Non-Fregean Logic

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2022

- Characterization of identity predicate in second-order logic:
  - *The principle of indiscernibility of identicals*:
    $\forall x \forall y \, (x = y \rightarrow \forall P \, (P(x) \leftrightarrow P(y)))$.
  - *The principle of identity of indiscernibles*:
    $\forall x \forall y \, (\forall P \, (P(x) \leftrightarrow P(y)) \rightarrow x = y)$.
- Axioms characterizing the identity predicate in first-order logic. Example – set theory:
  - Reflexivity, symmetry and transitivity of $=$.
  - $\forall x \forall y \forall z \, ((x \in y \wedge x = z) \rightarrow z \in y)$.
  - $\forall x \forall y \forall z \, ((x \in y \wedge y = z) \rightarrow x \in z)$.
- How could we characterize identity of *situations described by sentences*?

$$\begin{array}{ccccc} \boldsymbol{s}(\varphi) & \longleftarrow & \varphi & \longrightarrow & \boldsymbol{t}(\varphi) = 1 \text{ or } 0, \quad \text{if } \varphi \text{ is a sentence} \\ & {}_f\searrow & \downarrow & \nearrow_g & \\ & & \boldsymbol{r}(\varphi) & & \end{array}$$

- One can associate with any sentence φ:
  - its sense $\mathbf{s}(\varphi)$
  - its referent $\mathbf{r}(\varphi)$
  - its logical value $\mathbf{t}(\varphi)$.
- Functional dependencies between sense, referent and logical value are presented at the diagram above.
- $\mathbf{r}(\varphi)$ is the *situation* described by φ.
- Roman Suszko *Abolition of the Freagean axiom*.

- A binary connective $\equiv$ is called the *identity connective* in a logical system $(\mathbf{S}, C)$ iff it satisfies the following conditions (here C is a structural consequence and $X \vdash \alpha$ means that $\alpha \in C(X)$):
  1. $\vdash (\alpha \equiv \alpha)$
  2. $\alpha, \alpha \equiv \beta \vdash \beta$
  3. for any n-argument functor F_i in $(\mathbf{S}, C)$: $\alpha_1 \equiv \beta_1, \alpha_2 \equiv \beta_2, \ldots, \alpha_n \equiv \beta_n \vdash F_i(\alpha_1, \alpha_2, \ldots, \alpha_n) \equiv F_i(\beta_1, \beta_2, \ldots \beta_n)$.
- The first rule is an axiomatic rule, the second is specific for the identity connective, and the remaining rules are rules of invariance.
- $\equiv$ is the identity connective in a logical system $(\mathbf{S}, C)$ iff each C-theory is closed w.r.t. the above rules.

- **Theorem**. If $\equiv$ is the identity connective in a logical system $(\mathbf{S}, C)$, T is any invariant theory in this system, then the relation $\sim_T$ on S defined by $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$ is an invariant congruence and the algebra $\mathbf{S}/\sim_T$ is freely generated by the set of all $\sim_T$-equivalence classes of propositional variables of $\mathbf{S}$. $\square$
- **Theorem**. If $\equiv$ is the identity connective in a logical system $(\mathbf{S}, C)$, then the relation $\sim_{C(\emptyset)}$ defined by $\alpha \sim_{C(\emptyset)} \beta$ iff $(\alpha \equiv \beta) \in C(\emptyset)$ is the greatest congruence of the system $(\mathbf{S}, C)$.
- **Proof**. Let θ be a congruence of $(\mathbf{S}, C)$ and let $\alpha\theta\beta$.
- Since $\alpha\theta\beta$ and $\beta\theta\beta$, we have $(\alpha \equiv \beta)\theta(\beta \equiv \beta)$.
- Since θ is a logical congruence and $(\beta \equiv \beta) \in C(\emptyset)$, we have $(\alpha \equiv \beta) \in C(\emptyset)$, which means that $\alpha \sim_{C(\emptyset)} \beta$. $\square$

