Lewis Carroll's Resolution and Tableaux

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AALCS XII

The goal for today

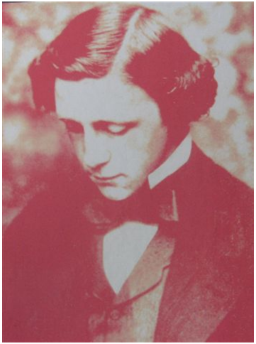
Resolution rule and analytic tableaux are not the XXth century invention. They were used already around 1896, in the logical works of Lewis Carroll (i.e. Charles Lutwidge Dodgson).

This fact has been known since 1977, i.e. the year in which W.W. Bartley III has published the fragments of Part II of Carroll's *Symbolic Logic*, found after several years of investigations.

We will show, using a few examples from Carroll's *Symbolic Logic* (1896), how he used resolution and his prototype of analytic tableaux. The examples in question are the famous Carroll's soriteses.

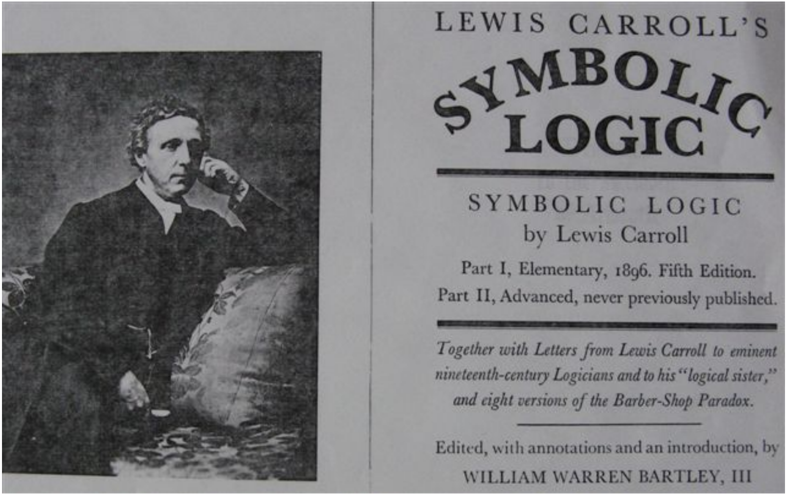
The Author and his work

Charles Lutwidge Dodgson



Lewis Carroll

The Author and his work


LEWIS CARROLL'S

SYMBOLIC LOGIC

SYMBOLIC LOGIC

by Lewis Carroll

Part I, Elementary, 1896. Fifth Edition.

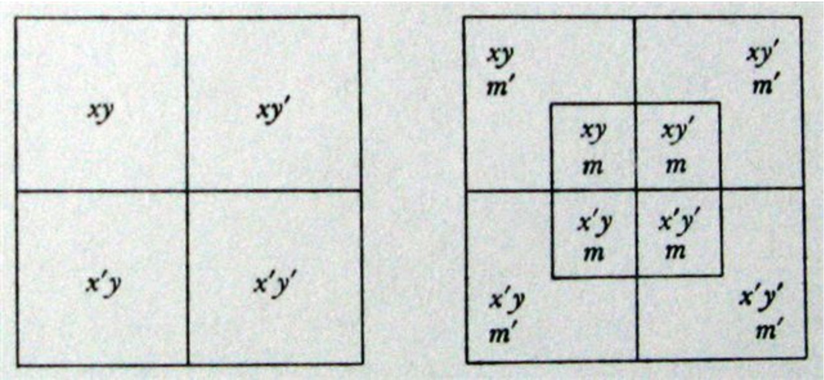
Part II, Advanced, never previously published.

Together with Letters from Lewis Carroll to eminent nineteenth-century Logicians and to his "logical sister," and eight versions of the Barber-Shop Paradox.

Edited, with annotations and an introduction, by

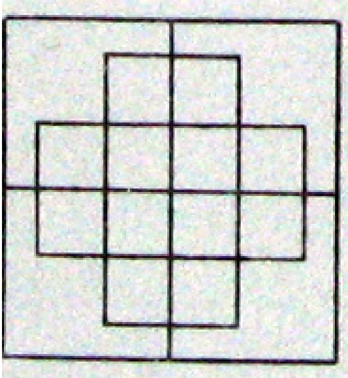
WILLIAM WARREN BARTLEY, III

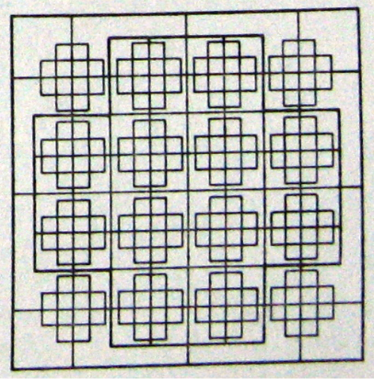
Carroll's diagrams



Carroll's diagrams for two and three sets.

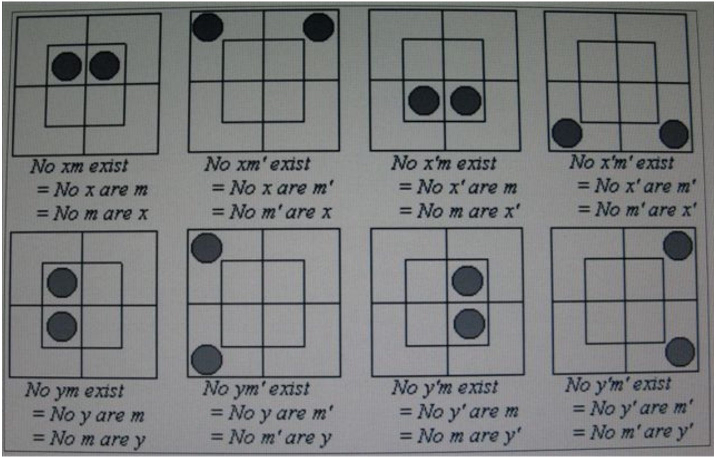
Carroll's diagrams





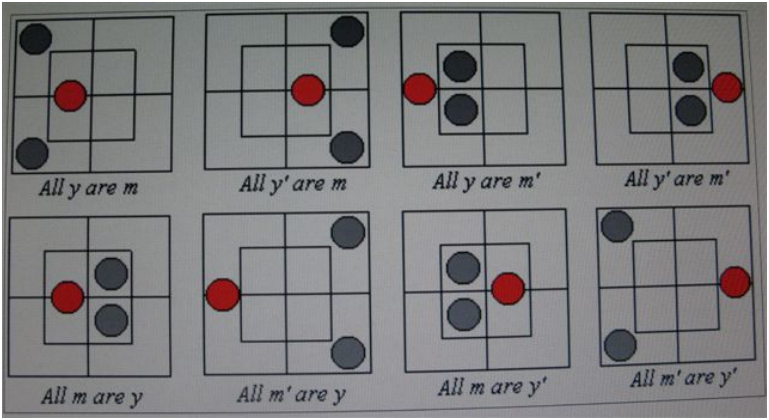
Carroll's diagrams for four and eight sets.

