Lewis Carroll's Resolution and Tableaux

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Resolution rule and analytic tableaux are not the XXth century invention. They were used already around 1896, in the logical works of Lewis Carroll (i.e. Charles Lutwidge Dodgson). This fact has been known since 1977, i.e. the year in which W.W. Bartley III has published the fragments of Part II of Carroll's *Symbolic Logic*, found after several years of investigations. We will show, using a few examples from Carroll's *Symbolic Logic* (1896), how he used resolution and his prototype of analytic tableaux. The examples in question are the famous Carroll's soriteses. We will also add a few metalogical and historical remarks.

1. A triviality from the algebra of sets

We use the standard notation. The complement of a set A (in a given universe U) is denoted by A'. Recall that $A \subseteq B$ is equivalent to $A \cap B' = \emptyset$. For any A, B and C we have:

$$(\bigstar) \quad (A \cap C = \emptyset \wedge B \cap C' = \emptyset) \to A \cap B = \emptyset.$$

($\bigstar$) is self-evident: the antecedent of ($\bigstar$) says that $A \subseteq C'$ and $B \subseteq C$. As an easy exercise the reader may try to represent the information given in the antecedent of ($\bigstar$) on the Carroll's diagram. What can be said about A, B and C from this diagram?

Here is (an algebraic) proof of ($\bigstar$):

1.	$A \cap C = \emptyset$	assumption
2.	$B \cap C' = \emptyset$	assumption
3.	$(A \cap C) \cup C' = C'$	$\cup C'$ to both sides
4.	$(B \cap C') \cup C = C$	$\cup C$ to both sides
5.	$(A \cup C') \cap (C \cup C') = C'$	3, calculation
6.	$(B \cup C) \cap (C \cup C') = C$	4, calculation
7.	$A \cup C' = C'$	5, $C \cup C' = U$
8.	$B \cup C = C$	6, $C \cup C' = U$
9.	$(A \cup C') \cap (B \cup C) = C \cap C'$	7,8 $\cap$ both sides
10.	$(A \cup C') \cap (B \cup C) = \emptyset$	9, $C \cap C' = \emptyset$
11.	$(A \cap B) \cup (B \cap C') \cup (A \cap C) \cup (C \cap C') = \emptyset$	10, calculation
12.	$A \cap B = \emptyset$	11, 1, 2, $C \cap C' = \emptyset$.

Q.E.D.

The reader may, if she wishes, try to find a simpler (but algebraic, without any use of quantifiers!) proof of ($\bigstar$).

Please notice that ($\bigstar$) is, as a matter of fact, a form of the ***resolution rule***.

2. Carroll's soriteses

This triviality from the algebra of sets suffices to find a conclusion of ***some*** soriteses, whose premises are general (affirmative or negative) sentences. Those names, which appear both: positively and negatively (i.e. as complements) can be eliminated, due to ($\bigstar$). The eliminated names are called *eliminands*, all the other names are called *retinends*. Only retinends can appear in the conclusion of a given sorites.

In order to find a conclusion from general sentences $\alpha_1, \alpha_2, \ldots, \alpha_n$ one should:

- (1) (re)formulate all premisses in a general negative form, using:

$$A \subseteq B \text{ is equivalent to } A \cap B' = \emptyset;$$

- (2) construct a *register of attributes*), i.e. a list of names appearing in the particular premisses in:
  - (a) a positive form
  - (b) a negative form;
- (3) rearrange the sequence of premisses so that the resolution rule ($\bigstar$) will be applied in a proper order;
- (4) applying ($\bigstar$) till all the eliminands will be eliminated;
- (5) formulate the conclusion (and, if one so wishes, transform it into a general affirmative sentence).

2.1. Carroll's algebraic notation

Carroll applied the following algebraic notation:

- X_0 for $X = \emptyset$, X_1 for $X \neq \emptyset$
- XY_0 for $X \cap Y = \emptyset$, XY_1 for $X \cap Y \neq \emptyset$
- † for conjunction, and ¶ in case when the premisses validate the conclusion.

An expression in the form XY_0 is called a *nullity* (and similarly for any finite number of sets with the empty intersection).

An expression in the form XY_1 is called a *entity* (and similarly for any finite number of sets with the non-empty intersection).

The subscript $_1$ was used, as a matter of fact, only in order to separate (sometimes compound) subjects from predicates in categorical sentences.

In Part I of *Symbolic Logic* Carroll accepted *existential import* for general affirmative sentences. Only later he seemed to accept that this assumption is not at all necessary.