- The quotient algebra $\mathbf{S}/\sim_{C(\emptyset)}$ is the Lindenbaum-Tarski algebra for $(\mathbf{S}, C)$.
- Let $\mathcal{K}_{\mathbf{S}}(C)$ be the class of algebras similar to $\mathbf{S}$ such that for every $\mathbf{A} \in \mathcal{K}_{\mathbf{S}}(C)$ any mapping from the set of $\sim_{C(\emptyset)}$-equivalence classes of propositional variables of $\mathbf{S}$ can be extended to a homomorphism from $\mathbf{S}/\sim_{C(\emptyset)}$ into $\mathbf{A}$.
- The class $\mathcal{K}_{\mathbf{S}}(C)$ can be used for developing semantics of logical systems with the identity connective.
- Note the difference between the identity connective and $\equiv$ and material equivalence $\leftrightarrow$ in $(\mathbf{S}, C)$:

  1. $(\alpha \leftrightarrow \beta) \in C(X)$ iff $C(X \cup \{\alpha\}) = C(X \cup \{\beta\})$
  2. $(\alpha \equiv \beta) \in C(X)$ iff $C(X \cup \{\varphi[p/\alpha]\}) = C(X \cup \{\varphi[p/\beta]\})$

(where $\alpha, \beta, \varphi \in S$, $X \subseteq S$, and p is a propositional variable of $\mathbf{S}$).

- $\mathcal{L} = (L, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \equiv)$ is the language of SCI.
- Modus ponens $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$ is the only rule of inference in SCI.
- TFA is the set of truth-functional axioms:

1. $\alpha \rightarrow (\beta \rightarrow \alpha)$
2. $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
3. $\neg\alpha \rightarrow (\alpha \rightarrow \beta)$
4. $(\alpha \rightarrow \beta) \rightarrow ((\neg\alpha \rightarrow \beta) \rightarrow \beta)$
5. $(\alpha \leftrightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$
6. $(\alpha \leftrightarrow \beta) \rightarrow (\beta \rightarrow \alpha)$
7. $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow (\alpha \leftrightarrow \beta))$
8. $(\alpha \wedge \beta) \leftrightarrow \neg(\alpha \rightarrow \neg\beta)$
9. $(\alpha \rightarrow \beta) \rightarrow (\neg\alpha \rightarrow \beta)$

- IDA is the set of identity axioms:
  1. $\alpha \equiv \alpha$
  2. $(\alpha \equiv \beta) \to (\neg\alpha \equiv \neg\beta)$
  3. $((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \to ((\alpha \circ \gamma) \equiv (\beta \circ \delta))$, where $\circ \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$
  4. $(\alpha \equiv \beta) \to (\alpha \leftrightarrow \beta)$.
- LA is the union of TFA and IDA.
- Consequence C in $\mathcal{L}$ is defined as follows: $\alpha \in C(X)$ iff α can be derived from $LA \cup X$ in a finite number of steps using modus ponens as the only rule of inference. From now on let C denote this consequence operator.
- Then C is finitary, structural, compact, and regular.
- $(\mathcal{L}, C)$ is called the *sentential calculus with identity* (SCI).
- The deduction theorems holds for C.
- $C(\emptyset)$ is the set of theorems of SCI.