Carroll's diagrams



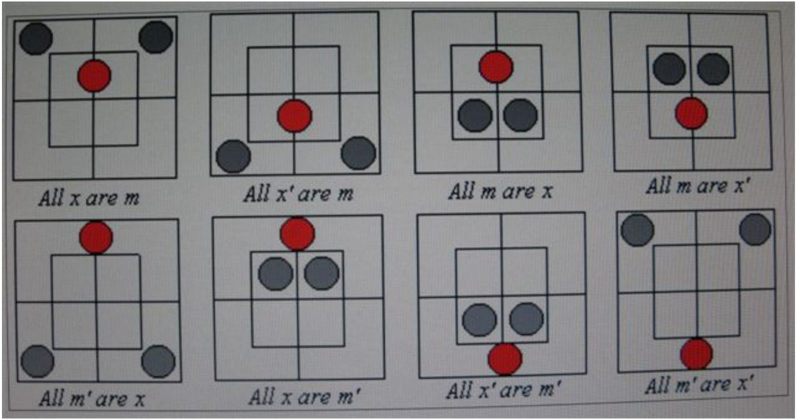
The gray circle marks empty areas.

Carroll's diagrams



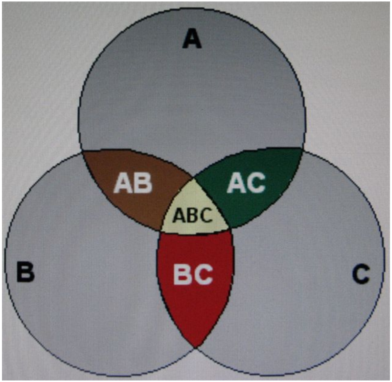
The red circle marks non-empty areas.

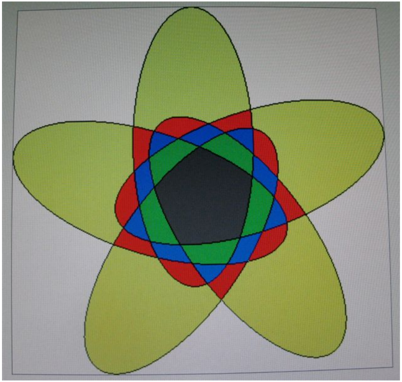
Carroll's diagrams



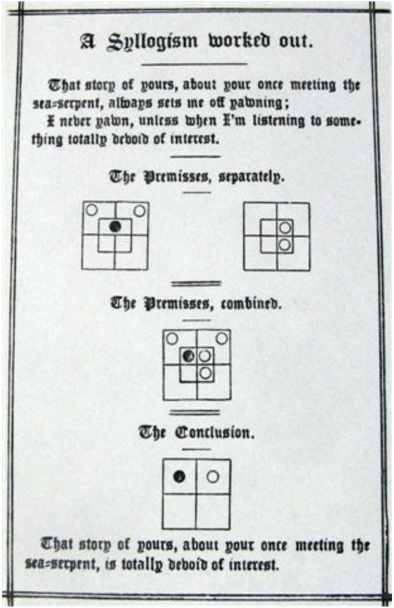
Carroll accepted existential import for general affirmative sentences.

Venn diagram





For comparison: Venn diagrams for three and five sets.



A Syllogism worked out.

That story of yours, about your once meeting the sea-serpent, always sets me off yawning;

I never yawn, unless when I'm listening to something totally devoid of interest.

The Premisses, separately.

The Premisses, combined.

The Conclusion.

That story of yours, about your once meeting the sea-serpent, is totally devoid of interest.

Let us begin to work

Now, let us begin some calculations.

A triviality from the algebra of sets

We use the standard notation. The complement of a set A (in a given universe U) is denoted by A'.

Recall that $A \subseteq B$ is equivalent to $A \cap B' = \emptyset$. For any A, B and C we have:

$$(\bigstar) \quad (A \cap C = \emptyset \wedge B \cap C' = \emptyset) \rightarrow A \cap B = \emptyset.$$

$(\bigstar)$ is self-evident: the antecedent of $(\bigstar)$ says that $A \subseteq C'$ and $B \subseteq C$.

Exercise: Represent the information given in the antecedent of $(\bigstar)$ on the Carroll's diagram. What can be said about A, B and C from this diagram?

For fun: an algebraic proof of (★)

1.	$A \cap C = \emptyset$	assumption
2.	$B \cap C' = \emptyset$	assumption
3.	$(A \cap C) \cup C' = C'$	$\cup C'$ to both sides
4.	$(B \cap C') \cup C = C$	$\cup C$ to both sides
5.	$(A \cup C') \cap (C \cup C') = C'$	3, calculation
6.	$(B \cup C) \cap (C \cup C') = C$	4, calculation
7.	$A \cup C' = C'$	5, $C \cup C' = U$
8.	$B \cup C = C$	6, $C \cup C' = U$
9.	$(A \cup C') \cap (B \cup C) = C \cap C'$	7,8 $\cap$ both sides
10.	$(A \cup C') \cap (B \cup C) = \emptyset$	9, $C \cap C' = \emptyset$
11.	$(A \cap B) \cup (B \cap C') \cup (A \cap C) \cup (C \cap C') = \emptyset$	10, calculation
12.	$A \cap B = \emptyset$	11, 1, 2, $C \cap C' = \emptyset$. Q.E.D.

Exercise: find a simpler (algebraic!) proof of (★).

Carroll's soriteses

This suffices to find a conclusion of ***some*** soriteses, whose premises are general (affirmative or negative) sentences. These names, which appear both: positively and negatively (i.e. as complements) can be eliminated, due to (★). The eliminated names are called *eliminands*, all the other names are called *retinends*. Only retinends can appear in the conclusion of a given sorites.
In order to find a conclusion from general sentences $\alpha_1, \alpha_2, \ldots, \alpha_n$ one should:

- (1) (re)formulate all premisses in a general negative form, using:

$$A \subseteq B \text{ is equivalent to } A \cap B' = \emptyset;$$

Carroll's soriteses

- (2) construct a *register of attributes*, i.e. a list of names appearing in the particular premisses in:
  - (a) a positive form
  - (b) a negative form;
- (3) rearrange the sequence of premisses so that the resolution rule (★) will be applied in a proper order;
- (4) applying (★) till all the eliminands will be eliminated;
- (5) formulate the conclusion (and, if one so wishes, transform it into a general affirmative sentence).