Here are some rules formulated by Carroll (*Symbolic Logic*, 126):

- Two Nullities, with Unlike Eliminands, yield a Nullity, in which both Retinends keep their Signs. A Retinend, asserted in the Premisses to exist, may be so asserted in the Conclusion.

$$XM_0 \dagger YM_0' \P XY_0$$

- A Nullity and an Entity, with Like Eliminands, yield an Entity, in which the Nullity-Retinend changes its Sign.

$$XM_0 \dagger YM_1 \P X'Y_1$$

- Two Nullities, with Like Eliminands asserted to exist, yield an Entity, in which both Retinends change their Signs.

$$XM_0 \dagger YM_0 \dagger M_1 \P X'Y_1'$$

2.2. „The Method of Underscoring”

The use of ($\bigstar$) was symbolized as follows. From premisses $XM_0 \dagger YM_0'$ we get (on the basis of ($\bigstar$)) the conclusion XY_0. We underscore the *eliminand* M in the first premiss once and we underscore the *eliminand* M' in the second premiss twice. Those names which are not underscored are *retinends* and form the conclusion XY_0. For a given sorites we apply this procedure for all *eliminand*. At the end we get a *nullity*, which is the conclusion of the sorites in question. Shortly: we underscore the first occurrence of an eliminand once, and its second occurrence twice.

For instance, for the premisses:

1	2	3	4	5	6	7
K_1L_0'	DH_0'	A_1C_0	B_1E_0'	$K'H_0$	$B'L_0$	$D_1'C_0'$

the method of underscoring gives:

1	5	2	6	4	7	3		
$\underline{K}\ \underline{L'}_0$	$\underline{\underline{K'}}\ \underline{H}_0$	$\underline{D}\ \underline{\underline{H'}}_0$	$\underline{B'}\ \underline{\underline{L}}_0$	$\underline{\underline{B}}\ E_0'$	$\underline{\underline{D'}}_1\ \underline{C'}_0$	$A_1\ \underline{\underline{C}}_0$	¶	$E'\ A_0 \dagger A_1$

Hence, the conclusion of 1.–7. is: $A \cap E' = \emptyset \wedge A \neq \emptyset$, i.e. (existential import!): $A \subseteq E$.

2.3. A very simple example: linear resolution

Consider the following categorical sentences (with their general-negative counterparts):

1.	$A \subseteq B$	$A \cap B' = \emptyset$
2.	$D \subseteq E$	$D \cap E' = \emptyset$
3.	$H \cap B = \emptyset$	$H \cap B = \emptyset$
4.	$C \cap E = \emptyset$	$C \cap E = \emptyset$
5.	$D' \subseteq A$	$D' \cap A' = \emptyset$

We construct the register of attributes:

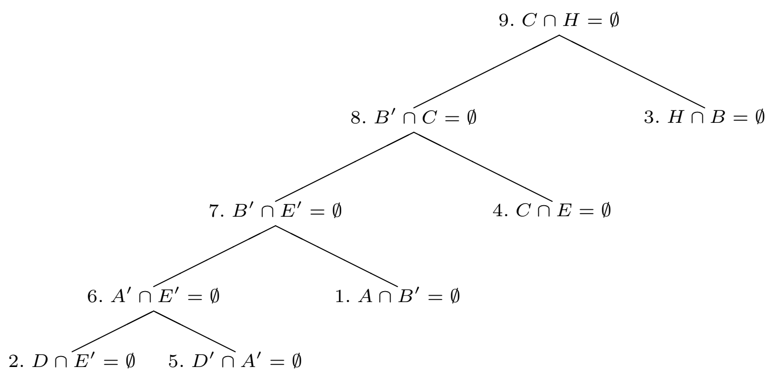
Name	Positively	Negatively
A	1	5
B	1	3
C	4	
D	2	5
E	4	2
H	3	

This table *suggests* that: A, B, D and E will be eliminated and that the conclusion could be: $C \cap H = \emptyset$.

We build the (resolution) proof:

1.	$A \cap B' = \emptyset$	assumption
2.	$D \cap E' = \emptyset$	assumption
3.	$H \cap B = \emptyset$	assumption
4.	$C \cap E = \emptyset$	assumption
5.	$D' \cap A' = \emptyset$	assumption
6.	$A' \cap E' = \emptyset$	($\bigstar$): 2,5, D
7.	$B' \cap E' = \emptyset$	($\bigstar$): 1,6, A
8.	$B' \cap C = \emptyset$	($\bigstar$): 4,7, E
9.	$C \cap H = \emptyset$	($\bigstar$): 3,8, B

Here is the corresponding proof tree:



Notice that:

- we can begin the resolution proof from any premiss
- the proof trees in the cases considered here always have the above simple form: they are determined by the sequence of pairs (C_i, A_i) ($0 \leqslant i \leqslant n$), where C_0 and all A_i are premisses or elements of some clause C_j for $j < i$, and each C_{i+1} ($i < n$) is the resolvent of C_i and A_i.

