Odyssey of the Mathematical Mind

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Motivation

- Lectures: *Puzzles* (2013–2015) for students of *cognitive science* (Adam Mickiewicz University).
- Text: *Odyssey of the Mathematical Mind* (in Polish); in preparation.
- English excerpts: *Entertaining Math Puzzles*, available on line at the web site of the *Group of Logic, Language and Information* (University of Opole).

- Main Goal: training in creative *problem solving* with special emphasis put on *paradox resolution*.
- Secondary Goal: analysis of *mistakes* caused by *common sense* intuitions.

Nothing as it seems

Main Topics:

- The Infinite
- Numbers and magnitudes
- Motion and change
- Space and shape
- Orderings
- Patterns and structures
- Algorithms and computation
- Probability

- Logic puzzles
- Paradoxes
- Sophisms
- Puzzles in: physics, linguistics, philosophy.

Miracles happen but they do not repeat themselves

- Zeno's paradoxes (arrow, dichotomy, Achilles and Tortoise).
- Speedy fly (and Polish State Railways, Inc.).

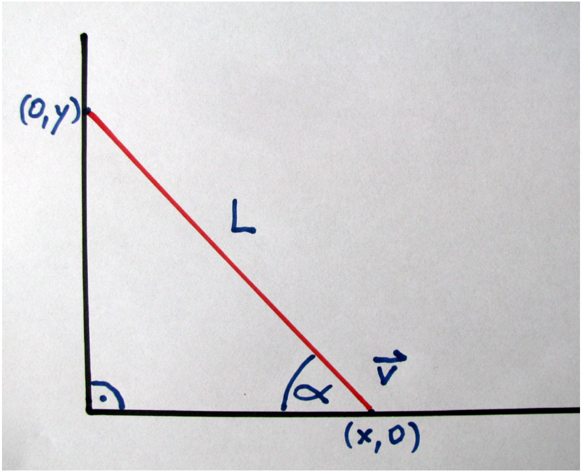
- Thomson's lamp is a device consisting of a lamp and a switch set on an electrical circuit. If the switch is *on*, then the lamp is lit, and if the switch is *off*, then the lamp is dim. Suppose that: at time $t = 0$ the switch is on, at $t = \frac{1}{2}$ it is off, at $t = \frac{3}{4}$ it is on, at $t = \frac{7}{8}$ it is off etc. What is the state of the lamp at time $t = 1$?
- Laugdogoitia's „beautiful supertask". Incompleteness of Newtonian mechanics and clash with general relativity theory.
- Hypercomputations. Is it possible to execute an infinite number of calculation steps in a finite time?

Ignorance means strength!

- S knows only the sum and P knows only the product of two numbers x and y and they are both aware of these facts. They both know that $x > 1$, $y > 1$, $x + y \leqslant 100$. The following dialogue takes place:
  - P: I do not know the two numbers.
  - S: I knew that you do not know them.
  - P: Now I know these numbers.
  - S: Now I know them, too.
- Find these numbers x and y.

- The above statements (in this order) imply several arithmetic facts which enable us to find the (unique) solution.
- This is Freudenthal's puzzle from 1969, popularized later by Gardner.

Sliding ladder

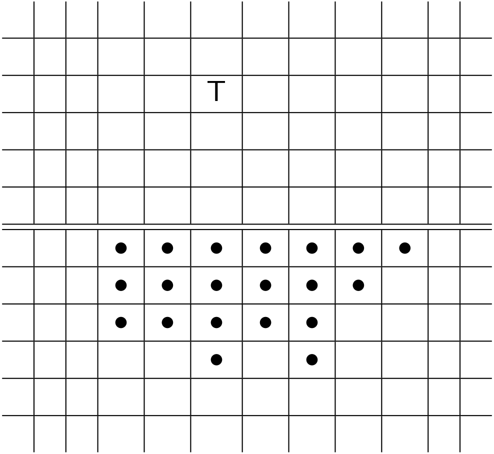


Incidentally created black hole

- A ladder of length L is leaning against a vertical wall. The bottom of the ladder is being pulled away from the wall horizontally at a uniform rate v. Determine the velocity with which the top of the ladder crashes to the floor. Bottom: $(x, 0)$, top: $(0, y)$.
$x^2 + y^2 = L^2$ and hence $\frac{dy}{dt} = -v \cdot \frac{x}{y}$. Thus, $\frac{dy}{dt} \to \infty$ when $y \to 0$.
- Assuming that the top of the ladder maintains contact with the wall we obtain an absurdity: the velocity in question becomes infinite!

- Actually, at a certain moment the ladder looses contact with the wall. After that, the motion of the ladder is described by the pendulum equation.
- More accurate descriptions of this problem involve friction, pressure force, etc.

Example: the accessible fourth level



The inaccessible fifth level

- The game is played on an infinite board – just imagine the whole Euclidean plane divided into equal squares and with a horizontal border somewhere. You may gather your army of checkers below the border. The goal is to reach a specified line above the border. The checkers move only vertically or horizontally. Thus diagonal moves are excluded. As in the genuine checkers, your soldier jumps (horizontally or vertically) over a soldier on the very next square (which means that he kills him) provided that it lands on a non-occupied square next to the square occupied previously by the killed soldier.

- It is easy to show that one can reach the first, second, third and fourth line above the border. However, no finite amount of soldiers gathered below the border can ever reach (by at least one surviving soldier) the fifth line above the border!