Carroll's algebraic notation

Carroll applied the following algebraic notation:

- X_0 for $X = \emptyset$, X_1 for $X \neq \emptyset$
- XY_0 for $X \cap Y = \emptyset$, XY_1 for $X \cap Y \neq \emptyset$
- † for conjunction, and ¶ in case when the premisses validate the conclusion.

An expression of the form XY_0 is called a nullity (and similarly for any finite number of sets with the empty intersection).
An expression of the form XY_1 is called an entity (and similarly for any finite number of sets with the non-empty intersection).

Carroll's algebraic notation

Note. The subscript $_1$ was used, as a matter of fact, only in order to separate (sometimes compound) subjects from predicates in categorical sentences.

Note. In Part I of *Symbolic Logic* Carroll accepted existential import for general affirmative sentences. Only later he seemed to accept that this assumption is not at all necessary.

Carroll's algebraic notation

Here are some rules formulated by Carroll:

Two Nullities, with Unlike Eliminands, yield a Nullity, in which both Retinends keep their Signs. A Retinend, asserted in the Premisses to exist, may be so asserted in the Conclusion.

$XM_0 \dagger YM'_0 \P XY_0$

A Nullity and an Entity, with Like Eliminands, yield an Entity, in which the Nullity-Retinend changes its Sign.

$XM_0 \dagger YM_1 \P X'Y_1$

Two Nullities, with Like Eliminands asserted to exist, yield an Entity, in which both Retinends change their Signs.

$XM_0 \dagger YM_0 \dagger M_1 \P X'Y'_1$

„The method of underscoring"

The use of (★) was symbolized as follows.

From premisses $XM_0 \dagger YM'_0$ we get (on the basis of (★)) the conclusion XY_0.
We underscore the *eliminand* M in the first premiss once and we underscore the *eliminand* M' in the second premiss twice.
Those names which are not underscored are *retinends* and form the conclusion XY_0.

For a given sorites we apply this procedure to all *eliminands*. At the end we get a *nullity*, which is the conclusion of the sorites in question.

Shortly: we underscore the first occurrence of an eliminand once, and its second occurrence twice.

„The method of underscoring"

For instance, for the premisses:

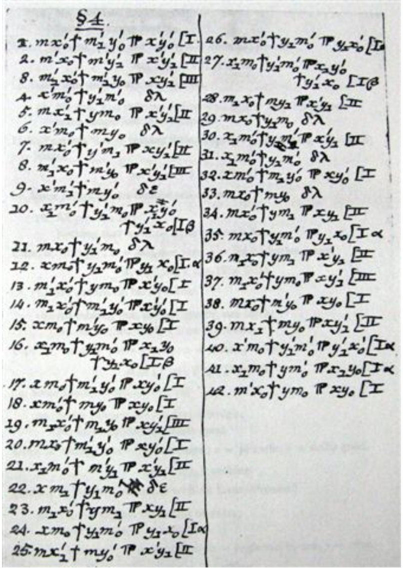
1	2	3	4	5	6	7
$K_1 L'_0$	DH'_0	$A_1 C_0$	$B_1 E'_0$	$K' H_0$	$B' L_0$	$D'_1 C'_0$

the method of underscoring gives:

1	5	2	6	4	7	3		
$\underline{K}\ \underline{L'}_0$	$\underline{\underline{K'}}\ \underline{H}_0$	$\underline{D}\ \underline{\underline{H'}}_0$	$\underline{B'}\ \underline{\underline{L}}_0$	$\underline{\underline{B}}\ E'_0$	$\underline{\underline{D'}}_1\ \underline{C'}_0$	$A_1\ \underline{\underline{C}}_0$	¶	$E'\ A_0 \dagger A_1$

Hence, the conclusion of 1.–7. is: $A \cap E' = \emptyset \wedge A \neq \emptyset$, i.e. (existential import!): $A \subseteq E$.

From Carroll's manuscript



Example 1

Consider the following categorical sentences (with their general-negative counterparts):

1.	$A \subseteq B$	$A \cap B' = \emptyset$
2.	$D \subseteq E$	$D \cap E' = \emptyset$
3.	$H \cap B = \emptyset$	$H \cap B = \emptyset$
4.	$C \cap E = \emptyset$	$C \cap E = \emptyset$
5.	$D' \subseteq A$	$D' \cap A' = \emptyset$

Example 1

We construct the register of attributes:

Name	Positively	Negatively
A	1	5
B	1	3
C	4	
D	2	5
E	4	2
H	3	

This table *suggests* that: A, B, D and E will be eliminated and that the conclusion could be: $C \cap H = \emptyset$.

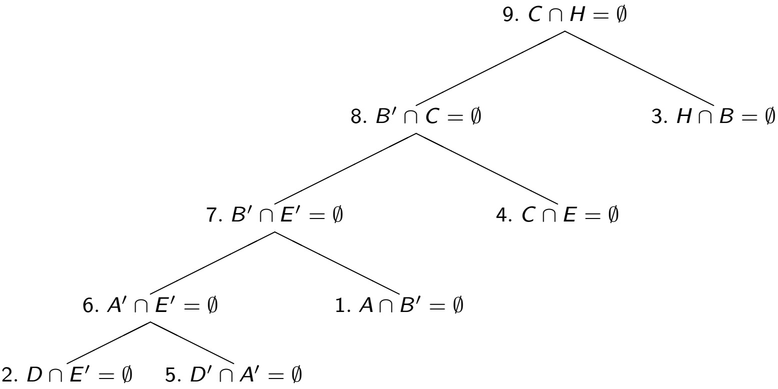
Example 1

We build the (resolution) proof:

1.	$A \cap B' = \emptyset$	assumption
2.	$D \cap E' = \emptyset$	assumption
3.	$H \cap B = \emptyset$	assumption
4.	$C \cap E = \emptyset$	assumption
5.	$D' \cap A' = \emptyset$	assumption
6.	$A' \cap E' = \emptyset$	$(\bigstar)$: 2,5, D
7.	$B' \cap E' = \emptyset$	$(\bigstar)$: 1,6, A
8.	$B' \cap C = \emptyset$	$(\bigstar)$: 4,7, E
9.	$C \cap H = \emptyset$	$(\bigstar)$: 3,8, B

Example 1

Here is the corresponding proof tree:



Example 1

Notice that:

- we can build the resolution proof starting from any premiss
- the proof trees in the cases considered here always have the above simple form: they are determined by a sequence of pairs (C_i, A_i) $(0 \leqslant i \leqslant n)$, where C_0 and all A_i are premisses or elements of some clause C_j for $j < i$, and each C_{i+1} $(i < n)$ is the resolvent of C_i and A_i.

Resolution of this form is called *linear*.