Such resolution is today called *linear*.

2.4. „The Method of Barred Premisses”

Let us recall Carroll’s *The Pigs and Ballons Problem*:

- 1. All, who neither dance on tight ropes nor eat penny-buns, are old.
- 2. Pigs, that are liable to giddiness, are treated with respect.
- 3. A wise balloonist takes an umbrella with him.
- 4. No one ought to lunch in public, who looks ridiculous and eats penny-buns.
- 5. Young creatures, who go up in balloons, are liable to giddiness.
- 6. Fat creatures, who look ridiculous, may lunch in public, provided they do not dance on tight ropes.
- 7. No wise creatures dance on tight ropes, if liable to giddiness.
- 8. A pig looks ridiculous, carrying an umbrella.
- 9. All, who do not dance on tight ropes, and who are treated with respect are fat.

What conclusion can we get from these premisses?
We find all the names occurring in the premisses:

A	—	balloonists
B	—	carrying umbrellas
C	—	dancing on tight ropes
D	—	eating penny-buns
E	—	fat
F	—	liable to giddiness
G	—	looking ridiculous
H	—	may lunch in public
J	—	old
K	—	pigs
L	—	treated with respect
M	—	wise.

We assume that *young* is the same as *not old*. Here are the schemes of the premisses:

1. $(C' \cap D') \cap J' = \emptyset$
2. $(K \cap F) \cap L' = \emptyset$
3. $(M \cap A) \cap B' = \emptyset$
4. $(G \cap D) \cap H = \emptyset$
5. $(J' \cap A) \cap F' = \emptyset$
6. $(E \cap G \cap C) \cap H' = \emptyset$
7. $(M \cap F) \cap C = \emptyset$
8. $(K \cap B) \cap G' = \emptyset$
9. $(C' \cap L) \cap E' = \emptyset$.

Observe that all these sentences have compound subjects. In what follows we are going to skip the parentheses.

We build the register of attributes:

Name	Positively	Negatively
A	3,5	
B	8	3
C	7	1,6,9
D	4	1
E	6	9
F	2,7	5
G	4,6	8
H	4	6
J		1,5
K	2,8	
L	9	2
M	3,7	

This table *suggests* that the conclusion could be: $K \cap M \cap A \cap J' = \emptyset$. Before we give the resolution proof, let us recall what Carroll meant by ***barred premisses***. If a name A occurs positively in a premiss P and the complementary name A' occurs in the premisses $Q_1, Q_2, \dots, Q_k$, then the premisses $Q_1, Q_2, \dots, Q_k$ should be considered first, before we take into account the premiss P (i.e. we should first apply ($\bigstar$) to $Q_1, Q_2, \dots, Q_k$ and only after that to P). Carroll wrote that in such a case that P ***is barred by*** $Q_1, Q_2, \dots, Q_k$.

In the case just considered we have exactly this situation:

- the premiss 5 is barred by the premisses 2 and 7;
- the premiss 7 is barred by the premisses 1, 6 and 9;
- the premiss 8 is barred by the premisses 4 and 6.

We give the corresponding resolution proof below. The steps are numbered twice and the boldface numbers refer to the premisses given above or to the consecutive resolvents obtained during the proof.

1.	**1.**	$C' \cap D' \cap J' = \emptyset$	assumption
2.	**4.**	$G \cap D \cap H = \emptyset$	assumption
3.	**10.**	$C' \cap J' \cap G \cap H = \emptyset$	(★): **1**,**4**, D
4.	**6.**	$E \cap G \cap C' \cap H' = \emptyset$	assumption
5.	**11.**	$C' \cap J' \cap G \cap E = \emptyset$	(★): **6**,**10**, H
6.	**8.**	$K \cap B \cap G' = \emptyset$	assumption
7.	**12.**	$C' \cap J' \cap E \cap K \cap B = \emptyset$	(★): **8**,**11**, G
8.	**9.**	$C' \cap L \cap E' = \emptyset$	assumption
9.	**13.**	$C' \cap J' \cap K \cap B \cap L = \emptyset$	(★): **9**,**12**, E
10.	**7.**	$M \cap F \cap C = \emptyset$	assumption
11.	**14.**	$J' \cap K \cap B \cap L \cap M \cap F = \emptyset$	(★): **7**,**13**, C
12.	**3.**	$M \cap A \cap B' = \emptyset$	assumption
13.	**15.**	$J' \cap K \cap L \cap M \cap F \cap A = \emptyset$	(★): **3**,**14**, B
14.	**2.**	$K \cap F \cap L' = \emptyset$	assumption
15.	**16.**	$J' \cap K \cap M \cap F \cap A = \emptyset$	(★): **2**,**15**, L
16.	**5.**	$J' \cap A \cap F' = \emptyset$	assumption
17.	**17.**	$J' \cap K \cap M \cap A = \emptyset$	(★): **5** ,**16**, F.

