence operation. For each $k \geqslant 1$, the following conditions are equivalent

(i) $C = C^{(k)}$,

(ii) There is a set of matrices $\mathbb{K}$ such that $C = Cn_{\mathbb{K}}$ and each $M \in \mathbb{K}$ is generated by a set of elements of cardinality $\leqslant k$.

PROOF. (i) $\rightarrow$ (ii). Put $\mathbb{K} = \{(\mathcal{S}(k), C(X) \cap S(k)) : X \subseteq L\}$.

(ii) $\rightarrow$ (i). We have $C \leqslant C^{(k)}$ for all $k \geqslant 1$. Let $\alpha \notin C(X)$ for some X and α. Under the assumption $C = Cn_{\mathbb{K}}$ we have $h(X) \subseteq \bar{M}$ and $h\alpha \notin \bar{M}$, for some $M \in \mathbb{K}$. Since M is generated by some of its k elements we have $h(eX) \subseteq \bar{M}$ and $h(e\alpha) \notin \bar{M}$ for suitably defined $e \in Hom(\mathcal{S}, \mathcal{S}(k))$, which gives $\alpha \notin C^{(k)}(X)$. ■

58.3. LEMMA. (W. Dziobiak [1979a], W. Sachwanowicz, unpublished). Let C' be SF and $C \in [C')_0$. Then the following two conditions are equivalent

(i) C is finitely approximable,

(ii) $C = \inf\{C^{(k)} : k \geqslant 1\}$.

PROOF. (i) $\rightarrow$ (ii). Assume (i). Let $C = Cn_{\mathbb{K}}$, all $M \in \mathbb{K}$ being finite. Assume that $\alpha \notin C(X)$. Then $\alpha \notin Cn_M(X)$ for some $M \in \mathbb{K}$. M is finite and by Lemma 58.2 we have $Cn_M = Cn_M^{(n)}$, where n is the cardinality of M, which yields $C^{(n)} \leqslant Cn_M$. Hence, $\alpha \notin \inf\{C^{(k)} : k \geqslant 1\}(X)$, and we obtain $C \geqslant \inf\{C^{(k)} : k \geqslant 1\}$. Of course the converse is true as well, and (ii) is established.

(ii) $\rightarrow$ (i). It will be enough to prove that each $C^{(k)}$ is SF. By the assumption C' is SF. Hence, by 56.4, each algebra $\mathcal{S}(k)C'$ is finite. From this and the fact that $C' \leqslant C^{(k)}$ we conclude that $\mathcal{S}(k)|C^{(k)}$ is finite for all $k \geqslant 1$. But $C^{(k)} = (C^{(k)})^{(k)}$ and by 56.3 we conclude that each $C^{(k)}$ is SF.

Let us remind that a lattice $(A, \leqslant)$ is said to satisfy *descending chain condition, DCC,* iff there is no sequence a_i, $i \in \omega$, of elements of A such that $a_i > a_{i+1}$, for all $i \in \omega$. $(A, \leqslant)$ satisfies *ascending chain condition ACC* iff there is no sequence $a_i, i \in \omega$, of elements of A such that $a_i < a_{i+1}$, for all $i \in \omega$.

58.4. THEOREM. (W. Dziobak[1980]). Let C be SF. The following conditions are equivalent

(i) C is hereditarily SF,

(ii) $([C)_0, \leqslant)$ satisfies DDC.

PROOF. Assume that $C = Cn_{\mathbb{K}}$, $\mathbb{K}$ being finite set of finite matrices. (i) $\rightarrow$ (ii). Assume (i) and suppose that there is a descending chain $C_1 >$

$C_2 > \ldots$ of $C_i \in [C)_0$. By 44.7, for each C_i there exists $\mathbb{K}_i \subseteq SP(\mathbb{K})$ such that $C_i = Cn_{\mathbb{K}_i}$. Put $C^+ = \inf\{C_i : i \geqslant 1\} = Cn_{\bigcup\{K_i : i \geqslant 1\}}$. From (i) it follows that there is a finite set $\mathbb{K}^+$ of finite matrices such that $C^+ = Cn_{\mathbb{K}^+}$. Let $M \in \mathbb{K}^+$. M is finite and $C^+ \geqslant Cn_M$. By 47.9a we obtain, $M \in \overleftarrow{H}_S H_S SP(\bigcup \mathbb{K}_i : i \geqslant 1)$. Hence, for some matrix homomorphisms f, h and some matrices $N_j \bigcup\{K_i : i \geqslant 1\}, j \in J$ we have

$$M \ \underset{f}{\xrightarrow{\text{onto}}}\ M' \ \underset{h}{\xleftarrow{\text{onto}}}\ M\text{''} \ \in \ S(\sqcap\{N_j : \ j \ \in \ J\}).$$

Because $\mathbb{K}$ is finite and so are all matrices in $\mathbb{K}$, M including, the set J and both M' and M'' are finite, and hence $C_i \leqslant Cn_M$, for some i. $\mathbb{K}^+$ is finite, and thus we have $C_i \leqslant Cn_{\mathbb{K}^+}$ for some i. Since $C^+ = Cn_{\mathbb{K}^+}$ then $C^+ \geqslant C_i$, for some i and hence, beginning from some i, $C_i = C_{i+k}$, for all $k \geqslant 0$.

(ii) $\to$ (i). Suppose that for some $C_0 \in [C)$, C_0 is not SF. Since C is SF then $\mathcal{S}(k)|C$ is finite for all $k \geqslant 1$, cf. 56.4. Hence, for no $k \geqslant 1$, $C_0 \leqslant C_0^{(k)}$. Each $C_0^{(k)}$ is SF and $C_0^{(n)} \leqslant C_0^{(k)}$ when $n \geqslant k$ (cf. 56.3b). By the definition of $C_0^{(k)}$ we have $C_0(X) = \inf\{C_0^{(k)} : k \geqslant 1\}(X)$ for all finite X. All this implies that for each n there is $k > n$ such that $C_0^{(k)} \leqslant C_0^{(n)}$, and this implies non–(ii). ■

58.5. We shall say that a matrix M is *critical* iff

$$C_M \ \neq \ \inf\{Cn_{M^+} : M^+ \ \text{ is a finite proper submatrix of } M\}$$

(W. Dziobak who defined this notion (cf. [1980]) and showed its usefulness in logical analyses, mentioned the notion of *s*–critical algebra (cf. S. Oates McDonald, M.R. Vangham–Lee [1978]) as the source of his inspiration).

58.6. Lemma. Let $\mathbb{K}$ be a set of similar non–trivial matrices. Then

$$Cn_{\mathbb{K}} \ = \ \inf\{Cn_M : \ M \ \text{ is critical and } \ M \ \in \ S(\mathbb{K})\}.$$

Proof. Obviously, we have $\leqslant$. Now, in order to prove the $\geqslant$ part of the lemma, assume that $\alpha \notin Cn_{\mathbb{K}}(X)$ and $\alpha \in Cn_M(X)$ for all critical matrices $M \in S(\mathbb{K})$. Under the assumptions made, $\alpha \notin Cn_N(X)$ for some N in $\mathbb{K}$ which is not critical. But if N has no finite and proper submatrices then N is critical ($\inf\emptyset = Cn_\emptyset$) contrary to the assumption and hence N must have finite and proper submatrices and hence critical. Among them there must be N^+ such that $\alpha \notin Cn_{N^+}(X)$ which again renders contradiction. ■

Let $\mathbb{K}$ and $\mathbb{K}^+$ be sets of similar matrices. We shall say that $\mathbb{K}^+$ is *locally closed* in $\mathbb{K}$ iff for all $M \in SP(\mathbb{K})$, if all finitely generated submatrices of M are in $\mathbb{K}^+$, then M is in $\mathbb{K}^+$.