Example 2: The Pigs and Ballons Problem

- 1. All, who neither dance on tight ropes nor eat penny-buns, are old.
- 2. Pigs, that are liable to giddiness, are treated with respect.
- 3. A wise balloonist takes an umbrella with him.
- 4. No one ought to lunch in public, who looks ridiculous and eats penny-buns.
- 5. Young creatures, who go up in balloons, are liable to giddiness.
- 6. Fat creatures, who look ridiculous, may lunch in public, provided they do not dance on tight ropes.
- 7. No wise creatures dance on tight ropes, if liable to giddiness.
- 8. A pig looks ridiculous, carrying an umbrella.
- 9. All, who do not dance on tight ropes, and who are treated with respect are fat.

What conclusion can we get from these premisses?

Example 2: The Pigs and Ballons Problem

We find all the names occurring in the premisses:

A	—	balloonists
B	—	carrying umbrellas
C	—	dancing on tight ropes
D	—	eating penny-buns
E	—	fat
F	—	liable to giddiness
G	—	looking ridiculous
H	—	may lunch in public
J	—	old
K	—	pigs
L	—	treated with respect
M	—	wise.

Example 2: The Pigs and Ballons Problem

We assume that *young* is the same as *not old*. Here are the schemes of the premisses:

1. $(C' \cap D') \cap J' = \emptyset$
2. $(K \cap F) \cap L' = \emptyset$
3. $(M \cap A) \cap B' = \emptyset$
4. $(G \cap D) \cap H = \emptyset$
5. $(J' \cap A) \cap F' = \emptyset$
6. $(E \cap G \cap C) \cap H' = \emptyset$
7. $(M \cap F) \cap C = \emptyset$
8. $(K \cap B) \cap G' = \emptyset$
9. $(C' \cap L) \cap E' = \emptyset$.

Observe that all these sentences have compound subjects. In what follows we are going to skip the parentheses.

Example 2: The Pigs and Ballons Problem

We build the register of attributes:

Name	Positively	Negatively
A	3,5	
B	8	3
C	7	1,6,9
D	4	1
E	6	9
F	2,7	5
G	4,6	8
H	4	6
J		1,5
K	2,8	
L	9	2
M	3,7	

Example 2: The Pigs and Ballons Problem

This table *suggests* that the conclusion could be: $K \cap M \cap A \cap J' = \emptyset$.

Before we give the resolution proof, let us recall what Carroll meant by *barred premisses*. If a name A occurs positively in a premiss P and the complementary name A' occurs in the premisses $Q_1, Q_2, \ldots, Q_k$, then the premisses $Q_1, Q_2, \ldots, Q_k$ should be considered first, before we take into account the premiss P (i.e. we should first apply ($\bigstar$) to $Q_1, Q_2, \ldots, Q_k$ and only after that to P). Carroll wrote that in such a case that P *is barred by* $Q_1, Q_2, \ldots, Q_k$.

In the case just considered we have exactly this situation:

- the premiss 5 is barred by the premisses 2 and 7;
- the premiss 7 is barred by the premisses 1, 6 and 9;
- the premiss 8 is barred by the premisses 4 and 6.

Example 2: The Pigs and Ballons Problem

1.	**1.**	$C' \cap D' \cap J' = \emptyset$	assumption
2.	**4.**	$G \cap D \cap H = \emptyset$	assumption
3.	**10.**	$C' \cap J' \cap G \cap H = \emptyset$	(★): **1**,**4**, D
4.	**6.**	$E \cap G \cap C' \cap H' = \emptyset$	assumption
5.	**11.**	$C' \cap J' \cap G \cap E = \emptyset$	(★): **6**,**10**, H
6.	**8.**	$K \cap B \cap G' = \emptyset$	assumption
7.	**12.**	$C' \cap J' \cap E \cap K \cap B = \emptyset$	(★): **8**,**11**, G
8.	**9.**	$C' \cap L \cap E' = \emptyset$	assumption
9.	**13.**	$C' \cap J' \cap K \cap B \cap L = \emptyset$	(★): **9**,**12**, E
10.	**7.**	$M \cap F \cap C = \emptyset$	assumption
11.	**14.**	$J' \cap K \cap B \cap L \cap M \cap F = \emptyset$	(★): **7**,**13**, C
12.	**3.**	$M \cap A \cap B' = \emptyset$	assumption
13.	**15.**	$J' \cap K \cap L \cap M \cap F \cap A = \emptyset$	(★): **3**,**14**, B
14.	**2.**	$K \cap F \cap L' = \emptyset$	assumption
15.	**16.**	$J' \cap K \cap M \cap F \cap A = \emptyset$	(★): **2**,**15**, L
16.	**5.**	$J' \cap A \cap F' = \emptyset$	assumption
17.	**17.**	$J' \cap K \cap M \cap A = \emptyset$	(★): **5** ,**16**, F.

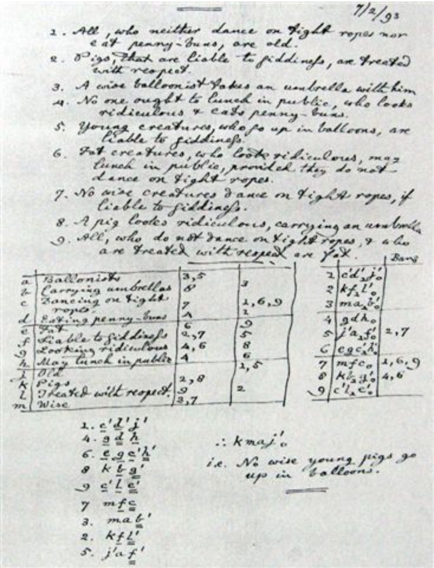
Example 2: The Pigs and Ballons Problem

Thus, $K \cap M \cap A \cap J' = \emptyset$ is the conclusion from 1.–9. It may be read as: ***No wise young pigs go up in balloons***.