Thus, $K \cap M \cap A \cap J' = \emptyset$ is the conclusion from 1.–9. It may be read as: ***No wise young pigs go up in balloons***.

Carroll's algebraic notation enabled him to represent the proof in a very concise way (we omit the subscript $_0$):

1.	**1.**	$\underline{C'}\ \underline{D'}\ J'$	6.	**7.**	$M\ \underline{F}\ \underline{\underline{C}}$
2.	**4.**	$\underline{G}\ \underline{\underline{D}}\ \underline{H}$	7.	**3.**	$M\ A\ \underline{\underline{B'}}$
3.	**6.**	$\underline{E}\ \underline{G}\ \underline{C'}\ \underline{\underline{H'}}$	8.	**2.**	$K\ \underline{F}\ \underline{\underline{L'}}$
4.	**8.**	$K\ \underline{B}\ \underline{\underline{G'}}$	9.	**5.**	$J'\ A\ \underline{\underline{F'}}$
5.	**9.**	$\underline{C'}\ \underline{L}\ \underline{\underline{E'}}$	10.	$\therefore$	$K\ M\ A\ J'$

2.5. „The Library Problem”

As the reader has probably guessed already, Carroll's resolution rule (★) is not sufficient for dealing properly with any sorites. He was aware of this fact himself. And this was probably one of the reasons for which he turned to a more general method, i.e. his „Method of Trees”. But before we discuss that method let us look at a problem which does not have a solution using methods described so far.

Consider the following sentences:

- 1. All the old books are Greek.
- 2. All the quartos are bound.
- 3. None of the poets are old quartos.

The general names occurring here are:

A	—	bound
B	—	Greek
C	—	old
D	—	poetry
E	—	quartos.

The schemes of the premisses are:

1. $C \cap B' = \emptyset$
2. $E \cap A' = \emptyset$
3. $D \cap C \cap E = \emptyset$.

Here is the register of attributes:

Name	Positively	Negatively
A		2
B		1
C	1,3	
D	3	
E	2,3	

It is clear that in this case ***none*** of the general names involved can be eliminated (using (★)). Carroll claims that the remedy is to accept the following additional assumption asserting the fact that the union of all the considered names exhausts the universe of discourse:

$$4. \quad A' \cap B' \cap C' \cap D' \cap E' = \emptyset.$$

Then, he says, the conclusion should be: $A' \cap B' = \emptyset$. But ***he is wrong***.

Consider the following simple counterexample. Let $A = B = C = E = \{x\}$, $D = \{y\}$, $x \neq y$, and the universe is $\{x, y\}$. Then 1.–4. hold, but $A' \cap B' = \{y\} \neq \emptyset$. The book x may be e.g. an old Greek *in quarto* edition of the *Prior Analytics* (which by no means is any poetry), and let us take for y e.g. a heap of unbound new *in folio*, say Polish, poems.

The correspondence between Carroll and John Cook Wilson concerning this problem contains Carroll's remarks about syllogisms with negated conjunctions of names. It may be also observed that Carroll uses not only the De Morgan laws but also the laws of distribution.

2.6. An open (?) problem

Let $\mathbb{S}$ be a family of nullities of the form:

$$\begin{array}{l} X^{11} \cap X^{12} \cap \ldots \cap X^{1n_1} = \emptyset \\ X^{21} \cap X^{22} \cap \ldots \cap X^{2n_2} = \emptyset \\ \ldots \\ X^{m1} \cap X^{m2} \cap \ldots \cap X^{mn_m} = \emptyset \end{array}$$

where $\mathbf{R}(\mathbb{S})$, the set of all retinends of $\mathbb{S}$, is non-empty.

It seems that the solution to the following problem (exercise?) will provide a proper metalogical setting for Carroll's method of resolving soriteses with the rule (★):

(★★) Find necessary and sufficient conditions under which the nullity

$$\bigcap \mathbf{R}(\mathbb{S}) = \emptyset$$

logically follows from the set of nullities $\mathbb{S}$.