58.7 Theorem. (W. Dziobiak[1980]). Let $\mathbb{K}$ be a finite set of finite matrices. Then the following conditions are equivalent:

(i) $Cn_{\mathbb{K}}$ is hereditarily SF,

(ii) $Cn_{\mathbb{K}}$ is hereditarily finitely approximable,

(iii) There is no infinite critical matrix in $SP(\mathbb{K})$

(iv) $SP(\mathbb{K}) \cap Matr(C)$ is locally closed in $\mathbb{K}$, for all $C \geqslant Cn_{\mathbb{K}}$.

PROOF. (i) $\to$ (ii). Let $C \in [Cn_{\mathbb{K}})_0$. Then (cf. the part (ii)–(i) of the proof of 58.3) $C(X) = \inf\{C^{(k)} : k \geqslant 1\}(X)$, for all finite X and thus (i) implies (ii).

(ii) $\to$ (iii). Suppose that M is an infinite critical matrix in $SP(\mathbb{K})$. Under the assumptions imposed on $\mathbb{K}$ all finitely generated matrics in $SP(\mathbb{K})$ are finite. On the other hand, M is infinite and critical. This yields that

$$Cn_M \ \leqslant\ \inf\{Cn_{M^+} : M^+ \text{ is finitely generated submatrix of } M\}$$

does not hold. Observe that $Cn_M^{(k)} = \inf Cn_N$; N is a k–generated submatrix of M, for all $k \geqslant 1$. Hence, by Lemma 58.3, Cn_M is not finitely approximable.

(iii) $\to$ (iv). Assume (iii). Let $C \in [Cn_{\mathbb{K}})_0$ and let $M \notin SP(\mathbb{K})$. Assume that all finitely generated submatrices of M belong to $SP(\mathbb{K}) \cap Matr(C)$. If M is finite then, of course, $M \in SP(\mathbb{K}) \cap Matr(C)$, itself. Suppose that M is not finite. In that case $M \in SP(\mathbb{K}) \cap Matr(C)$ again, for otherwise M would be an infinite critical matrix, contrary to (iii).

(iv) $\to$ (i). Let $C \in [Cn_{\mathbb{K}})_0$ and let Q be the set of all standard rules of C. We have $Cn_{\mathbb{K}} \leqslant Cl_Q \leqslant C$. Hence, $SP(\mathbb{K}) \cap Matr(C) \subseteq SP(\mathbb{K}) \cap Matr(Cl_Q)$ and in view of 44.7 it is enough to show the converse inclusion, since then we shall shaw that $C = Cl_Q$ and thus we shall prove that C is finitary (cf. 10.4).

Let $N \in SP(\mathbb{K}) \cap Matr(Cl_Q)$ be a finitely generated matrix and suppose that $N \notin SP(\mathbb{K}) \cap Matr(C)$. Then, for some X and some α, $\alpha \in C(X)$ and $\alpha \notin Cn_N(X)$. Assume that N is generated by k elements, then we have $e\alpha \notin Cn_N(eX)$ for some $e \in Hom(\mathcal{S}, \mathcal{S}(k))$, $\mathcal{S}$ being the language of C. Observe that the rule X/α is $Cn_{\mathbb{K}}$–equivalent to some standard rule, i.e. there is a finite set Y and a formula β such that

$$Cn_{\mathbb{K}(+X/\alpha)} = Cn_{\mathbb{K}(+Y/\beta)}.$$

Now, the fact that $e\alpha \in C(eX)$, and the fact that C is structural imply $Y/\beta \in Q$, which yield $\alpha \in Cn_N(Y)$. But this contradicts the assumption we made. Hence, all finitely generated matrices in $SP(\mathbb{K}) \cap Matr(Cl_Q)$ are in $SP(\mathbb{K}) \cap Matr(Cl_Q)$ which, by (iv), implies that all matrices of the former set are in the latter, exactly what we have had to prove. ■

59. Degree of maximality

59.1. The cardinality of the class of all invariant systems of a given logic C has been called the *degree of completeness* of C. (cf. Łukasiewicz and A. Tarski [1930]). For example the degree of completeness of 3–valued Łukasiewicz logic $Ł_3$ is 3; all its invariant systems are: $Ł_3(\emptyset)$, $K(\emptyset)$, L (cf. M. Wajsberg, 1930).

From the inferential point of view, the degree of completeness of C is the cardinality of all axiomatic strengthenings of C in $[C)_0$. The cardinality of $[C)_0$, i.e. the cardinality of all structural strengthening of C has been called (cf. R. Wójcicki [1974a]) the *degree of maximality* of C, and denote by $dm(C)$. Of course, the degree of completeness and that of maximality need not coincide. For instance, $dm(Ł_3) = 4$. The following logics: $Ł_3, Ł_{3(+(p\vee\neg p)\to(p\wedge\neg p)/q)}$, K, $Cn_\emptyset$ are all structural logics stronger than $Ł_3$. The result was established mainly by syntactical methods in R. Wójcicki [1974]. A semantic approach to the problem of the degree of maximality of various logics, SF in the first place, was open by G. Malinowski's paper in which the result concerning $Ł_3$ was extended on $Ł_4$ and all $Ł_n$ with n prime. But only W. Dziobiak result that we are going to present in this Section put the investigations into proper perspective.

59.2. Recall, that an element a of a lattice $(A, \leqslant)$ is called *compact*, cf. e.g. G.G. Grätzer [1978] if $a \leqslant \sup B$, $B \subseteq A$ implies that for some finite $B_f \subseteq B$, $a \in \sup B_f$. The following theorems of lattice theory are known.

a. The following two conditions are equivalent

(i) All elements of a lattice $(A, \leqslant)$ are compact,

(ii) $(A, \leqslant)$ satisfies ACC.

(cf. 58.3 for the definition of ACC).

b. Let $(A, \leqslant)$ be a poset. All chains in $(A, \leqslant)$ are finite iff $(A, \leqslant)$ satisfies both ACC and DCC.

(Cf. P. Crawley, R.P. Dilworth [1973], for the proofs of the theorems).

59.3. Theorem. W. Dziobiak [1980]). Let $\mathbb{K}$ be a finite set of finite matrices of the same similarily type. Then the following conditions are equivalent:

(i) There is finitely many non–isomorphic critical matrices in $SP(\mathbb{K})$ and all of them are finite.

(ii) card$\{Cn_M : M \in SP(\mathbb{K})$ and M is finite $\}_\cong < \aleph_0$.

(iii) $dm(Cn_\mathbb{K}) < \aleph_0$.

(iv) All structural strengthenings of $Cn_\mathbb{K}$ are finitely based relative to $Cn_\mathbb{K}$ and SF.

(v) The lattice $([Cn_\mathbb{K})_0, \leqslant)$ satisfies both ACC and DCC.

(vi) All chains in $([Cn_\mathbb{K})_0, \leqslant)$ are finite.

Proof. (i) → (ii). By Lemma 58.6.