- Algebras similar to the algebra of SCI-language are called *SCI-algebras*. The class of all such algebras includes the class of all B-algebras.
- A subset F of the universe of a SCI-algebra $\mathbf{A}$ is called a *SCI-filter* iff for any homomorphism h of the algebra of SCI-language into $\mathbf{A}$ the set $h^{-1}(F)$ is a SCI-theory.
- $(\mathbf{A}, F)$ is a *SCI-matrix* iff $\mathbf{A}$ is a SCI-algebra and F is a SCI-filter. The fact that C is a structural consequence implies that for any set X the Lindenbaum matrix $(\mathcal{L}, C(X))$ is a SCI-matrix (because $h^{-1}[C(X)]$ is a SCI-theory for any homomorphism $h : \mathcal{L} \to \mathbf{A}$).
- $(\mathbf{A}, D)$ is called a *SCI-model* iff $\mathbf{A}$ is a SCI-algebra, and $D \subseteq A$ is such that for any $a, b \in A$:
  1. $\neg^{\mathbf{A}} a \in D$ iff $a \notin D$
  2. $a \wedge^{\mathbf{A}} b \in D$ iff $a \in D$ oraz $b \in D$
  3. $a \vee^{\mathbf{A}} b \in D$ iff $a \in D$ lub $b \in D$
  4. $a \rightarrow^{\mathbf{A}} b \in D$ iff $a \notin D$ or $b \in D$
  5. $a \leftrightarrow^{\mathbf{A}} b \in D$ iff $a, b \in D$ or $a, b \notin D$
  6. $a \circ^{\mathbf{A}} b \in D$ iff $a = b$.

- If $(\mathbf{A}, D)$ is a SCI-model, then D is a normal ultrafilter in $\mathbf{A}$, and $\mathbf{A}$ is called a *semi-model* of $\mathcal{L}$. If $\mathbf{A}$ is a semi-model, then the intersection of all its normal ultrafilters is non-empty, because the sets $\{a \vee^{\mathbf{A}} -a : a \in A\}$, $\{a \rightarrow^{\mathbf{A}} a : a \in A\}$, $\{a \leftrightarrow^{\mathbf{A}} a : a \in A\}$, $\{a \circ^{\mathbf{A}} a : a \in A\}$, $\{\neg^{\mathbf{A}}(a \circ^{\mathbf{A}} a) : a, b \in A, a \neq b\}$ are included in every normal ultrafilter in $\mathbf{A}$.
- A Boolean ultrafilter U is called *normal* in $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ iff for any $a, b \in A$: $a \circ b \in U$ iff $a = b$.
- **Theorem**. There exists a normal ultrafilter in B-algebra $\mathbf{A} = (A, \wedge, \vee, -, \triangleright, \div, \circ)$ iff for any n and m and every finite sequences $c_1, \ldots, c_n$, $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ of A the following condition holds: $(*)$ if $\bigwedge\limits_{i=1}^{n}(c_i \circ c_i) \leqslant \bigvee\limits_{j=1}^{m}(a_j \circ b_j)$, then $a_j = b_j$ for some $1 \leqslant j \leqslant m$. $\square$

- Let $\mathfrak{M} = (\mathbf{A}, D)$ be a SCI-matrix and $h : \mathcal{L} \to \mathbf{A}$ be a homomorphism. We recall that:
  - $Sat_h(\mathfrak{M}) = \{\alpha \in L : h(\alpha) \in D\}$ and hence $Sat_h(\mathfrak{M}) = h^{-1}(D)$.
  - $E(\mathfrak{M}) = \bigcap\limits_{h \in Hom(\mathcal{L},\mathbf{A})} Sat_h(\mathfrak{M})$ and hence $E(\mathfrak{M}) = \bigcap\limits_{h \in Hom(\mathcal{L},\mathbf{A})} h^{-1}(D)$.
- **Theorem**. T is a complete theory iff there exists a SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ and a homomorphism $h : \mathcal{L} \to \mathbf{A}$ such that $T = Sat_h(\mathfrak{M})$.
- Note that if T is a complete theory, then the Lindenbaum-Tarski matrix $\mathfrak{M}(T) = (\mathcal{L}/\sim_T, T/\sim_T)$ is a SCI-model and $T = \mathrm{Sat}_{k_{\sim_T}}\mathfrak{M}(T)$, where $k_{\sim_T}(a) = a/\sim_T$ and $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$. □
- **Completeness theorem for SCI**. For any $X \subseteq L$ and $\alpha \in L$: $\alpha \in C(X)$ iff for every SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ and for every homomorphism $h : \mathcal{L} \to \mathbf{A}$ we have: if $X \subseteq Sat_h(\mathfrak{M})$, then $\alpha \in Sat_h(\mathfrak{M})$. □

In particular, $\alpha \in C(\emptyset)$ iff $\alpha \in C_{\mathfrak{M}}(\emptyset)$ for every SCI-model $\mathfrak{M}$.