Observe that this problem can also be formulated in terms of covers of the universe of discourse U. We simply reformulate the assumptions of the problem using De Morgan laws.

The solution of (★★) would give us a kind of ***Completeness Theorem*** for Carroll's *Method of Underscoring*. Though (★) alone is not sufficient for resolving all soriteses in the manner suggested at first by Carroll, we can reasonably ask for which soriteses is this method sound and complete.

Notice also that the solution of (★★) should be given in purely algebraic terms (form the Algebra of Sets), if one thinks of being faithful to the methods used by Carroll.

3. Carroll's „Method of Trees"

On July 16, 1894 Carroll wrote in his *Diary*:

> *Today has proved to be an epoch in my Logical work. It occurred to me to try a complex Sorites by the method I have been using for ascertaining what cells, if any, survive for possible occupation when certain nullities are given. I took one of 40 premisses, „pairs within pairs"* & *many bars,* & *worked it like a genealogy, each term providing all its descendents. It came out beautifully,* & *much shorter than the method I have used hitherto — I think of calling it the „Genealogical Method".* (*Symbolic Logic*, 279)

Carroll used also the name ***The Method of Trees***. Actually, it is just a prototype of the modern method of ***analytic tableaux***. Let us show briefly how this method worked.

3.1. A tree without branching

Consider the sentences:

1. $D' \cap N' \cap M' = \emptyset$
2. $K \cap A' \cap C' = \emptyset$
3. $L \cap E \cap M = \emptyset$
4. $D \cap H \cap K' = \emptyset$
5. $H' \cap L \cap A' = \emptyset$
6. $H \cap M' \cap B' = \emptyset$
7. $A' \cap B \cap N = \emptyset$
8. $A \cap M' \cap E = \emptyset$.

We build the register of attributes:

Name	Positively	Negatively
A	8	2,5,7
B	7	6
C		2
D	4	1
E	3,8	
H	4,6	5
K	2	4
L	3,5	
M	3	1,6,8
N	7	1

This table *suggests* that the conclusion could be: $C' \cap E \cap L = \emptyset$. Instead of a resolution proof we will give a proof by contradiction. Suppose that $C' \cap E \cap L = \emptyset$ ***does not hold***.

Then:

$$(\dagger) \quad C' \cap E \cap L \neq \emptyset$$

i.e. $C' \cap E \cap L$ contains some element.

We will show that this supposition leads to a contradiction and hence should be abandoned.

Let $x \in C' \cap E \cap L$. Since $x \in E \cap L$, and (see 3.) $(L \cap E) \cap M = \emptyset$, we have $x \notin M$, i.e. $x \in M'$. Thus, $x \in E \cap M'$. Hence, because $A \cap (M' \cap E) = \emptyset$ (see 8.), $x \notin A$, i.e. $x \in A'$. From $x \in C'$ (see ($\dagger$)) and $x \in A'$ we get $x \notin K$ (see 2.: $K \cap (A' \cap C') = \emptyset$). Therefore $x \in K'$. From $x \in E$ and $x \in A'$ we obtain (see 5.: $H' \cap (L \cap A') = \emptyset$) $x \notin H'$, i.e. $x \in H$. From $x \in H$ i $x \in K'$, to (see 4.: $D \cap (H \cap K') = \emptyset$) $x \notin D$, i.e. $x \in D'$. From $x \in M'$ oraz $x \in H$, to (see 6.: $(H \cap M') \cap B' = \emptyset$) $x \notin B'$, i.e. $x \in B$. From $x \in D'$ and $x \in M'$ we get (see 1.: $(D' \cap M') \cap N' = \emptyset$) $x \notin N'$, i.e. $x \in N$. Finally, from $x \in A'$ and $x \in B$, we obtain (see 7.: $(A' \cap B) \cap N = \emptyset$) $x \notin N$, i.e. $x \in N'$. Because $N \cap N' = \emptyset$, we get a ***contradiction***: $x \in N$ and $x \in N'$. Thus, we should reject ($\dagger$) and we obtain the conclusion $C' \cap E \cap L = \emptyset$.

This was a proof for those who demand (as my dear female students in the Humanities do) that we should talk in full declarative sentences and not omit anything. Now let us take a look at a simplified proof, presented in the usual manner.