(ii) → (iii). Assume (ii). Let $C \in [Cn_{\mathbb{K}})_0$. In order to prove (iii) it is enough to show that there is a finite set $\mathbb{K}^+ \subseteq SP(\mathbb{K})$ such that $C = Cn_{\mathbb{K}^+}$. By 44.7 we have $C = Cn_{\mathbb{K}}$, for some $\mathbb{K}' = \{M_t : t \in T\} \subseteq SP(\mathbb{K})$. Denote by $\mathbb{K}_t^n$, $n \geqslant 0$, $t \in T$, the set of all submatrices of M_t generated by sets of the cardinality $\leqslant n$. Verify, that all matrices in $\mathbb{K}_t^n$ must be finite, because $\mathbb{K}$ is a finite set of finite matrices. We shall show that

$$(1) \qquad Cn_{M_t} = \inf\{Cn_{\mathbb{K}_t^n} : n \geqslant 0\},$$

which, when established, yields (iii).

The part $\leqslant$ of (1) is obvious. Then, assume that $\alpha \notin Cn_{M_t}(X)$ for some X and α. Let X_f be a finite subset of X. Of course, $\alpha \notin Cn_{M_t}(X_f)$. Hence, for some valuation h in M_t, $hX \subseteq \bar{M}_t$ and $h\alpha \notin \bar{M}_t$. Let $\{p_1, \ldots, p_m\} = Var(X, \alpha)$ and let M be the submatrix of M_t generated by $hp_1, \ldots, hp_m$. We have $\alpha \notin Cn_M(X_f)$ and $M \in \mathbb{K}_t^n$. Hence $\alpha \in \inf\{Cn_{\mathbb{K}_t^n} : n \geqslant 0\}(X_f)$ for no $X_f \subseteq X$. From (ii) it follows that $\inf\{Cn_{\mathbb{K}_t^n} : n \geqslant 0\}$ is SF and thus finitary (cf. 45.2). Hence $\alpha \notin \inf\{Cn_{\mathbb{K}_t^n} : n \geqslant 0\}(X)$, and we have got (1).

(iii) → (iv). Since $dm(Cn_{\mathbb{K}}) < \aleph$ then $([Cn_{\mathbb{K}})_0, \leqslant)$ satisfies DCC. Hence, by 58.4 and 57.2, from (iii) we arrive at (iv).

(iv) → (v). By 58.4 and 59.2a.

(v) ⟷ (vi). By 59.2b.

(vi) → (i). Assume (vi) and non–(i). Put

$$(2) \qquad \mathbb{Q} = \{Cn_{\mathbb{K}^+} : \mathbb{K}^+ \subseteq SP(\mathbb{K}) \text{ and } SP(\mathbb{K}^+)$$

has infinitely many non–isomorphic critical matrices}.

By the assumption $Cn_{\mathbb{K}} \in \mathbb{Q}$. Condition (v), 59.2b and Zorn's lemma imply that $(\mathbb{Q}, \leqslant)$ has some maximal element. Let C be one of them and let $C = Cn_{\mathbb{K}^+}$ where $\mathbb{K}^+$ is the class that by (2) defines$C$ as an element of $\mathbb{Q}$. Hence, $SP(\mathbb{K}^+)$ has infinitely many critical matrices. By (v) and 58.4 we conclude that C is SF. From this, by 57.2, we conclude that $([C)_0, \leqslant)$ is atomic and has finitely many atoms. Let they be $C_1, ..., C_k$.

Now, we shall prove that the class $\{M \in SP(\mathbb{K}^+) : M$ is critical and $C = Cn_M\}$ has infinitely many non–isomorphic critical matrices. We shall show this if we show that the set $\{M \in SP(\mathbb{K}^+) : C \neq Cn_M$ and M is critical $\}$ has finitely many non–isomorphic critical matrices because there are infinitely many of them in $SP(\mathbb{K}^+)$. Suppose that the set is not finite modulo isomorphism. Then, for some $i_0, 1 \leqslant i_0 \leqslant k$, the class $\mathbb{K} = \{M \in SP(\mathbb{K}^+) : C_{i_0} \leqslant Cn_M \text{and} M \text{is critical}\}$ has infinitely many non–isomorphic critical matrices. This yields $C_{i_0} \leqslant Cn_{\mathbb{K}}$, which implies that C is not maximal in $(\mathbb{Q}, \leqslant)$, contrary to the assumption.

Let M_i, $i \in I$ be the family of all non–isomorphic critical matrices in $SP(\mathbb{K}^+)$ that determine a consequence coinciding with C. We know already that I is infinite. Divide I into two parts:

(1) $I_1 = \{i \in I : M_1 \text{has no proper submatrix}\}$

(2) $I_2 = I \setminus I_1$,

If $i \in I_1$, M_i is 1–generated. But $\mathbb{K}$ is a finite set of finite matrices and hence I_1 is finite. Suppose then that $i \in I_2$. M_i is critical, hence $C_j \leqslant \inf\{C_N : N \text{is a finite and proper submatrix of} M_i\}$ for some atom C_j. Let n be the least natural number for which 56.4 holds true when this theorem, it follows that $C_j = C_{+X/\alpha}$ for some finite set X and some α such that $X \cup \{\alpha\} \subseteq \mathcal{S}(n)$. This yields $\alpha \in Cn_N(X)$ for all finite and proper submatrices N of M_i, and $\alpha \notin Cn_{M_i}(X)$. Thus M_i must be generated by a set of cardinality $\leqslant n$, which implies that I_2 is finite. Hence, I is finite. Contradiction. Now, that the critical matrices in $SP(\mathbb{K})$ are finite follows from (vi), 58.4 and 58.7. ■

60. Some applications of Theorem 59.3

60.1 A class of algebras $\mathbb{A}$ is said to be *product representable* iff there is a finite set $\mathbb{A}' \subseteq \mathbb{A}$ of finite algebras such that each algebra in $\mathbb{A}$ is isomorphic with a direct product of some algebras from $\mathbb{A}'$. A matrix M for a language $\mathcal{S}$ is said to be *φ-definable* iff (a) φ is a formula of $\mathcal{S}_{PCI}$ (cf.46.1) of the form $\alpha(p) = \beta(p)$, where α, β are formulas of $\mathcal{S}$ in one variable p, and (b) $\bar{M} = \{a : \varphi(a)\}$. Observe that φ is an open positive Horn's formula.

From 59.3 it follows

COROLLARY. Let $\mathbb{K}$ be a finite class of finite matrices and let the variety $v(A_{\mathbb{K}})$ generated by the set $A_{\mathbb{K}}$ of all algebras of the matrices in $\mathbb{K}$ be product representable. Then, if for some φ, all matrices in $\mathbb{K}$ are φ–definable then $dm(Cn_{\mathbb{K}}) \leqslant \aleph_0$.

PROOF. Assume that all matrices in $\mathbb{K}$ are φ–definable. Since φ is an open positive Horn's formula then (cf. A. I. Malcev [1970]) all matrices in $SP(\mathbb{K})$ are also φ–definable. Now let $A_{\mathbb{K}}$ be product representable by $A_0 \subseteq A_{\mathbb{K}}$. Select any matrix $M \in SP(\mathbb{K})$. Let $M = (\mathring{A}, D)$. We have $\mathring{A} \sim= \mathring{A}_1 \times \ldots \times \mathring{A}_n$ for some $\mathring{A}_1, \ldots, \mathring{A}_n \in A_0$. Let f be the isomorphism between $\mathring{A}$ and the product of $\mathring{A}_i$. But then f is an isomorphism between M and $(\mathring{A}_1 \times \ldots \times \mathring{A}_n, f(D))$. Since φ is a positive Horn's formula than

$$(\mathring{A}_1 \times \ldots \times \mathring{A}_n, f(D)) \cong ((\mathring{A}_1, (f(D))_1) \times \ldots \times ((\mathring{A}_n, (f(D))_n)).$$

Hence, the consequence operation defined by the right hand product of matrices coincides with Cn_M. This, the finiteness of the set A, and Lemma 41.5 imply that condition (ii) of 59.3 is satisfied. ■

60.2 For each $n \in \AA$, Łukasiewicz truth–table algebra $\mathcal{L}_n$ generates the variety $v(\mathcal{L}_n)$ which is product representable. This claim is neither obvious nor even easy to prove, still the theorem is of rather algebraic than logical kind and we shall leave it without proof.