- We say that a theory T in SCI-language is *quasi-complete* iff:
  1. T is consistent
  2. T is invariant
  3. for any formulas α and β, if $Var(\alpha) \cap Var(\beta) = \emptyset$ and $\alpha \vee \beta \in T$, then $\alpha \in T$ or $\beta \in T$.
- **Theorem**. A SCI-theory T is quasi-complete iff there exists a SCI-model $\mathfrak{M}$ such that $E(\mathfrak{M}) = T$. □

- Axiomatic extensions of C. If $A \subseteq L$ is a set of additional axioms, then let $C^A(X) = C(A \cup X)$ for any $X \subseteq L$. Obviously, $C^A(\emptyset) = C(A)$.
- Note that if $C(A) \neq C(B))$, then $C^A \neq C^B$.
- **Theorem**. C^A is a structural consequence iff A is invariant. $\square$
- Invariant theories in the SCI-language are called *theories of situations*.
- If there exists a model $\mathfrak{M}$ such that $T = E(\mathfrak{M})$, then $\mathfrak{M}$ is called *adequate for* T. If a model $\mathfrak{M}$ is such that for all $\alpha \in L$ and $X \subseteq L$: $\alpha \in C^T(X)$ iff $\alpha \in C^{\mathfrak{M}}(X)$, then $\mathfrak{M}$ is called *adequate for the system* $(\mathcal{L}, C^T)$ *(for consequence* C^T*)*.
- **Theorem**. If T is a consistent theory, then $\mathfrak{M} = (\mathbf{A}, D)$ is adequate for C^T iff
  1. $\mathfrak{M}$ is adequate for T.
  2. For every complete theory T_i such that $T \subseteq T_i$ there exists a homomorphism $h : \mathcal{L} \to \mathbf{A}$ such that $T_i = h^{-1}(D)$. $\square$
- **Theorem**. If $\mathbf{M}$ is adequate for T, then $C^T = C_{\mathbf{M}}$ iff $C_{\mathbf{M}}$ is a finitary consequence. $\square$

- Let AB denote the set of all substitutions of the formulas:
  1. $((p \wedge q) \vee r) \equiv ((q \vee r) \wedge (p \vee r))$
  2. $((p \vee q) \wedge r) \equiv ((q \wedge r) \vee (p \wedge r))$
  3. $(p \vee (q \wedge \neg q)) \equiv p$
  4. $(p \wedge (q \vee \neg q)) \equiv p$
  5. $(p \rightarrow q) \equiv (\neg p \vee q)$
  6. $(p \leftrightarrow q) \equiv ((p \rightarrow q) \wedge (q \rightarrow p))$.
- Let $WB = C(AB)$. Then $WB = C(\{\alpha \equiv \beta : \alpha \leftrightarrow \beta \in TFT\})$.
- WB is an invariant theory and it determines a structural consequence C^{WB} defined by: $\alpha \in C^{WB}(X)$ iff $\alpha \in C(WB \cup X)$.
- Theories in SCI-language containing the theory WB are called *Boolean (theories of situations)*.
- $(\mathbf{A}, U)$ is called a *B-model* iff $\mathbf{A}$ is a B-algebra and U is a normal ultrafilter in $\mathbf{A}$.

- **Theorem**. WB is exactly the set of all SCI-formulas which are true in every B-model.
- **Proof**. Let $\mathfrak{M} = (\mathbf{A}, U)$ be a B-model. Of course, $E(\mathfrak{M})$ is closed with respect to the modus ponens rule. In order to prove that $AB \subseteq E(\mathfrak{M})$ one should calculate the value of any axiom from AB under an arbitrary homomorphism $h : \mathcal{L} \to \mathbf{A}$. It is easy to check that this value is always an element of U.
- Let us suppose now that $\alpha \notin WB$. We are going to show that then $\alpha \notin E(\mathfrak{M})$ for some B-model $\mathfrak{M}$.
- If $\alpha \notin WB$, then there exists a complete theory T such that $WB \subseteq T$ but $\alpha \notin T$.
- The quotient model $(\mathcal{L}/ \sim_T, T/ \sim_T)$ is a B-model for T and therefore also for WB. We have: $\alpha \notin E((\mathcal{L}/ \sim_T, T/ \sim_T))$.
- We thus proved that $WB = \bigcap\limits_{\mathfrak{N}} E(\mathfrak{N})$. □