1.	$D' \cap N' \cap M' = \emptyset$	assumption
2.	$K \cap A' \cap C' = \emptyset$	assumption
3.	$L \cap E \cap M = \emptyset$	assumption
4.	$D \cap H \cap K' = \emptyset$	assumption
5.	$H' \cap L \cap A' = \emptyset$	assumption
6.	$H \cap M' \cap B' = \emptyset$	assumption
7.	$A' \cap B \cap N = \emptyset$	assumption
8.	$A \cap M' \cap E = \emptyset$	assumption
9.	$x \in C' \cap E \cap L$	supposition
10.	$x \in M'$	3,9
11.	$x \in A'$	8,9,10
12.	$x \in K'$	2,9,11
13.	$x \in H$	5,9,11
14.	$x \in D'$	4,12,13
15.	$x \in B$	6,10,13
16.	$x \in N$	1,10,14
17.	$x \in N'$	7,11,15
18.	$\perp$	CONTRADICTION: 16,17.

Thus, we have proven: $C' \cap E \cap L = \emptyset$.

3.2. A tree with branching

Consider the following sentences:

1. $H \cap M \cap K = \emptyset$
2. $D' \cap E' \cap C' = \emptyset$
3. $H \cap K' \cap A' = \emptyset$
4. $B \cap L \cap H' = \emptyset$
5. $C \cap K \cap M' = \emptyset$
6. $H \cap C' \cap E = \emptyset$
7. $B \cap A \cap K' = \emptyset$.

We build the register of attributes:

Name	Positively	Negatively
A	7	3
B	4,7	
C	5	2,6
D		2
E	6	2
H	1,3,6	4
K	1,5	3,7
L	4	
M	1	5

This table *suggests* that the conclusion could be: $B \cap D' \cap L = \emptyset$.

Let us suppose that

$$(\ddagger) \quad B \cap D' \cap L \neq \emptyset.$$

We will show that this supposition leads to a contradiction and hence should be rejected.

Let $x \in B \cap D' \cap L$. Then, from $x \in B \cap L$ and the premiss 4. we get $x \notin H'$, i.e. $x \in H$. From $x \in H$ and the premiss 1.: $H \cap M \cap K = \emptyset$, we obtain $x \notin M \cap K$. And now the inference can not be linear any more, because $x \notin M \cap K$ means that:

- (1) $x \notin K$, czyli $x \in K'$ ***or***
- (2) $x \notin M$, czyli $x \in M'$.

Each case: (1) and (2) should now be considered separately. It is evident for us today: we simply apply the corresponding De Morgan law. Carroll has formulated this law by himself (in a letter to John Cook Wilson, dated November 11, 1896), without any hint to the contemporary works in the Algebra of Logic.

Let us come back to the proof. Carroll observes at this point that:

- (1') we can add $x \in M$ to (1); however this is of no further use, because M occurs in the premiss 1. only and we have used 1. already;
- (2') we can add $x \in K$ to (2); and this may be of some use, because K occurs in the premiss 5. which was not used yet.

The following ***rule*** is the justification of (1') and (2'), in Carroll's own words:

> *Thus, if we found a Premiss proving that the Thing* ***could not*** *have the Pair of Attributes $b'c$, we might say it* ***must*** *have b or c'. And we might afterwards tack on, at pleasure, either c to b, making the two headings bc and c', or b' to c', making them b and $c'b'$.* (*Symbolic Logic*, 287)

Thus, Carroll refers here to the following observation:

$$(A \cap B)' = (A' \cap B) \cup (A \cap B') \cup (A' \cap B').$$

One may guess that he got this observation from his diagrams.

(1) From $x \in K'$, $x \in H$ and the premiss 3. we get: $x \notin A'$, i.e. $x \in A$. From $x \in K'$, $x \in A$ and the premiss 7. we have $x \notin A$. Thus, we get a contradiction: $x \in A$ and $x \notin A$.

(2) From $x \in M'$, (2') and the premiss 5. we get $x \notin C$, i.e. $x \in C'$. From $x \in D'$ (see $(\ddagger)$), $x \in C'$ and the premiss 2. we obtain $x \notin E'$, i.e. $x \in E$. Finally, from $x \in H$ oraz $x \in C'$ and the premiss 6. we have $x \notin E$. Thus we get a contradiction: $x \in E$ and $x \notin E$.

We have shown that ***each*** of (1) and (2) leads to a contradiction. Hence we should reject the supposition $(\ddagger)$. Finally, we have:

$$B \cap D' \cap L = \emptyset.$$

This is equivalent to an affirmative sentence: $(B \cap L) \subseteq D$.