Carroll's algebraic notation enabled him to represent the proof in a very concise way (we omit the subscript $_0$ in all formulas below):

1.	**1.**	$\underline{C'}\ \underline{D'}\ J'$	6.	**7.**	$M\ \underline{F}\ \underline{\underline{C}}$
2.	**4.**	$\underline{G}\ \underline{\underline{D}}\ \underline{H}$	7.	**3.**	$M\ A\ \underline{\underline{B'}}$
3.	**6.**	$\underline{E}\ \underline{G}\ \underline{C'}\ \underline{\underline{H'}}$	8.	**2.**	$K\ \underline{F}\ \underline{\underline{L'}}$
4.	**8.**	$K\ \underline{B}\ \underline{\underline{G'}}$	9.	**5.**	$J'\ A\ \underline{\underline{F'}}$
5.	**9.**	$\underline{C'}\ \underline{L}\ \underline{\underline{E'}}$	10.	$\therefore$	$K\ M\ A\ J'$

From Carroll's manuscript



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1. All, who neither dance on tight ropes nor eat penny-buns, are old.
2. Pigs, that are liable to giddiness, are treated with respect.
3. A wise balloonist takes an umbrella with him.
4. No one ought to lunch in public, who looks ridiculous & eats penny-buns.
5. Young creatures, who go up in balloons, are liable to giddiness.
6. Fat creatures, who look ridiculous, may lunch in public, provided they do not dance on tight ropes.
7. No wise creatures dance on tight ropes, if liable to giddiness.
8. A pig looks ridiculous, carrying an umbrella.
9. All, who do not dance on tight ropes, & who are treated with respect are fat.

a	Balloonists	3,5	
b	Carrying umbrellas	8	3
c	Dancing on tight ropes	7	1,6,9
d	Eating penny-buns	4	1
e	Fat	6	9
f	Liable to giddiness	2,7	5
g	Looking ridiculous	4,6	8
h	May lunch in public	4	6
j	Old		1,5
k	Pigs	2,8	
l	Treated with respect	9	2
m	Wise	3,7	

		Bang
1	$c'd'_1 j'_0$	
2	$kf_1 l'_0$	
3	$ma_1 b'_0$	
4	gdh_0	
5	$j'a_1 f'_0$	2,7
6	$egc'_1 h'_0$	
7	mfc_0	1,6,9
8	$kb_1 g'_0$	4,6
9	$c'l_1 e'_0$	

1. $c'd'j'$
4. gdh
6. $egc'h'$
8 kbg'
9 $c'le'$
7 mfc
3. mab'
2. kfl'
5. $j'af'$

$\therefore kmaj'_0$

i.e. No wise young pigs go up in balloons.

Example 3: The Library Problem

Consider the following sentences:

- 1. All the old books are Greek.
- 2. All the quartos are bound.
- 3. None of the poets are old quartos.

The general names occurring here are:

A	—	bound
B	—	Greek
C	—	old
D	—	poetry
E	—	quartos.

Example 3: The Library Problem

The schemes of the premisses are:

1. $C \cap B' = \emptyset$
2. $E \cap A' = \emptyset$
3. $D \cap C \cap E = \emptyset$.

Here is the register of attributes:

Name	Positively	Negatively
A		2
B		1
C	1,3	
D	3	
E	2,3	

Example 3: The Library Problem

It is clear that in this case ***none*** of the general names involved can be eliminated (using (★)). Carroll claims that the remedy is to accept the following additional assumption asserting the fact that the union of all the considered names exhausts the universe of discourse, which corresponds to the following statement in the general-negative form:

$$4. \quad A' \cap B' \cap C' \cap D' \cap E' = \emptyset.$$

Then, he says, the conclusion should be: $A' \cap B' = \emptyset$. But **he is wrong**. Consider the following simple counterexample. Let $A = B = C = E = \{x\}$, $D = \{y\}$, $x \neq y$, and the universe is $\{x, y\}$. Then 1.–4. hold, but $A' \cap B' = \{y\} \neq \emptyset$. The book x may be e.g. an old Greek *in quarto* edition of the *Prior Analytics* (which by no means is any poetry), and let us take for y e.g. a heap of unbound new *in folio*, say Polish, poems.

Example 3: The Library Problem

The correspondence between Carroll and John Cook Wilson concerning this problem contains Carroll's remarks about syllogisms with negated conjunctions of names.

It may be also observed that Carroll uses not only the De Morgan laws but also the laws of distribution.

Example 4: from a letter to John Cook Wilson

Carroll invites Wilson to consider the following sorites:

- 1. $A \subseteq B \cup C \cup D$
- 2. $A \cap B \subseteq C \cup H$
- 3. $B \subseteq A \cup C \cup D$
- 4. $B \cap C \cap E \subseteq D$
- 5. $C \cap D \subseteq A \cup B$
- 6. $E \subseteq A \cup B \cup D$
- 7. $B \cap D \subseteq A \cup H$
- 8. $A \cap C \cap K \subseteq B$
- 9. $D \cap K \subseteq B \cup C$.

Example 4: from a letter to John Cook Wilson

The above affirmative sentences are transformed into the corresponding negative ones:

$$
\begin{array}{ll}
1. & A \cap B' \cap C' \cap D' = \emptyset \\
2. & A \cap B \cap C' \cap H' = \emptyset \\
3. & A' \cap B \cap C' \cap D' = \emptyset \\
4. & B \cap C \cap D' \cap E = \emptyset \\
5. & A' \cap B' \cap C \cap D = \emptyset \\
6. & A' \cap B' \cap D' \cap E = \emptyset \\
7. & A' \cap B \cap D \cap H' = \emptyset \\
8. & A \cap B' \cap C \cap K = \emptyset \\
9. & B' \cap C' \cap D \cap K = \emptyset.
\end{array}
$$

Example 4: from a letter to John Cook Wilson

We build the register of attributes:

Name	Positively	Negatively
A	1,2,8	3,5,6,7
B	2,3,4,7	1,5,6,8,9
C	4,5,8	1,2,3,9
D	5,7	1,3,4,6
E	4,6,9	
H		2,7
K	8,9	

This table *suggests* that the conclusion could be: $E \cap H' \cap K = \emptyset$.

However, this is a **wrong** suggestion. We will come back to this example later.

Carroll's ,,Method of Trees''

On July, 16, 1894 Carroll wrote in his *Diary*:

Today has proved to be an epoch in my Logical work. It occurred to me to try a complex Sorites by the method I have been using for ascertaining what cells, if any, survive for possible occupation when certain nullities are given. I took one of 40 premisses, ,,pairs within pairs'' & many bars, & worked it like a genealogy, each term providing all its descendents. It came out beatifully, & much shorter than the method I have used hitherto — I think of calling it the ,,Genealogical Method''.

Carroll used also the name ***The Method of Trees***. Actually, it is just a prototype of the modern method of ***analytic tableaux***.

Example 5

Consider the sentences:

1. $D' \cap N' \cap M' = \emptyset$
2. $K \cap A' \cap C' = \emptyset$
3. $L \cap E \cap M = \emptyset$
4. $D \cap H \cap K' = \emptyset$
5. $H' \cap L \cap A' = \emptyset$
6. $H \cap M' \cap B' = \emptyset$
7. $A' \cap B \cap N = \emptyset$
8. $A \cap M' \cap E = \emptyset$.