Now, in each $\mathcal{L}_n$ the designated element 1 is φ–definable by $p = p \to p$. The identity is satisfied only by 1.

As an immediate corollary to these two observations we have

a. (R. Wójcicki [1974]) For each finite n, $dm(\mathcal{L}_n) < \aleph_0$ This result can be improved as follows

b. (G. Malinowski [1977]) For all logics of the form $Cn_{(\mathcal{L}_n,D)}$ where n is finite, $D \subseteq \mathcal{L}_n$ and $D \cap \{0,1\} \neq \emptyset$, $dm(Cn_{(\mathcal{L}_n,D)})$ is finite.

With the help of McNaughton [1951] criterion one may verify that each matrix of the form described above is φ–definable.

Chapter 14

Finite Formalizations And Decidability

61. Two algebraic lattices

61.1. A lattice $(A, \leqslant)$ is said to be *algebraic* iff it is complete and for each $a \in A$ there is a set B of compact elements of A such that $a \in \sup B$. (cf. e.g. G. Grätzer [1978]).

61.2 THEOREM. Let C be a finitary consequence operation defined in $\mathcal{S}$. Then

a. $(Th_C, \subseteq)$ is a complete lattice

b. For each $X \in Th_C$, X is compact iff X is *finitely axiomatizable with respect to* C, i.e. $X = C(X_f)$ for some finite $X_f \subseteq X$.

c. There are denumerably many compact elements in $(Th_C, \subseteq)$

d. The lattice $(Th_C, \subseteq)$ is algebraic.

PROOF. The proof is easy. Still, let us present it in an outline. (a) Let $\mathbb{X} \subseteq Th_C$. Then $\inf \mathbb{X} = bigcap \mathbb{X}$ and $\sup \mathbb{X} = C(\bigcup \mathbb{X})$. (b) if $\mathbb{X} = C(\{a_1, ..., a_n\})$ and $X = \sup \mathbb{X}$, $\mathbb{X} \subseteq Th_C$, then $X = \sup\{X_1, ..., X_n\}$, $a_i \in X_i \in \mathbb{X}$. This gives the "only if" part of (b). To have the "if" part suppose that X is not finitely axiomatizable. Let $X = \{a_i : i \in \omega\}$. Define $X_n = \{a_i : i = 0, .., n\}$. $X = \sup\{C(X_i) : i \in \omega\}$ but $X \neq \sup \mathbb{X}$ for no finite $\mathbb{X} \subseteq \{C(X_i) : i \in \omega\}$. Hence, X is not compact. Both (c) and (d) are obvious, when (a) and (b) are proved. ■

61.3. THEOREM. Let $\mathcal{C}_0$ be the class of all standard logics in $\mathcal{S}$. Then

a. $(\mathcal{C}_0, \leqslant_f)$ is a complete lattice,

b. For each $C \in \mathcal{C}_0$, C is compact iff C is finitely based

c. There are denumerably many compact elements in $(\mathcal{C}_0, \leqslant_f)$,

d. The lattice $(\mathcal{C}_0, \leqslant_f)$ is algebraic.

PROOF. Modify in a suitable way the proof of 61.2. ■

62. Finitely axiomatizable theories and finitely based logics

62.1. Of two theorems we state below the first one belongs to the lattice theory, the second one is a kind of representation theorem.

a. Let $(A, \leqslant)$ be an algebraic lattice. If there are denumerably many compact elements in A then for each aA, the following conditions are equivalent:

(i) a is not compact,

(ii) There is an infinite sequence $a_0, a_1, \ldots$ of elements of A such that $a_i < a_{i+1}$ for all i and $a = \sup\{a_i : i \in \omega\}$.

b. Each algebraic lattice is isomorphic to a lattice of the form $(Th_C, \subseteq)$ where C is a finitary consequence.

Of the two theorems stated above the first one is provable by a rather easy argument, and the second one is a variant of a purely lattice theoretic representation theorem. Namely the following is known to be true (cf. e.g. G. Grätzer [1978]):

c. A lattice $(A, \leqslant)$ is algebraic iff it is isomorphic to the lattice of all ideals of a join–semilattice with the least element.

From 61.2, and 62.1 a it follows

62.2. COROLLARY (A. Tarski [1930a]). Let C be a finitary consequence and let $X \in Th_C$. Then the following two conditions are equivalent:

(i) X is finitely axiomatizable.

(ii) There is no sequence $X_0, X_1, \ldots, X_i, \ldots$ of theories of C such that for each $i \in \omega$, X_i is a proper subset of X_{i+1}, and

$$X = C(\bigcup X_i).$$

■

Observe that on the ground of 62.1b, theorem 62.1a and corollary 62.2 are equivalent. 62.1a is simple a lattice counterpart of logical theorem 62.2. Thus, the following can be considered as a corollary to 62.1a as well as a corollary to 62.2 (cf. S. Bloom [1975]).

62.3. Corollary. Let C be a standard logic. Then C is finitely based iff there is no sequence $C_0, C_1, \ldots$ of standard logics such that for each $i \in \omega$, C_i is properly weaker than C_{i+1} and $C = \sup\{C_i : i \in \omega\}$.

The two corollaries are syntactical but we have, of course, the following (cf. S. Bloom [1975]).

62.4. Theorem. A logic C defined in $\mathcal{S}$ is finitely based iff the class $Matr(C)$ is finitely axiomatizable by quasi–identities in $\mathcal{S}_{PCI}$.

Proof. A standard rule $\alpha_1, \ldots, \alpha_n/\beta$ is valid in C iff the universal closure of

$$(D(\alpha_1) \times \ldots \times D(\alpha_n)) \to D(\beta)$$

is valid in $Matr(C)$ (for the definition of $\mathcal{S}_{PCI}$ cf. 46.1 – 46.3). ■

Since there are some criteria of definability of models in sentences of particular kind, Theorem 62.4 may happen to be helpful in deciding whether a particular logic is finitely based or not.

There is no counterpart of 62.4 for the notion of axiomatizability relative to C. Even if we assume that C is structural (which is necessary for C to be determined by a class of matrices) and even if we restrict ourselves to invariant sets (again in order to deal with sets determined by matrices) there is no obvious way in which a suitable counterpart of 62.4 can be defined.

63. Axiomatizable theories and parafinitely based logics

63.1. A set of formulas X will be said to be *axiomatizable* iff $X = C(\emptyset)$ for some finitely based structural consequence C. Since there are denumerably many finitely based logics, there are denumerably many axiomatizable theories. On the other hand there are $2^{\aleph_0}$ invariant sets of formulas in all languages that involve a n–ary connective $n \geqslant 2$ or involve at least two unary connectives, hence some invariant sets of formulas in such languages are not axiomatizable.

There are not too many results concerning axiomatizable theories available, though the notion seems to be of some importance. Of some importance is also the following notion.