- Let $WT = C(\{\alpha \equiv \beta : \alpha \leftrightarrow \beta \in C(\emptyset)\})$. Any theory containing WT is called a *WT-theory*. Such theories are supposed to formalize thesis 5.141 of Wittgenstein's *Tractatus* (if two sentences entail one another, then they are the same sentence).
- Each WT-theory is a theory of the consequence C^{WT} defined by: $\alpha \in C^{WT}(X)$ iff $\alpha \in C(WT \cup X)$.
- If T is a WT-theory and $(\alpha \leftrightarrow \beta) \in C(\emptyset)$, then $\varphi[p/\alpha] \in T$ iff $\varphi[p/\beta] \in T$, for any formula φ and variable p.
- WT is the least Boolean theory in SCI-language which is closed with respect to the Gödel's rule: $\frac{\alpha,\beta}{\alpha\equiv\beta}$. Moreover:
    - There exists a translation f of $\mathcal{L}$ on the language of S_4-system: $f(\alpha) = \alpha$ if α does not contain the identity connective and $f(\alpha \equiv \beta) = \Box(\alpha \leftrightarrow \beta)$. Then $\alpha \in WT$ iff $f(\alpha) \in S_4$.
    - A converse translation is provided by the function g such that: $g(\alpha) = \alpha$ if $\Box$ does not occur in α and$g(\Box\alpha) = \alpha \equiv (\alpha \vee \neg\alpha)$. Then $\alpha \in S_4$ iff $g(\alpha) \in WT$.

- It is known that:
  - $\alpha \in S_4$ iff for any TB-algebra $\mathbf{A}$ and any homomorphism $h : \mathcal{L} \to \mathbf{A}$: $h(\alpha) = 1_{\mathbf{A}}$.
  - $(\Box\alpha \vee \Box\beta) \in S_4$ iff $\Box\alpha \in S_4$ or $\Box\beta \in S_4$.
  - Let $\alpha \sim_{S_4} \beta$ iff $\Box(\alpha \leftrightarrow \beta) \in S_4$. Then $\sim_{S_4}$ is a congruence and $\mathcal{L}/\sim_{S_4}$ is a well-connected Boolean algebra.
  - S_4 is quasi-complete.
- The existence of the translations mentioned above implies that:
  - $\alpha \in WT$ iff for any TB-algebra $\mathbf{A}$ and any homomorphism $h : \mathcal{L} \to \mathbf{A}$: $h(\alpha) = 1_{\mathbf{A}}$.
  - $\alpha \equiv \beta \vee \gamma \equiv \delta \in WT$ iff $\alpha \equiv \beta \in WT$ or $\gamma \equiv \delta \in WT$.
  - Algebra $\mathcal{L}/\sim_{WT}$ is a well-connected TB-algebra.
  - WT is a quasi-complete theory.

- There exists a SCI-model $\mathfrak{M}$ such that $WT = E(\mathfrak{M})$. This follows from the fact that WT is quasi-complete and that there exists a complete theory T such that WT is the largest invariant theory included in T.
- Let $\sim_T$ be a congruence defined by: $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$.
- Let $\mathfrak{M}_T = (\mathcal{L}/\sim_T, T/\sim_T)$. Then $E(\mathfrak{M}_T) = WT$ and $\mathfrak{M}_T$ is a countable model adequate for WT.
- Because C^{WT} is regular, the class of all Lindenbaum-Tarski models $(\mathcal{L}/\sim_T, T/\sim_T)$, where T is a complete WT-theory, is adequate for the system $(\mathcal{L}, C^{WT})$.