Example 5

We build the register of attributes:

Name	Positively	Negatively
A	8	2,5,7
B	7	6
C		2
D	4	1
E	3,8	
H	4,6	5
K	2	4
L	3,5	
M	3	1,6,8
N	1	

Example 5

This table *suggests* that the conclusion could be: $C' \cap E \cap L = \emptyset$.
Instead of a resolution proof we will give a proof by contradiction.
Suppose that $C' \cap E \cap L = \emptyset$ **does not hold**.

Then:

$$(\dagger) \quad C' \cap E \cap L \neq \emptyset$$

i.e. $C' \cap E \cap L$ contains some element.
We will show that this supposition leads to a contradiction and hence should be abandoned.

Example 5

Let $x \in C' \cap E \cap L$. Since $x \in E \cap L$, and (see 3.) $(L \cap E) \cap M = \emptyset$, we have $x \notin M$, i.e. $x \in M'$. Thus, $x \in E \cap M'$. Hence, because $A \cap (M' \cap E) = \emptyset$ (see 8.), $x \notin A$, i.e. $x \in A'$. From $x \in C'$ (see (†)) and $x \in A'$ we get $x \notin K$ (see 2.: $K \cap (A' \cap C') = \emptyset$). Therefore $x \in K'$. From $x \in E$ and $x \in A'$ we obtain (see 5.: $H' \cap (L \cap A') = \emptyset$) $x \notin H'$, i.e. $x \in H$. From $x \in H$ i $x \in K'$, to (see 4.: $D \cap (H \cap K') = \emptyset$) $x \notin D$, i.e. $x \in D'$. From $x \in M'$ and $x \in H$, to (see 6.: $(H \cap M') \cap B' = \emptyset$) $x \notin B'$, i.e. $x \in B$. From $x \in D'$ and $x \in M'$ we get (see 1.: $(D' \cap M') \cap N' = \emptyset$) $x \notin N'$, i.e. $x \in N$. Finally, from $x \in A'$ and $x \in B$, we obtain (see 7.: $(A' \cap B) \cap N = \emptyset$) $x \notin N$, i.e. $x \in N'$. Because $N \cap N' = \emptyset$, we get a ***contradiction***: $x \in N$ and $x \in N'$. Thus, we should reject (†) and we obtain the conclusion $C' \cap E \cap L = \emptyset$.

This was a proof for those who demand (as my dear female students in the Humanities do) that we should talk in full declarative sentences and not omit anything.
Now let us take a look at a simplified proof, presented in the usual manner.

1.	$D' \cap N' \cap M' = \emptyset$	assumption
2.	$K \cap A' \cap C' = \emptyset$	assumption
3.	$L \cap E \cap M = \emptyset$	assumption
4.	$D \cap H \cap K' = \emptyset$	assumption
5.	$H' \cap L \cap A' = \emptyset$	assumption
6.	$H \cap M' \cap B' = \emptyset$	assumption
7.	$A' \cap B \cap N = \emptyset$	assumption
8.	$A \cap M' \cap E = \emptyset$	assumption
9.	$x \in C' \cap E \cap L$	supposition
10.	$x \in M'$	3,9
11.	$x \in A'$	8,9,10
12.	$x \in K'$	2,9,11
13.	$x \in H$	5,9,11
14.	$x \in D'$	4,12,13
15.	$x \in B$	6,10,13
16.	$x \in N$	1,10,14
17.	$x \in N'$	7,11,15
18.	$\bot$	Contradiction: 16,17.

Thus, we have proven: $C' \cap E \cap L = \emptyset$.

Example 6

Consider the following sentences:

1. $H \cap M \cap K = \emptyset$
2. $D' \cap E' \cap C' = \emptyset$
3. $H \cap K' \cap A' = \emptyset$
4. $B \cap L \cap H' = \emptyset$
5. $C \cap K \cap M' = \emptyset$
6. $H \cap C' \cap E = \emptyset$
7. $B \cap A \cap K' = \emptyset$.

Example 6

We build the register of attributes:

Name	Positively	Negatively
A	7	3
B	4,7	
C	5	2,6
D		2
E	6	2
H	1,3,6	4
K	1,5	3,7
L	4	
M	1	5

This table *suggests* that the conclusion could be: $B \cap D' \cap L = \emptyset$.

Example 6

Let us suppose that

$$(\ddagger) \quad B \cap D' \cap L \neq \emptyset.$$

We will show that this supposition leads to a contradiction and hence should be rejected.

Let $x \in B \cap D' \cap L$. Then, from $x \in B \cap L$ and the premiss 4. we get $x \notin H'$, i.e. $x \in H$. From $x \in H$ and the premiss 1.: $H \cap M \cap K = \emptyset$, we obtain $x \notin M \cap K$. And now the inference can not be linear any more, because $x \notin M \cap K$ means that:

- (1) $x \notin K$, i.e. $x \in K'$ ***or***
- (2) $x \notin M$, i.e. $x \in M'$.

Example 6

Each case: (1) and (2) should now be considered separately. It is evident for us today: we simply apply the corresponding De Morgan law. Carroll has formulated this law by himself (in a letter to John Cook Wilson, dated November 11, 1896), without any hint to the contemporary works in the Algebra of Logic.

Let us come back to the proof. Carroll observes at this point that:

- (1') we can add $x \in M$ to (1); however this is of no further use, because M occurs in the premiss 1. only and we have used 1. already;
- (2') we can add $x \in K$ to (2); and this may be of some use, because K occurs in the premiss 5. which was not used yet.

Example 6

The following ***rule*** is the justification of (1') and (2'), in Carroll's own words:

Thus, if we found a Premiss proving that the Thing ***could not*** *have the Pair of Attributes $b'c$, we might say it* ***must*** *have b or c'. And we might afterwards tack on, at pleasure, either c to b, making the two headings bc and c', or b' to c', making them b and $c'b'$.*

Thus, Carroll refers here to the following observation:

$$(A \cap B)' = (A' \cap B) \cup (A \cap B') \cup (A' \cap B').$$

One may guess that he got this observation from his diagrams.

Example 6

(1) From $x \in K'$, $x \in H$ and the premiss 3. we get: $x \notin A'$, i.e. $x \in A$. From $x \in K'$, $x \in A$ and the premiss 7. we have $x \notin A$. Thus, we get a contradiction: $x \in A$ and $x \notin A$.

(2) From $x \in M'$, (2') and the premiss 5. we get $x \notin C$, i.e. $x \in C'$. From $x \in D'$ (see (‡)), $x \in C'$ and the premiss 2. we obtain $x \notin E'$, i.e. $x \in E$. Finally, from $x \in H$ oraz $x \in C'$ and the premiss 6. we have $x \notin E$. Thus we get a contradiction: $x \in E$ and $x \notin E$.

We have shown that **each** of (1) and (2) leads to a contradiction. Hence we should reject the supposition (‡). Finally, we have:

$$B \cap D' \cap L = \emptyset.$$

This is equivalent to an affirmative sentence: $(B \cap L) \subseteq D$.