A logic C will be said to be *para–finitely based* iff there is an inferential base for C of the form $C(\emptyset), \mathbb{Q})$, $C(\emptyset)$ is axiomatizable and $\mathbb{Q}$ is a finite set of standard rules of inference. Again, by comparing the cardinality of the set of all standard consequences in a given language with that of para–finitely based (of course, all para–finitely based consequences are standard) we conclude that some standard consequences are not para–finitely based.

63.2. All finitely based logics are para–finitely based. This is obvious. The converse does not hold true. We shall show this. To see this consider the following example.

Let K_0 be the natural extension of K into $\mathcal{L}_\Box$. Define K_n, $n > 0$ to be the least modal logic based on K such that $K_n(\emptyset)$ is closed under all rules of the form

$$\alpha/\Box\alpha$$

where $\alpha \in K_{n-1}$.

Verify that no formula of the form $\Box\alpha \in K_0(\emptyset)$ and verify by induction that no formula of the form $\Box^n\alpha \in K_n(\emptyset)$. Hence, we have

$$K_0 \ < \ K_1 \ < \ K_2 \ < \ \ldots,$$

i.e. K_i form an infinite sequence of consequences increasing in strength.

Now define K_ω to be the least modal logic based on K such that $K_\omega(\emptyset)$ is closed under Necessitation Rule $p/\Box p$. We verify easily that

$$K_\omega \ = \ \sup\{K_i : i \ \in \ \omega\}$$

and thus, in virtue of 62.3, K_ω is not finitely based; on the other hand $(K_\omega(\emptyset), MP)$ is an inferential base for K_ω, $K_\omega(\emptyset) = K_{0(+p/\Box p)}(\emptyset)$ and thus is axiomatizable and we conclude that K_ω is para–finitely based.

63.3. The example discussed above is artificial. But the argument we have presented can easily be transformed into an argument showing that quite many modal logics are para–finitely based though they are not finitely based. For instance the following holds true:

a. None of the following logics: M_E, M_C, M_K, M_T, B is finitely based

b. All logics mentioned above are para–finitely based.

64. A generalized version of Herrop's theorem and some problems concerning decidability

64.1. A logic C is said to be *decidable* iff there is an effective procedure that enables us, given any formula α, to decide in a finite number of steps whether $\alpha \in C(\emptyset)$ or $\alpha \notin C(\emptyset)$. Actually, this is not the logic but the set of its theorems that is decidable or not.

We say that C has *finite modal property, f. m. p.* , iff there exists a class $\mathbb{K} \subseteq Matr(C)$ of finite matrices such that $C(\emptyset) = xi(\mathbb{K})$. The notion of f. m. p. was defined by Harrop, who applied it to establish a certain criterion of decidability. The theorem we state below is a rather far going generalization of Harrop's result, though the proof of it is merely an obvious modification of Harrop's argument.

64.2. Theorem. Each logic C that is both para-$^{\mathrm{a}}$finitely based (finitely based in particular) and has finite model property is decidable.

Proof. Let $C(\emptyset) = C'(\emptyset)$ where C' is finitely based, and let Q be a finitely inferential base of C'. Since proofs are finite sequents of formulas, there are denumerably many of proofs. Let

$$\pi_1, \pi_2, \ldots, \pi_i, \ldots$$

be all proofs by means of Q from the empty set of premises. Of course, $\alpha \in C(\emptyset)$ iff for some π_i, α is the conclusion of the proof.

Now let,

$$M_1, M_2, \ldots, M_i, \ldots$$

be all (up to isomorphisms) finite matrices of C. Since, by the assumption of the theorem, C has finite model property, $\alpha \notin C(\emptyset)$ iff $\alpha \notin Cn_{M_i}(\emptyset) = \zeta(M_i)$, for some M_i. Obviously, for each M_i, $\zeta(M_i)$ is decidable.

Given any α, start with π_1 and verify whether α is the conclusion of π_1. If not verify whether $\alpha \in \zeta(M_1)$. If it is, verify whether α is the conclusion of π_2. If it is not, verify whether $\alpha \in \zeta(M_2)$, etc. By continuing this procedure, one will find in a finite number of steps either a proof π_i of α or a matrix M_i that shows that $\alpha \notin C(\emptyset)$. ■

64.3. We are not going to examine the notion of finite model property in any systematic way. The only thing we want to do is to discuss briefly how the very well known technique of filtration applies to referential matrices.

Suppose that a selfextensional logic C is adequate with respect to a class $\mathbb{K}$ of referential matrices, i.e. $C = Cn_{\mathbb{K}}$. Now suppose that $\alpha \notin C(\emptyset)$. Then, for some $\mathcal{R} \in \mathbb{K}$, $\alpha \notin Cn(\mathcal{R})(\emptyset)$. Let $p_1, \ldots, p_n$ be all propositional variables appearing in α, and let h be a valuation in $(\mathcal{R})$ that falsifies α, i.e. $h_t\alpha = 0$ for some $t \in T_{\mathcal{R}}$.

What we want to get is to have a procedure of transforming matrices $\mathbb{K}$ in a set $\mathbb{K}'$ of finite C–matrices such that $C = Cn_{\mathbb{K}'}$. Observe, that a referential matrix is infinite only if the set of its reference points is infinite. Would it be possible to restrict the set $T_{\mathcal{R}}$ of the reference points of $\mathcal{R}$ to a finite set in such a way that the resulting matrix suits our purposes? There is no general procedure that can be applied to form such a restriction. Still, for some logics and some classes of referential matrices such a procedure is available. It is the *filtration* method invented by J.C.A. McKinsey [1941], and then developed by E.J. Lemmon and D. Scott [1966]. Cf. also K. Segerberg [1968].

Denote by $Subf(\alpha)$ the set of subformulas of α. Now, let $\equiv_\alpha$ be the relation defined on $T_{\mathcal{R}}$ by

$$t_1 \equiv\ t_2(\alpha) \text{ iff for all } \beta \in Subf(\alpha),\ (h\beta)_{t_1} = (h\beta)_{t_2}$$

h being the valuation falsifying α in $\mathcal{R}$.

One easily verifies that $\equiv_\alpha$ is an equivalence relation. Moreover, $T_{\mathcal{R}}|\alpha$ is finite. Well, what we have to do now is to consider the quotient set $\mathcal{R}|\alpha$ just to realize that in general $\equiv \alpha$ is not a congruence and thus, though there exists the set $\mathcal{R}|\alpha$, there need not be the quotient algebra of the form $\mathcal{R}|\alpha$. And this is the very moment at which we cannot go any further without specifying both the logic we want to deal with and the class $\mathbb{K}$ that determines it.

Assume that $C = \vec{M}_E$ and $\mathbb{K}$ is the class of all K–standard referential matrices. In that case $\equiv \alpha$ is a congruence on the standard part of $\mathcal{R}$. Let $\mathcal{R}_\alpha$ be the algebra such that

(i) $\mathcal{R}|\alpha$ is the set of elements of

(ii) $\mathcal{R}_\alpha \restriction \{\wedge, \vee, \rightarrow, \neg\} = (\mathcal{R} \restriction \{\wedge, \vee, \rightarrow, \neg\})|\alpha$

(iii) For each $\underline{r} \in \mathcal{R}_\alpha$ and each $\underline{t} \in T_{\mathcal{R}}|\alpha, (\Box\underline{r})_{\underline{t}} = 1$ iff $\{\underline{t}' \in T_{\mathcal{R}}|\alpha : \underline{r}(\underline{t}' = 1\} \in N_\alpha(\underline{t})$, where N_α is defined by the following condition. For all $\underline{t} \in T$

$$\{|t'|_\alpha : t' \in T_{\mathcal{R}}|\alpha \;\text{ and }\; (h\beta)_{t'} \;=\; 1\} \;\in\; N_\alpha(\underline{t}) \;\text{ iff }\; \Box\beta \;\in\; Subf(\alpha),$$
$$\text{and }\; (h\Box\beta)_t \;=\; 1 \;\text{ for all }\; t \;\in\; \underline{t}\}.$$

Obviously, $\mathcal{R}_\alpha$ is a K–standard referential matrix, $\mathcal{R}_\alpha$ is finite, and it is an easy exercise to verify, that any valuation h_α in $\mathcal{R}_\alpha$ such that

$$h_\alpha \; p_i \;=\; |r_i|_\alpha \;, \quad i \;=\; 1, \ldots, n$$

falsifies α. In fact, $(h_\alpha\beta)_{\underline{t}} = (h\beta)_t$ for all $t \in \underline{t}$, and all $\beta \in Subf(\alpha)$. To establish this apply an inductive argument.