- We extend the SCI-language by introducing two sentential constants: $1 \equiv (p \lor \neg p) \quad 0 \equiv (p \land \neg p)$.
- Let AH be the set including AB, all substitutions of the above two definitions and all SCI-formulas of the form $(\alpha \equiv \beta) \equiv 0 \lor (\alpha \equiv \beta) \equiv 1$.
- Let $WH = C(AH)$. Theories including WH are called *WH-theories*.
- The theory WH is invariant and it is based on equational axioms AB together with the schemas:
  1. $1 \equiv (\alpha \lor \neg\alpha)$
  2. $0 \equiv (\alpha \land \neg\alpha)$
  3. $(\alpha \equiv \beta) \equiv ((\alpha \equiv \beta) \equiv 1)$
  4. $\neg(\alpha \equiv \beta) \equiv ((\alpha \equiv \beta) \equiv 0)$.
- Let $\alpha \sim_{WH} \beta$ iff $(\alpha \equiv \beta) \in WH$. Then $\sim_{WH}$ is a congruence.

- Theorem. For any *WH*-theory T the algebra $\mathcal{L}/\sim_T = (L/\sim_T, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \circ)$ satisfies the following conditions:
  1. $\mathcal{L}/\sim_T$ is a *TB*-algebra.
  2. For any $\alpha, \beta \in L$: $\neg(|\alpha| \circ |\beta|) = (|\alpha| \circ |\beta| \circ 0)$.
  3. If T is a complete theory, then $\mathcal{L}/\sim_T$ is a Henle algebra.
  4. $\alpha \in WH$ iff for every *TB*-algebra $\mathbf{A} = (A, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \circ)$ and for any $a, b \in A$: $\neg(a \circ b) = ((a \circ b) \circ 0)$; moreover, for any homomorphism $h : \mathcal{L} \rightarrow \mathbf{A}$: $h(\alpha) = 1_{\mathbf{A}}$. □
- Elements of the form $a \circ b$ are open elements in *TB*-algebras, and if $\neg(a \circ b) = ((a \circ b) \circ 0)$, then each closed element is also open. All open elements of the algebra form a Boolean algebra. *TB*-algebras in which $\neg(a \circ b) = ((a \circ b) \circ 0)$ are called *self-dual TB-algebras*.
- Systems S_5 and *WH* are mutually translatable, because $\circ$ and interior operation are mutually definable in *TB*-algebras:
  $\Box\alpha \mapsto \alpha \equiv (\alpha \vee \neg\alpha)$,
  $\alpha \equiv \beta \mapsto \Box(\alpha \leftrightarrow \beta)$.

- Because S_5 is quasi-complete, so is WH. Therefore there exists a complete WH-theory T such that WH is the largest invariant theory contained in T.
- Let $\alpha \sim_T \beta$ iff $(\alpha \equiv \beta) \in T$). The $\sim_T$-quotient of the Lindenbaum matrix $(\mathcal{L}, T)$ is a SCI-model $\mathfrak{M}_T$ such that $E(\mathfrak{M}_T) = WH$.
- $\mathfrak{M}_T$ is a countable model strongly adequate for WH.
- A SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ is called a *Henle model* iff $\mathbf{A}$ is a Henle algebra.
- **Theorem**. For any $\alpha \in L$ and $X \subseteq L$: $\alpha \in C^{WH}(X)$ iff $\alpha \in C^{\mathfrak{M}}(X)$ for all Henle models $\mathfrak{M}$.
- **Proof**. Suppose that $\alpha \notin C^{WH}(X)$.
- It follows from regularity of SCI that there exists a complete theory T such that $X \subseteq T$ and $\alpha \notin T$.