Example 4 again

And how did Carroll proceeded in the cases when no contradiction arose? We find an example in his letter to John Cook Wilson, dated November 18, 1896 (this is exactly the example 4 mentioned before):

$$\begin{array}{ll}
1. & A \cap B' \cap C' \cap D' = \emptyset \\
2. & A \cap B \cap C' \cap H' = \emptyset \\
3. & B \cap A' \cap C' \cap D' = \emptyset \\
4. & B \cap C \cap E \cap D' = \emptyset \\
5. & C \cap D \cap A' \cap B' = \emptyset \\
6. & E \cap A' \cap B' \cap D' = \emptyset \\
7. & B \cap D \cap A' \cap H' = \emptyset \\
8. & A \cap C \cap K \cap B' = \emptyset \\
9. & D \cap K \cap B' \cap C' = \emptyset.
\end{array}$$

Example 4 again

Recall the register of attributes:

Name	Positively	Negatively
A	1,2,8	3,5,6,7
B	2,3,4,7	1,5,6,8,9
C	4,5,8	1,2,3,9
D	5,7	1,3,4,6
E	4,6,9	
H		2,7
K	8,9	

The table *suggests* that the conclusion could be: $E \cap H' \cap K = \emptyset$. We will show that this is a ***wrong*** suggestion.

Example 4 again

Let us suppose that:

$$E \cap H' \cap K \neq \emptyset.$$

Let $x \in E \cap H' \cap K$. From $x \in E$ and the premiss 4. we have: $x \notin B \cap (C \cap D')$ and this means that the following disjunction holds:

- (1) $x \in B'$ ***or***
- (2) $x \in (C \cap D')'$.

Let us take (1) first. From $x \in E$ and $x \in B'$ and the premiss 6. we have $x \notin (A' \cap D')$ and thus the following disjunction holds:

- (1.1.) $x \in A$ ***or***
- (1.2.) $x \in D$.

Example 4 again

Let us consider (1.1.). From $x \in A$, $x \in B'$ and $x \in K$ and the premiss 8. we have $x \notin C$, i.e. $x \in C'$. From $x \in A$, $x \in B'$, $x \in C'$ and the premiss 1. we get: $x \notin D'$, i.e. $x \in D$. Finally, from $x \in K$, $x \in B'$ and $x \in C'$ and the premiss 1. we have: $x \notin D$. Thus we have obtained a contradiction: $x \in D$ and $x \notin D$. The case (1.1.) is excluded.

We come back now to (1.2.). Carroll makes at this point an additional assumption $x \in A'$, justifying it by the rule mentioned above.
From $x \in B'$, $x \in D$, $x \in A'$ and the premiss 5. we get $x \notin C$, i.e. $x \in C'$.
From $x \in K$, $x \in B'$, $x \in D$ and the premiss 9. we have $x \notin C'$. Hence, we have obtained a contradiction: $x \in C'$ and $x \notin C'$. The case (1.2.) is excluded.

Example 4 again

We come back to (2). At this point Carroll again uses his **rule** mentioned above and accepts an additional assumption: $x \in B$. From $x \in (C \cap D')'$ we get that the following disjunction holds:

- (2.1.) $x \in C'$ (and $x \in B$) ***or***
- (2.2.) $x \in D$ (and $x \in B$).

Let us take (2.1.) first. From $x \in H'$, $x \in C'$, $x \in B$ and the premiss 2. we get $x \notin A$, i.e. $x \in A'$. From $x \in B$, $x \in A'$, $x \in C'$ and the premiss 3. we have $x \notin D'$, i.e. $x \in D$. Finally, from $x \in H'$, $x \in B$, $x \in A'$ and the premiss 7. we obtain $x \notin D$ and this contradicts $x \in D$. The case (2.1.) is excluded.

Example 4 again

Finally, let us consider (2.2.). From $x \in H'$, $x \in B$, $x \in D$ and the premiss 7. we have: $x \notin A'$, i.e. $x \in A$. And at this point we can not make any use of the premisses in order to exclude the case (2.2.).

Therefore our supposition has been confirmed and this means that $E \cap H' \cap K = \emptyset$ ***is not*** a logical consequence of 1.–9.

Thus the case in which all the premisses (of the sorites in question) are true and its alleged conclusion is false has not been excluded. There may exist x such that $x \in E \cap H' \cap K$ and all of 1.–9. hold. It follows from our analysis of (2.2.) that in such a case we have: $x \in A \cap B \cap C \cap D$.

Example 4 again

The analysis of (2.2.) also shows what conclusions **do follow** from 1.–9. Any of the following sentences is such a legitimate conclusion:

- $E \cap H' \cap K \cap A' = \emptyset$
- $E \cap H' \cap K \cap B' = \emptyset$
- $E \cap H' \cap K \cap C' = \emptyset$
- $E \cap H' \cap K \cap D' = \emptyset$.

Observe that each of the above sentences, together with 1.–9. implies a contradiction.

Example 4 again

Furthermore, the analysis of (2.2.) shows that 1.–9. together with any of the following sentences:

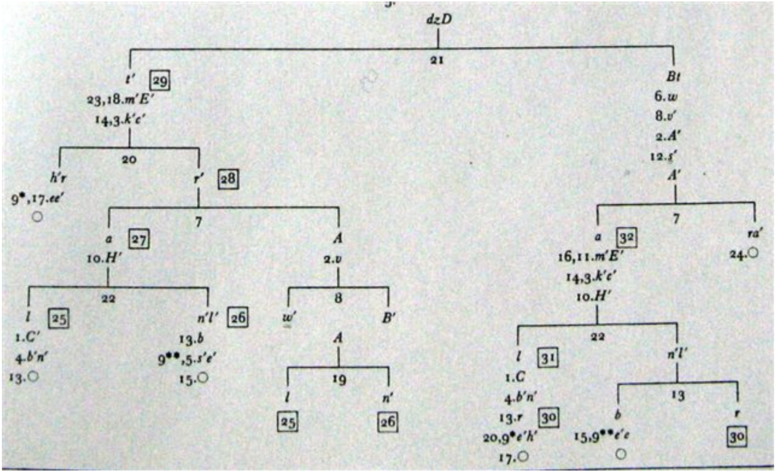
- 10. $A = \emptyset$
- 11. $B = \emptyset$
- 12. $C = \emptyset$
- 13. $D = \emptyset$

do imply $E \cap H' \cap K = \emptyset$.

Obviously, such modifications „break the symmetry" in Carroll's original example. We add them in order to show that the „Method of Trees" always provides an answer in the analysis of soriteses.

Obviously, Carrroll did not make any use of the relation $\in$. His proofs were purely algebraic.