Let $\mathbb{K}_\alpha = \{\mathcal{R}_\alpha : \alpha \notin M_E\}$. Since M_E is complete with respect to the class of all neighborhood matrices, cf. 26.2 and 26.3, and all matrices in $\mathbb{K}_\alpha$ are neighborhood matrices, hence $M_E \subseteq \zeta(\mathbb{K}_\alpha)$. We have constructed thematrices so that we have the converse, and we have proved that M_E has f. m. p.

64.4. The case of E is the simplest one because the only thing we have to take care of when defining $\mathcal{R}_\alpha$ (just this matrix is called the *filtration of* $\mathcal{R}$ *through* α) is that $\mathcal{R}_\alpha$ is a neighborhood matrix. If a modal system is determined by referential matrices of certain specific kind (say, normal, as it is the case of M_K) then we should be sure that all filtration should be again of this kind.

Though it cannot be done mechanically, the filtration method can be applied (not always successfully !) to all logics that have referential semantics. The method was designated for frames of various kind, but of course, it can always be adopted to referential matrices that correspond to a given class of frames, e.g. relational K–standard matrices corresponding to relational frames or epistemic matrices corresponding to epistemic frames.

65. Finite approximability and finite model property

65.1. Recall, that a logic C is said to be finitely approximable iff $C = Cn_{\mathbb{K}}$ for some class $\mathbb{K}$ of finite matrices. Of course, finite approximability implies f.m.p., and of course, there is no reason to expect the converse to hold true. It does not.

Curiously enough, finite approximability is a property not too common among well–known logics with an outstanding exception of K, of course. A partial explanation of this phenomenon is suggested by the following example. Let

$$X = \{(p_i \longleftrightarrow p_j) \to p_0 : i \neq j, i, j > 0\}.$$

We have $p_0 \notin J(X)$ which can be proved quite easily, for instance, by defining either an epistemic frame or a pseudo–boolean algebra in which $X \vdash p_0$ is not satisfied. Such a frame (or algebra) must be infinite however, since otherwise the valuations defined with respect to it would assign different values only to finitely many different variables. And this is exactly why J is not finitely approximable. For any class $\mathbb{K}$ of finite J–matrices $p_0 \in Cn_{\mathbb{K}}(X)$.

Already this example suggest that finite approximability is not a property easy to find among the logics that are not strongly finite. The argument we have produced in order to show that J is not finitely approximable, applies to rather large number of logics (of course, those which are not SF).

The following theorem sheds some more light on the problem.

65.2. Theorem. (W. Dziobiak [1981]). Let C be either an intermediate (i.e. a well determined logic stronger than J) or a well determined normal modal logic based on K. Then the following conditions are equivalent

(i) C is tabular,

(ii) C is strongly finite,

(iii) C is finitely approximable.

We shall omit the proof of this theorem. The part of it concerning intermediate logics was established by A. Wroński (unpublished).

Chapter 15

Comparing Different Logics Via Definability Relation

66. Definitional extensions

66.1. A logic C' is said to be a *definitional extension of* C iff

(i) The language $\mathcal{S}'$ of C' results from that of $\mathcal{S}$ by adding some new connectives $\S_1, \ldots, \S_n, \mathcal{S}' = (\mathcal{S}, \S_1, \ldots, \S_n)$

(ii) C' is a conservative extension of C, i. e. $C' \restriction \mathcal{S} = C$.

(iii) Let $\S_i$ be an r_i–ary connective. For each new connective $\S_i$, $i = 1, \ldots, n$, there exists a formula φ_i and there are propositional variables $p_1, \ldots, p_{r_i}$ such that for all $\alpha_1, \ldots, \alpha_{r_i}$

$$(D_i) \qquad \S_i(\alpha_1, \ldots, \alpha_{r_i}) = \varphi_i(\alpha_1/p_1, \ldots, \alpha_{r_i}/p_{r_i})(C')$$

(For the definition of congruence $\equiv_{C'}$ cf. 50.1)

66.2. If C' is a definitional extension of C for which conditions (i) — (iii) are satisfied, then C' will be referred to as the *definitional extension of* C *determined by condition* (D_i) or, alternatively, as the *definitional extension if* C *determined by conventions*

$$(D_i^*) \qquad \S_i(\alpha_1, \ldots, \alpha_{r_i}) =_{df} \ \varphi_i(\alpha_1/p_1, \ldots, \alpha_{r_i}/p_{r_i}).$$

The following remark is in order here. Under definitions (D_i^*), $\S_i(\alpha_1, \ldots, \alpha_r)$ serve as abbreviations for $\varphi_i(\alpha_1/p_1, \ldots, \alpha_{r_i}/p_{r_i})$. For instance, the familiar conventions

$$\alpha \longleftrightarrow \beta =_{df} (\alpha \to \beta) \wedge (\beta \to \alpha),$$

defines $\alpha \longleftrightarrow \beta$ as an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$ and

$$\Diamond\alpha \ =_{df} \ \neg\Box\neg\alpha$$

defines $\Diamond\alpha$ as an abbreviation for $\neg\Box\neg\alpha$, Thus (D_i^*) introduce $\S_1, \ldots, \S_n$ merely as certain auxiliary symbols whose role is similar to that of parentheses; though convenient they dispensable. By introducing new symbols via conventions (D_i^*) we do not change the algebra of formulas and thus, under the definition of a propositional language to which we subscribe, we do not change the language. The language $\mathcal{S}$ and the language $\mathcal{S}'$ extended by auxiliary symbols $\S_1, \ldots, \S_n$ whose syntactical role is defined by conventions (D_i^*), are exactly the same languages.

This is not quite the case covered by definition 66.1. Under this definition, the definitional extension C' of C is defined in the language in which $\S_1, \ldots, \S_n$ are "fully–fledged" connectives and thus the language $\mathcal{S}' \ = (\mathcal{S}, \S_1, \ldots, \S_n)$ differs from $\mathcal{S}$. Still, as one can easily verify, we have

66.3. Theorem. Let C' be a logic defined in a language $(\mathcal{S}, \S_1, \ldots, \S_n)$ and let C' be a definitional extension of $C = C' \restriction \mathcal{S}$, determined by conditions (D_i), $i = 1, \ldots, n$. Then, for each $X \subseteq S'$

$$C'(X) \ = \ C(X),$$

under the assumption that C applied to X via conventions (D_i^*). ■

Just in view of this theorem the definitional extension determined by (D_i) can be viewed as the extension corresponding to conventions (D_i^*).