3.3. A tree with no contradiction

And how did Carroll proceeded in the cases when no contradiction arose? We find an example in his letter to John Cook Wilson, dated November 18, 1896:

1. $A \cap B' \cap C' \cap D' = \emptyset$
2. $A \cap B \cap C' \cap H' = \emptyset$
3. $B \cap A' \cap C' \cap D' = \emptyset$
4. $B \cap C \cap E \cap D' = \emptyset$
5. $C \cap D \cap A' \cap B' = \emptyset$
6. $E \cap A' \cap B' \cap D' = \emptyset$
7. $B \cap D \cap A' \cap H' = \emptyset$
8. $A \cap C \cap K \cap B' = \emptyset$
9. $D \cap K \cap B' \cap C' = \emptyset$.

We build the register of attributes:

Name	Positively	Negatively
A	1,2,8	3,5,6,7
B	2,3,4,7	1,5,6,8,9
C	4,5,8	1,2,3,9
D	5,7	1,3,4,6
E	4,6,9	
H		2,7
K	8,9	

The table *suggests* that the conclusion could be: $E \cap H' \cap K = \emptyset$. We will show that this is a ***wrong*** suggestion.

Let us suppose that:

$$E \cap H' \cap K \neq \emptyset.$$

Let $x \in E \cap H' \cap K$. From $x \in E$ and the premiss 4. we have: $x \notin B \cap (C \cap D')$ and this means that the following disjunction holds:

- (1) $x \in B'$
- (2) $x \in (C \cap D')'$.

Let us take (1) first. From $x \in E$ and $x \in B'$ and the premiss 6. we have $x \notin (A' \cap D')$ and thus the following disjunction holds:

- (1.1.) $x \in A$
- (1.2.) $x \in D$.

Let us consider (1.1.). From $x \in A$, $x \in B'$ and $x \in K$ and the premiss 8. we have $x \notin C$, i.e. $x \in C'$. From $x \in A$, $x \in B'$, $x \in C'$ and the premiss 1. we get: $x \notin D'$, i.e. $x \in D$. Finally, from $x \in K$, $x \in B'$ and $x \in C'$ and the premiss 1. we have:

$x \notin D$. Thus we have obtained a contradiction: $x \in D$ and $x \notin D$. The case (1.1.) is excluded.

We come back now to (1.2.). Carroll makes at this point an additional assumption $x \in A'$, justifying it by the rule mentioned above.

From $x \in B'$, $x \in D$, $x \in A'$ and the premiss 5. we get $x \notin C$, i.e. $x \in C'$. From $x \in K$, $x \in B'$, $x \in D$ and the premiss 9. we have $x \notin C'$. Hence, we have obtained a contradiction: $x \in C'$ and $x \notin C'$. The case (1.2.) is excluded.

We come back to (2). At this point Carroll again uses his rule mentioned above and accepts an additional assumption: $x \in B$. From $x \in (C \cap D')'$ we get that the following disjunction holds:

- (2.1.) $x \in C'$ (and $x \in B$)
- (2.2.) $x \in D$ (and $x \in B$).

Let us take (2.1.) first. From Skoro $x \in H'$, $x \in C'$, $x \in B$ and the premiss 2. we get $x \notin A$, i.e. $x \in A'$. From $x \in B$, $x \in A'$, $x \in C'$ and the premiss 3. we have $x \notin D'$, i.e. $x \in D$. Finally, from $x \in H'$, $x \in B$, $x \in A'$ and the premiss 7. we obtain $x \notin D$ and this contradicts $x \in D$. The case (2.1.) is excluded.

Finally, let us consider (2.2.). From $x \in H'$, $x \in B$, $x \in D$ and the premiss 7. we have: $x \notin A'$, i.e. $x \in A$. And at this point we can not make any use of the premisses in order to exclude the case (2.2.).

Therefore our supposition has not been confirmed and this means that

$$E \cap H' \cap K = \emptyset$$

is not a logical consequence of 1.–9.

Thus the case in which all the premisses (of the sorites in question) are true and its alleged conclusion is false has not been excluded. There may exist x such that $x \in E \cap H' \cap K$ and all of 1.–9. hold. It follows from our analysis of (2.2.) that in such a case we have: $x \in A \cap B \cap C \cap D$.

The analysis of (2.2.) also shows what conclusions ***do follow*** from 1.–9. Any of the following sentences is such a legitimate conclusion:

- $E \cap H' \cap K \cap A' = \emptyset$
- $E \cap H' \cap K \cap B' = \emptyset$
- $E \cap H' \cap K \cap C' = \emptyset$
- $E \cap H' \cap K \cap D' = \emptyset$.

Observe that each of the above sentences, together with 1.–9. implies a contradiction.

Furthermore, the analysis of (2.2.) shows that 1.–9. together with any of the following sentences:

- 10. $A = \emptyset$

- 11. $B = \emptyset$
- 12. $C = \emptyset$
- 13. $D = \emptyset$

do imply $E \cap H' \cap K = \emptyset$.