Carroll's trees



Some trees considered by Carroll were complicated, indeed.

A few metalogical remarks

As we have seen, the resolution rule (★) alone is not sufficient for a proper analysis of soriteses. On the other hand, the „Method of Trees" appears to be sufficient and adequate for such a goal. Moreover, this method can be applied to **arbitrary** categorical sentences. As we know from the result of Löwenheim (1915) the Monadic Predicate Calculus is **decidable**. Hence, given any set of categorical sentences X and a categorical sentence α exactly one of the following cases holds:

- (1) from X and $\neg\alpha$ we get a ***contradiction***; then α ***logically follows*** from X;
- (2) from X and $\neg\alpha$ we do not get a ***contradiction***; then α ***does not logically follow*** from X.

A few metalogical remarks

We should, of course, prove that the „Method of Trees" is correct, i.e. that it is ***sound*** and ***complete***.

Such a proof exists for (analytic tableaux for) Predicate Calculus and hence also for Carroll's „Method of Trees".

Another possibility is to construct tableaux for (classical) syllogistic and prove soundness and completeness of the system — cf. e.g. Simons 1989.

A few metalogical remarks

Carroll did not prove the correctness of his „Method of Trees". Observe that such a proof is possible only in metalogic. And (systematic) metalogic was born a few decades later.

Carroll did not think of proving the correctness of his resolution method („the Method of Underscoring"). Observe that also the proof of the correctness of the resolution consequence needs a fairly developed metalogic, inaccessible at that time.

But Carroll was fully aware of the fact that one can not limit oneself to the „suggestions" given by the registers of attributes.
In the case of each of his soriteses he always provides a proof that the conclusion is validated by the premisses (either by the resolution method or by his „Method of Trees").

A few metalogical remarks

May we say that Carroll's *Symbolic Logic* is based on some ***logical system***? Bartley observes that Carroll was using the truth tables already around 1894 and hence long before the proposals of Emil Post and Ludwig Wittgenstein. All the classical syllogistic is included in *Symbolic Logic*.

As we have seen above, Carroll was using also a prototype of the analytic tableaux method. Usually, one attributes the first uses of this method to the works of Beth, Hintikka, Kanger and Schütte from the fifties of the XXth century. The method has been further elaborated by Smullyan, Lis and Jeffrey. And now it is one of the most important methods e.g. in the automated theorem proving.

A few historical remarks

Carroll's logical works were written in the second half of XIX century. At that time a new approach in logic was developed, called The Algebra of Logic. It has begun with the works of George Boole and Augustus De Morgan and culminated in the work of Ernst Schröder. Meanwhile, such prominent logicians as e.g. MacColl, Peirce, Jevons and Venn also participated in the development of the algebraic approach.

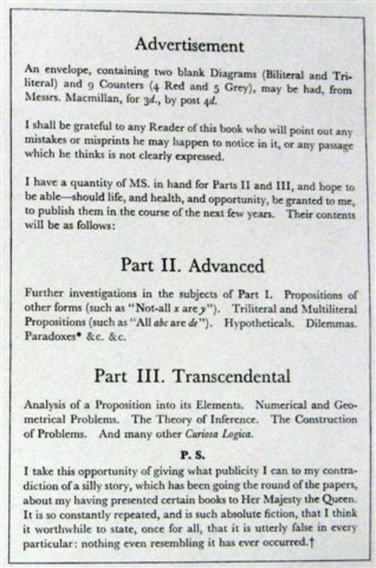
Carroll certainly did know these works. But he had not attempted at the construction of a *system of logic* in his *Symbolic Logic*. Rather, the book was thought of as a teaching manual. It was Carroll's ambition to present logic in a popular way, accessible to a common reader. Remember that he taught mathematics and logic at The Christ College and his students were the innocent girls of the Victorian Epoch.
Without any doubts, Carroll has achieved his goal. His *Symbolic Logic* is still (after more than one hundred years!) used by the lecturers of logic. This is caused partly by his literary talent: almost all of his examples are vivid and have some subtle intricacy, both logical and linguistic.

A few historical remarks

Part I of *Symbolic Logic* is very elementary, it is simply an introduction to classical syllogistic. It also contains the discussion of Carroll's diagrams and his „Method of Underscoring" (i.e. a prototype of linear resolution).

Part II, found after 70 years by W.W. Bartley contains more difficult problems: e.g. categorical sentences with compound subjects and negated predicates. Here Carroll also introduces his „Method of Trees". Six chapters of this part have never been found.

Carroll did advertise also *Part III: Transcendental*. According to his own words, he collected a lot of notes concerning it. Two of the planned chapters were supposed to be entitled: *Analysis of a Proposition into its Elements* and *The Theory of Inference*.



Advertisement

An envelope, containing two blank Diagrams (Biliteral and Triliteral) and 9 Counters (4 Red and 5 Grey), may be had, from Messrs. Macmillan, for 3*d*., by post 4*d*.

I shall be grateful to any Reader of this book who will point out any mistakes or misprints he may happen to notice in it, or any passage which he thinks is not clearly expressed.

I have a quantity of MS. in hand for Parts II and III, and hope to be able—should life, and health, and opportunity, be granted to me, to publish them in the course of the next few years. Their contents will be as follows:

Part II. Advanced

Further investigations in the subjects of Part I. Propositions of other forms (such as "Not-all *x* are *y*"). Triliteral and Multiliteral Propositions (such as "All *abc* are *de*"). Hypotheticals. Dilemmas. Paradoxes* &c. &c.

Part III. Transcendental

Analysis of a Proposition into its Elements. Numerical and Geometrical Problems. The Theory of Inference. The Construction of Problems. And many other *Curiosa Logica*.

P. S.

I take this opportunity of giving what publicity I can to my contradiction of a silly story, which has been going the round of the papers, about my having presented certain books to Her Majesty the Queen. It is so constantly repeated, and is such absolute fiction, that I think it worthwhile to state, once for all, that it is utterly false in every particular: nothing even resembling it has ever occurred.†

Carroll advertised Part III of *Symbolic Logic*. It has been probably never written in a form ready for publication.

A few historical remarks

As it is known, after Carroll's death almost all of his carefully collected notes

HAVE BEEN PUT ON FIRE.

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Lewis Carroll in his own eyes

Alice Liddell (Photo by LC)

The end

We are still borrowing examples from Carroll's *Symbolic Logic* in order to make our serious (and sometimes boring) lectures in logic more interesting.

The common opinion claims that *Symbolic Logic* is just a collection of more or less funny examples.

This opinion is, it seems, very unjust to the Author. We were trying to bring the attention to some Carroll's ideas which show that he was not only a giant of the world literature but also a brilliant logician who discovered important methods of proof long before others did.