66.4. The following should be made clear. While each definitional extension can be viewed as a definitional extension determined by some definitional conventions of the form (D_i^*), the conventions of the form (D_i^*) need not define any definitional extension. This, perhaps somewhat surprising observation, explains as follows. No restriction has been imposed on variables that may appear in the formulas φ_i. Suppose that a formula φ_i involves a variable p different from all variables $p_1, \ldots, p_{r_i}$. For instance, let $\S_i$ be nullary connective, say T, and let $\varphi_i = p \to p$. Now the convention

$$(T) \qquad T \ =_{df} \ p \ \to \ p,$$

is perfectly acceptable in any propositional language that involves $\to$. The only somewhat peculiar thing about it is that it assigns a distinguished role to the variable p; under this convention T and $p \to p$ are the same formulas, but T and $q \to q$, q being different from p, are not!

Let (T) be a notational convention accepted for a language $\mathcal{S}$, let C be a logic in that language. Now, consider the language of the form $(\mathcal{S}, T)$, i. e. an extension of $\mathcal{S}$ with a new connective (not merely a new auxiliary symbol) T and ask whather there is a definitional extension C' of C such that

$$T \equiv p \rightarrow p(C')$$

holds true.

As we shall show below, (cf. 65.5), $\alpha \equiv \beta(C')$ implies $e\alpha \equiv e\beta(C')$ for all substitution e. Hence, we have to have

$$T \equiv \alpha \rightarrow \alpha\ (C')$$

for all formulas α of $(\mathcal{S}, T)$. But this requirement implies that we must have as well

$$\alpha \rightarrow \alpha \equiv \beta \rightarrow \beta\ (C)$$

for all α, β of the language $\mathcal{S}$. The latter condition is a condition that concerns the logic C, and in general (i.e. for some C) need not be true.

66.5 LEMMA. Let $\alpha \equiv \beta(C)$. Then, for each substitution e (in the language of C), $e\alpha \equiv e\beta(C)$.

PROOF. Recall that $\alpha \equiv \beta(C)$ iff for any formula γ and for any variable p, $C(\gamma(\alpha/p)) = C(\gamma(\beta/p))$.

Let e be a substitution and let γ be a formula such that $Var(\gamma) = p_1, \ldots, p_n$. Let $q_1, \ldots, q_n$ be some variables that occur neither in α nor in β. Put $\gamma^+ = \gamma(q_1/p_1, \ldots, q_n/p_n)$ and define the new substitution e^+ as follows :

$$e^+(p) = \begin{cases} p_i, & \text{when } p = q_i \\ e(p), & \text{otherwise.} \end{cases}$$

Observe that $e^+(\gamma^+(\alpha/q_i)) = \gamma(e\alpha/q_i)$ and $e^+(\gamma^+(\beta/q_i)) = \gamma(e\beta/q_i)$. Since $\alpha \equiv \beta(C)$, $C(\gamma^+(\alpha/p)) = C(\gamma^+(\beta/p))$. By structurality of C we obtain $C(e^+\gamma(\alpha/p)) = C(e^+\gamma(\beta/p))$ which yields $C(\gamma(e\alpha/p)) = C(\gamma(e\beta/p))$.

The observation we made in 66.4 can be easily transformed into the proof of the following

66.6 THEOREM. Let $\mathcal{S}$, $\mathcal{S}' = (\mathcal{S}, \S_1, \ldots, \S_n)$, and $\varphi_1, \ldots, \varphi_n$ be as in 66.1. In order for a definitional extension C' of C determined by conditions (D_i) (or equivalently by conventions (D_i^*)) to exist it is necessary and sufficient that for each φ_i and for each substitution e such that $e_{p_k} = p_k$ for all p_i, $k = 1, \ldots, r_i$,

$$\varphi_i = e\,\varphi_i\ (C)$$

. ■

66.7 Let Ψ be all connectives of a language $\mathcal{S}$ and let C be a logic defined in $\mathcal{S}$. We say that a connective $\S \in \Psi$ is *definable* in C in terms of connectives $\Psi' \in \Psi$ iff $C \upharpoonright \Psi' \cup \{\S\}$ is a definitional extension of $C \upharpoonright \Psi'$.

Just, for an illustration, let us mention the following well known facts

a. Let $\S \in \{\wedge, \vee, \rightarrow\}$. Each of the connectives $\S' \in \{\wedge, \vee, \rightarrow\}/\{\S\}$ is definable in K in terms of $\{\S, \neg\}$,

b. None of the connectives of $\mathcal{L}$ is definable in J in term of the remaining ones.

66.8 Apart from definability in the sense defined above, the following "weaker" notion of definability is of some importance. Let us discuss the matter in somewhat loose manner.

Let C' be a logic defined in a language $\mathcal{S}' = (\mathcal{S}, \S_1, \ldots, \S_n)$. If each theorem $\alpha' \in C'(\emptyset)$ can be translated (in the sense that is obvious enough to be safely left undefined) with the help of conventions of the form (D_i^*) onto a theorem $\alpha \in C(\emptyset)$, the connectives $\S_1, \ldots, \S_n$ are said to be *definable with respect to* $C'(\emptyset)$ *in terms of the connectives of* $\mathcal{S}$ or, alternatively, they will be said to be *weakly definable in* C' *in terms of the connectives of* $\mathcal{S}$.

Of course, definability implies weak definability but not vice versa. Thus, for instance, proposition a. of 66.7 implies

a'. Let $\S \in \{\wedge, \vee, \rightarrow\}$. Each of the connectives $\S' \in \{\wedge, \vee, \rightarrow\}/\S$ is definable with respect to $K(\emptyset)$ in terms of $\{\S, \neg\}$.

Now, the following holds true

b'. None of the connectives of $\mathcal{L}$ is definable with respect to $J(\emptyset)$ in terms of remaining ones.

Still, it is proposition b of 66.7 that is derivable from b', not b' from b.

67. Definability

67.1. A logic C will be said to be *definable* in a logic C' iff there exists a definitional extension C'' of C' such that C coincides (up to isomorphism) with a fragment of C''.

67.2. For an illustration consider the following example. Given nay finitely many valued Łukasiewicz logic $Ł_n$ denote by $Ł_n^*$ the definitional extension of $Ł_n$ determined by conditions

$(\Rightarrow)$ $\alpha \Rightarrow \beta = \alpha \rightarrow_{n-1} \beta (Ł_n^*)$

$(=\!\!\mid)$ $=\!\!\mid \alpha \equiv \alpha \Rightarrow \neg(\alpha \rightarrow \alpha)(Ł_n^*)$

(cf. 13.1 for the definition of $\to_{n-1}$). For each $n \geqslant 3$, $Ł_n^*\{\wedge, \vee, \Rightarrow, =\!\shortmid\}$ coincides, up to isomorphism, with K. Thus K is definable in each $Ł_n$, n finite $\geqslant 3$. For the details of the proof, cf. M. Tokarz [1971]. See M. Tokarz and R. Wójcicki [1971]).

67.3. The result we have just mentioned seems to be of considerable philosophical interest. Each $Ł_n$, $n \geqslant 3$ is essentially weaker than K. But it turns out that nevertheless "the expressive power" of Łukasiewicz calculi is greater than that of K. Whatever can be said in terms of connectives of K, it can be said in terms of connectives of any $Ł_n$, $n \geqslant 3$. Incidentally, K is not definable in $Ł_\omega$. This, rather sophisticated result, was established by P. Wojtylak [1979b]

67.4. To have one more example of definability of a calculus in another one, verify that the connective $\neg$ of N is definable in N in terms of $\to$ and $\sim$ by

$$(\neg) \qquad \neg\alpha =_{df} \alpha \to \sim (\alpha \to \alpha)$$

On the other hand, N is a conservative extension of J, hence J is definable in $N \restriction \{\wedge, \vee, \to, \sim\}$.