Obviously, such modifications „break the symmetry" in Carroll's original example. We add them in order to show that the „Method of Trees" always provides an answer in the analysis of soriteses.

Let us also recall that Carroll obviously did not make any use of the relation $\in$. He worked with purely algebraic terms.

4. A few metalogical remarks

As we have seen, the resolution rule ($\bigstar$) alone is not sufficient for a proper analysis of soriteses. On the other hand, the „Method of Trees" appears to be sufficient and adequate for such a goal. Moreover, this method can be applied to ***arbitrary*** categorical sentences. As we know from the result of Löwenheim (1915) the Monadic Predicate Calculus is ***decidable***. Hence, given any set of categorical sentences X and a categorical sentence α exactly one of the following cases holds:

- (1) from X and $\neg\alpha$ we get a ***contradiction***; then α ***logically follows*** from X;
- (2) from X and $\neg\alpha$ we do not get a ***contradiction***; then α ***does not logically follow*** from X.

We should, of course, prove that the „Method of Trees" is correct, i.e. that it is ***sound*** and ***complete***. Such a proof exists for (analytic tableaux for) Predicate Calculus and hence also for Carroll's „Method of Trees". Another possibility is to construct tableaux for (classical) syllogistic and prove soundness and completeness for the system — cf. e.g. Simons 1989.

Carroll did not prove the correctness of his „Method of Trees". Observe that such a proof is possible only in metalogic. And metalogic was born a few decades later. Carroll did not prove the correctness of his resolution method („the Method of Underscoring"). Also this proof needs a fairly developed metalogic, inaccessible at the

But Carroll was fully aware of the fact that one can not limit oneself to the „suggestions" given by the registers of attributes. In the case of each of his soriteses he always provides a proof that the conclusion is validated by the premisses (either by the resolution method or by his „Method of Trees").

May we say that Carroll's *Symbolic Logic* is based on some ***logical system***? Bartley observes that Carroll was using the truth tables already around 1894 and hence long before the proposals of Emil Post and Ludwig Wittgenstein. All the classical syllogistic is included in *Symbolic Logic*.

As we have seen above, Carroll was using also a prototype of the analytic tableaux method. Usually, one attributes the first uses of this method to the works of Beth, Hintikka, Kanger and Schütte from the fifties of the XXth century. The method has been further elaborated by Smullyan, Lis and Jeffrey. And now it is one of the most important methods e.g. in the automated theorem proving.

5. A few historical remarks

Carroll's logical works were written in the second half of XIX century. At that time a new approach in logic was developed, called The Algebra of Logic. It has begun with the works of George Boole and Augustus De Morgan and culminated in the work of Ernst Schröder. Meanwhile, such prominent logicians as e.g. MacColl, Peirce, Jevons and Venn also participated in the development of the algebraic approach.

Carroll certainly did know these works. But he had not attempted at the construction of a ***system of logic*** in his *Symbolic Logic*. Rather, the book was thought of as a teaching manual. It was Carroll's ambition to present logic in a popular way, accessible to a common reader. Remember that he taught mathematics and logic at The Christ College and his students were the innocent girls of the Victorian Epoch.

Without any doubts, Carroll has achieved his goal. His *Symbolic Logic* is still (after more than one hundred years!) used by the lecturers of logic. This is caused partly by his literary talent: almost all of his examples are vivid and have some subtle intricacy, both logical and linguistic.

Part I of *Symbolic Logic* is very elementary, it is simply an introduction to classical syllogistic. It also contains the discussion of Carroll's diagrams and his „Method of Underscoring" (i.e. the prototype of linear resolution).

Part II, found after 70 years by W.W. Bartley contains more difficult problems: e.g. categorical sentences with compound subjects and negated predicates. Here Carroll also introduces his „Method of Trees". Six chapters of this part have never been found.

Carroll did advertise also PART III: TRANSCENDENTAL. According to his own words, he collected a lot of notes concerning it. Two of the planned chapters were supposed to be entitled: *Analysis of a Proposition into its Elements* and *The Theory of Inference*.

As it is known, after Carroll's death almost all of his carefully collected notes

HAVE BEEN PUT ON FIRE.

* * *

We are still borrowing examples from Carroll's *Symbolic Logic* in order to make our serious (and sometimes boring) lectures in logic more interesting. The common opinion claims that *Symbolic Logic* is just a collection of more or less funny examples. This opinion is, it seems, very unjust to the Author. We were trying to bring the attention to some Carroll's ideas which show that he was not only a giant of the world literature but also a brilliant logician who discovered important methods of proof long before others did.

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