- Let $\alpha \sim_T \beta$ iff $\alpha \equiv \beta \in T$.
- The quotient matrix $\mathfrak{M}(T)(\mathcal{L}/\sim_T, T/\sim_T)$ is then a Henle model.
- Let $k_{\sim_T}$ be the canonical homomorphism. We have: $k_{\sim_T}(\alpha) = \alpha/\sim_T$, $X \subseteq Sat_{k_{\sim_T}}(\mathfrak{M}(T))$ and $\alpha \notin Sat_{k_{\sim_T}}(\mathfrak{M}(T))$.
- Suppose, in turn, that for some Henle model $\mathfrak{M} = (\mathbf{A}, D)$ and for some homomorphism $h : \mathcal{L} \to \mathbf{A}$ we have: $h[X] \subseteq D$ i $h(\alpha) \notin D$.
- It follows from the fact that $h^{-1}[D]$ is a complete WH-theory that there exists a complete theory T such that $WH \cup X \subseteq T$ and $\alpha \notin T$, and therefore $\alpha \notin C^{WH}(X)$. □

- A theory T in SCI-language is called *Fregean* iff it contains all formulas from $\mathcal{L}$ represented by a schema $(\alpha \equiv \beta) \equiv (\alpha \leftrightarrow \beta)$.
- Let AF be the set of all such formulas and let $WF = C(AF)$. Each Fregean theory is an invariant B-theory.
- The two-element Boolean algebra $\mathbf{B}_2$ is a model of each Fregean theory.
- The identity connective is truth-functional in any Fregean theory. Material equivalence in such theories has all properties of the identity connective.
- For any $\alpha \in L$ and $X \subseteq L$ let $\alpha \in C^{WF}(X)$ iff for any homomorphism $h : \mathcal{L} \to \mathbf{B}_2$: if $h[X] \subseteq \{1\}$, then $h(\alpha) = 1$.

- **Theorem**. For any natural numbers $n \geqslant 2$, $1 \leqslant t < n$ there exists a SCI-model $\mathfrak{M} = (\mathbf{A}, D)$ such that $|A| = n$ and $|D| = t$. □
- **Theorem**. For any natural number n there exists a finite SCI-algebra $\mathbf{A}$ which contains n distinct subsets $D_1, \ldots, D_n$ such that for $1 \leqslant i \leqslant n$ the pair $(\mathbf{A}, D_i)$ is a SCI-model and $E((\mathbf{A}, D_i)) \neq E((\mathbf{A}, D_j))$ for $i \neq j$. □
- SCI is decidable, because it has the finite model property:
- **Theorem**. If α is satisfiable in some SCI-model, then it is satisfiable in some finite SCI-model. □
- $C(\emptyset) = \bigcap E(\mathfrak{M})$, where the intersection concerns all SCI-models. We have also $C(\emptyset) = \bigcap E(\mathfrak{M})$, where the intersection concerns all finite SCI-models.
- Nevertheless, there is no single finite SCI-model $\mathfrak{M}$ such that $C(\emptyset) = E(\mathfrak{M})$.

- **Theorem**. There exists a countable model $\mathbf{M} = (\mathbf{A}, D)$ such that $C(\emptyset) = E((\mathbf{A}, D))$. □
- **Theorem**. Each model adequate for $C(\emptyset)$ is infinite. □
- **Theorem**. There exists a model $\mathfrak{M}$ of the power of continuum such that $C = C_{\mathfrak{M}}$. □
- **Theorem**. Each model $\mathfrak{M}$ such that $C = C_{\mathfrak{M}}$ is uncountable. □
- **Theorem**. There exists a countable model $\mathfrak{M}$ such that $C^{WH} = C_{\mathfrak{M}}$. □
- **Theorem**. Each matrix adequate for the system $(\mathcal{L}, C^{WT})$ is uncountable, and hence each model adequate for this system is uncountable. □

- α, β, γ,... sentential formulas
- ξ, η, ζ, ... nominal formulas
- two types of functors (sentential as well as nominal): binding variables and not binding them
- $\sigma(F) = (k, m, n)$, where $k = 0$ (if F is a sentential) lub $k = 1$ (if F is a nominal functor), and m (number of sentential arguments) and n (number of nominal arguments):
  - if $\sigma(F) = (0, m, 0)$, then F is a m-argument connective
  - if $\sigma(F) = (1, 0, n)$, then F is a n-argument predicate
- $\alpha[v/\varphi]$: the result of substitution of φ for variable v in formula α
- generalization of α: the result of adding a quantifier prefix to α; *rule of generalization*: $\frac{\alpha(v)}{\forall v\ \alpha(v)}$
- $Gn(A)$: the set of all generalizations of formulas from A
- X is an *invariant* set of formulas iff $Gn(X) \subseteq X$.