67.5. Let us examine briefly how J and K are related to each other. As known, J admits a definitional extension J^* with $\Rightarrow$ and $=\!\shortmid$ being new connectives such that

$$J^*(\emptyset) \restriction \{\wedge, \vee, \Rightarrow, =\!\shortmid\} = K(\emptyset).$$

This result is sometimes interpreted as definability of the classical logic K in the intuitionistic one. But, of course, in order to have such definability in the sense which we accepted here it does not suffice to have the identity mentioned above. It would be necessary to have

$$J^* \restriction \{\wedge, \vee, \Rightarrow, =\!\shortmid\} = K$$

As it has been proved in R. Wójcicki [1970] there is no definitional extension J^* of J under which the latter identity holds true. Thus K is not definable in J. Curiously enough (this is P. Wojtylak's result [1979b]), given any strengthenings J_1^+, J_2^+ of J, J_1^+ is definable in J_2^+ iff $J_1^+ = J_2^+$.

68. Definitional variants

68.1. In view of Theorem 66.3, given any logic C, we may safely treat C as essentially the same logic as any of its definitional extensions. The latters will be referred to as *definitional variants* of C. More generally, a logic C' will be said to be a *definitional variant* of C iff the two logics have a common definitional extension.

Thus for instance, both $K \restriction \{\wedge, \neg\}$ and $K \restriction \{\vee, \neg\}$ are definitional variants of K and, of course, of each other. $N\{\wedge, \vee, \to, \sim\}$ is a definitional variant of N. As known, as the primitive connectives of J may serve $\wedge$, $\vee$, $\to$ and "absurd" $\perp$, the last one being a nullary connective. Thus a definitional variant of J can be defined in the language determined by the connectives just mentioned.

Of course, we have

68.2. THEOREM. Let C_1, C_2 be a definitional variants of each other. Then C_1 is definable in C_2 and vice versa. ■

68.3. Curiously enough, the converse does not hold true. An example to this was given by P. Wojtylak (unpublished).

Let $\mathcal{S}_\Box$ be the language determined by only one unary connective $\Box$, and $\mathcal{S}_\Diamond$ the language determined by $\Diamond$, again being a unary connective.

Denote by $\Box_M$, $\Diamond_N$ the operations on the set ω of all natural numbers defined by

$$\Box_M(k) = k+1,$$
$$\Diamond_N(k) = k+2$$

and put

$$I_k = \{n \in \omega : k \leqslant n\}, \quad k \neq 0$$
$$I_0 = \{5k : k \in \omega\} \cup \{5k+1 : k \in \omega\}$$

Now let

$$M = (\omega, \Box_M, \{I_k : k \in \omega\}),$$
$$N = (\omega, \Diamond_N, \{I_k : k \in \omega\}),$$

be two ramified matrices, $(\omega, \Box_M)$ and $(\omega, \Diamond_N)$ being their algebras, respectively. In what follows, instead of $\Box_M$ and $\Diamond_N$ we shall write $\Box$ and $\Diamond$, respectively.

Consider the logics Cn_M and Cn_N. Since

$$\Diamond k = \Box\Box\, k \; (= \Box^2 k)$$

for all $k \in \omega$, the logic Cn_N is definable in C_M by the convention

$$\Diamond\alpha =_{df} \Box^2\alpha.$$

The argument to this effect, involving a certain, rather obvious, matrix criterion of definability, is straightforward.

Now define an operation $\boxtimes : \omega \to \omega$ by

$$\boxtimes k = k+6 = \Diamond\Diamond\Diamond\alpha (= \Diamond^3\alpha)$$

and denote by $Cn_{M'}$ the operation defined in $\mathcal{S}_\Box$ by the ramified matrix

$$M' = (\omega, \boxtimes, \{I_k : k \in \omega\}).$$

We shall leave to the reader the part of the proof that consist in showing that $Cn_M = Cn_{M'}$. But if $Cn_M = Cn_{M'}$, we conclude that Cn_M is definable in Cn_N for, quite clearly Cn_M, is definable in Cn_N, by the convention

$$\boxtimes =_{df} \Diamond^3\alpha$$

Suppose that there exists a common definitional extension C' of both Cn_M and Cn_N. Then, of course, there exists a common definitional extension C of Cn_M and Cn_N defined in the language determined by the connectives $\Box$ and $\Diamond$ only. Moreover, the two connectives must be related to each other in such a way that for some k and some s,

(1) $$\Box\alpha \equiv \Diamond^k\alpha(C),$$

(2) $$\Diamond\alpha \equiv \Box^s\alpha(C).$$

Now, if $k = 0$ we would have

(3) $$\Box\alpha \equiv \Box^2\alpha(C).$$

Similarly $s = 0$ implies

(4) $$\Diamond\alpha \equiv \Diamond^2\alpha(C).$$

But (3) yields

(5) $$\Box\alpha \equiv \Box^2\alpha(Cn_M).$$

And (4) yields

(6) $$\Diamond\alpha \equiv \Diamond^2\alpha(Cn_N).$$

Now we rather easily verify that neither (5) nor (6) holds true.

A bit more involved argument is necessary in order to show that neither $k = 1$ nor $s = 1$. The assumption to the contrary disagrees with the fact that

(7) $$\Box^2\alpha \notin Cn_M(\alpha,\ \Box\alpha)$$

but au the same time

(8) $$\Diamond^2\alpha \ \in\ Cn_M(\alpha,\ \Diamond\alpha).$$

We leave the argument to the effect that (7) and (8) hold true to the reader.

Thus $k, s \geqslant 2$, and once (1) and (2), we conclude that

$$\Box\alpha \equiv \Box^{ks}\alpha(C), \tag{9}$$

and hence

$$\Box\alpha \equiv \Box^{l}\alpha(Cn_M), \tag{10}$$

for some $l > 1$. But, as one may verify (10) is false for all $l > 1$. The contradiction at which we arrive concludes the proof.

Bibliography

Abbreviations

AL	—	Algebra i Logika (Novosibirsk)
BAP	—	Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Phisiques (Warsaw)
BSL	—	Bulletin of the Section of Logic of the Institute of Philosophy and Sociology, Polish Academy of Sciences (Wrocław, Łódź)
CR-V	—	Comptes Rendus des Seances de la Société des Sciences et les Lettres de Varsovie, d. III
DAN	—	Doklady Akademii Nauk SSSR, (Moscow)
FM	—	Fundamenta Mathematicae (Warsaw)
IM	—	Indagationes Mathematicae (Koninklijke Nederlandse Akademie van Wettenschappen. Proceedings. Series A. Mathematical Sciences (Amsterdam)
JSL	—	The Journal of Symbolic Logic (Pasadena, CA)
NDJFL	—	Notre Dame Journal of Formal Logic (Notre Dame, Ind.)
RML	—	Reports on Mathematical Logic (Kraków)
SL	—	Studia Logica (Wrocław, Łódź)
SM	—	Soviet Mathemetics. Doklady (A translation of the mathematics section of *DAN* (Providence, R.I.)
TAMS	—	Transactions of the Americal Mathematical Society (Providence, R.I.)
ZML	—	Zeitschrift für Mathematische Logik und Grundlagen des Mathematik (Berlin)

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INDEX

(Note. The symbol ≈ reads 'is usually abbreviated as')