König's Lemma in action

Suppose you have an infinite number of balls, more exactly: you have an infinite number of balls numbered with 1, an infinite number of balls numbered with 2, an infinite number of balls numbered with 3, etc. – an infinite number of balls numbered with any positive integer. You have also a box, in which at the start of the game there is a certain finite number of such numbered balls. Your goal is to get the box empty, according to the following rule. At each move you are permitted to replace any of the balls inside the box by an arbitrary finite number of balls with numbers less than the number on the ball removed. Of course, balls with number 1 on them are simply removed from the box, because you can not replace them by balls numbered with a positive integer smaller than 1. Is it possible to make the box empty in a finite number of steps?

The answer to the puzzle is certainly affirmative. However, you can not *in advance* predict the number of steps required to finish the game.

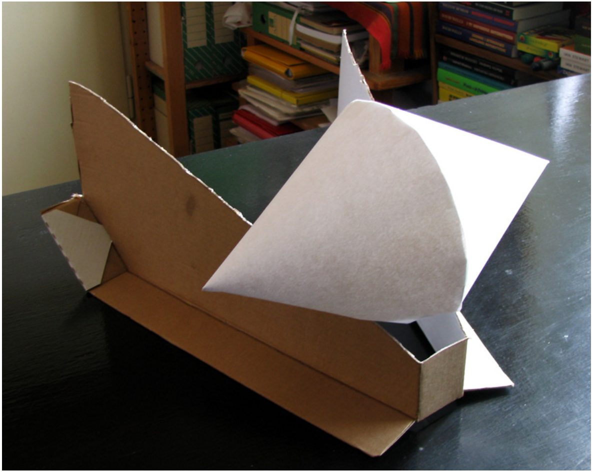
Ant on a rubber rope

Ant on a rubber rope

- An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rope it is crawling on), starting from its left fixed end. At the same time, the whole rope starts to stretch with the speed 1 km per second (both in front of and behind the ant, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc). Will the ant ever reach the right end of the rope?
- The answer is positive. The key to solution is the divergence of the harmonic series. Important hint: replace continuous process by a discrete one.
- The main question is: which part of the rope is crawled by the ant in each consecutive second?

- Other puzzles involving harmonic series: lion and man, jeep problem, best candidate, maximum possible overhang etc.

Double cone rolling up

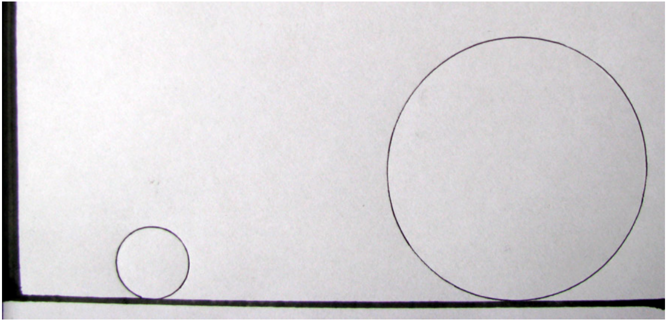


Defying gravity?!

- Imagine a double cone (two identical cones joined at their bases). It is put on the inclined rails which, in turn, are placed on a table. The rails have a common point and are diverging. The common point is the lowest point of the rails – they are directed upwards.
- Let the angle of inclination of the rails equals α, the angle between horizontal surface of the table and the up-going rails equals β and the angle at the apex of each cone equals γ. Finally, let the radius of the circle forming the common base of the cones equals r.

- Puzzle: can we determine α, β, γ and r in such a way that the double cone will be rolling upwards on the rails?
- This can be done. Obviously, the center of gravity of the cone will move downwards, obeying the law of gravity.

Bouncing balls



Watch the video Billiard Balls Count π at YouTube.

Catch π

- Let the mass of two balls be M and m, respectively. Suppose that $M = 100^n m$ ($n \geqslant 0$). We roll the ball M towards ball m which is near the wall. Thus M hits m which bounces off the wall. How many times do the balls touch each other before the ball M changes direction?
- The answer depends on n, of course.

- A surprising fact is that the number of balls' collisions is equal *precisely* to the first $n+1$ digits of π. Actually, one can obtain $n+1$ first digits of any real number, suitably modifying the ratio $\frac{M}{m}$.
- It is worth noticing that the result is purely *deterministic* and not based on probability, as in the well-known Buffon's needle puzzle.

Pathologies reveal the vitality of mathematics!

- Objects called *pathological* in mathematics are either unexpected, contradicting the old intuitions or specially created, e.g. just to show the limitations of intuition.
- Pathologies may thus appear at the border between Known and Unknown at a given epoch (e.g.: negative and imaginary numbers).
- Pathologies may also be constructed on purpose, e.g. to show a conflict between particular intuitions (say, cardinality and measure).

- Important: usually, pathologies become *domesticated*.
- Paradox resolution and domestication of pathologies may thus be crucial for the development of new mathematical theories.

Math is sexy! Logic is fun!

- Students are active during classes and seem to be really interested in this enterprize. They are encouraged to use their imagination and to be creative.
- Besides, it is much more easier for them to acquire small, concise chunks of dissipated knowledge rather than to listen to lengthy expositions of entire theories (which they can get from textbooks anyway).

- I hope you did enjoy the puzzles shortly presented today. If you wish, I can show the solutions in full detail after this talk.
- If you know of an interesting puzzle, please let me know, I will show it to my students.

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