- Functors not binding variables: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\equiv_0$ (identity connective), $\equiv_1$ (identity predicate).
- *Alphabet of a W-language J*: any sequence $A(J) = (V_0, V_1, \mathbf{F}, Q, \sigma)$ such that:
  1. $V_0, V_1, \mathbf{F}, Q$ are disjoint sets (sentential variables, nominal variables, functors not binding variables, quantifiers).
  2. V_0, V_1 are infinite (usually countable) sets.
  3. $\mathbf{F}$ is a finite or countable set such that $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\equiv_0$, $\equiv_1$ are elements of $\mathbf{F}$.
  4. $Q = \{\forall, \exists\}$.
  5. σ is the function defined above.

- The sets $S(J)$ (sentential formulas of J) and $N(J)$ (nominal formulas of J) are defined inductively:
  1. $V_0 \subseteq S(J)$, $V_1 \subseteq N(J)$
  2. If $F \in \mathbf{F}$ and $\sigma(F) = (k, m, n)$, then for any $\alpha_1, \ldots, \alpha_m \in S(J)$ and $\eta_1, \ldots, \eta_n \in N(J)$:
     1. $F(\alpha_1, \ldots, \alpha_m, \eta_1, \ldots, \eta_n) \in S(J)$, if $k = 0$
     2. $F(\alpha_1, \ldots, \alpha_m, \eta_1, \ldots, \eta_n) \in N(J)$, if $k = 1$
  3. If $\alpha \in S(J)$ and $v \in V_0 \cup V_1$, then $\forall v\ \alpha \in S(J)$ and $\exists v\ \alpha \in S(J)$.
- Let J_0 denote the open fragment of J, S_0 sentential formulas of J_0 and N_0 nominal formulas of J_0.
- $\mu = (\mathbf{F}, \sigma)$ is called the *syntax* of J.

- Consequence in W-languages is defined by the axioms given below and the rule modus ponens $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$ as the only rule of inference.
- *A*1 *Axioms for sentential functors*. All generalizations of the axiom schemes in TFA.
- *A*2 *Axioms for quantifiers*. All generalizations of the following schemes:
  1. $\forall v\ \alpha[v/\varphi]$
  2. $\forall v\ (\alpha \rightarrow \beta) \rightarrow (\forall v\ \alpha \rightarrow \forall v\ \beta)$
  3. $\alpha \rightarrow \forall v\ \alpha$ (if v is not free in α)
  4. $\exists v\ \alpha \leftrightarrow \neg \forall v\ \neg \alpha$.

- *A3 Axioms for the identity connective and identity predicate*. All generalizations of the following schemes:
  1. $\varphi_1 \equiv \varphi_2$, if φ_1 i φ_2 differ at most w.r.t. bounded variables
  2. $(\alpha \equiv \beta) \rightarrow (\alpha \rightarrow \beta)$
  3. For each $F \in \mathbf{F}$ a schema of an invariance axiom:
     $$(\varphi_1 \equiv \psi_1) \wedge (\varphi_2 \equiv \psi_2) \wedge \ldots \wedge (\varphi_m \equiv \psi_m) \rightarrow F(\varphi_1, \varphi_2, \ldots, \varphi_m) \equiv F(\psi_1, \psi_2, \ldots, \psi_m)$$
  4. $\forall v\ (\alpha \equiv \beta) \rightarrow (\forall v\ \alpha \equiv \forall v\ \beta)$
  5. $\forall v\ (\alpha \equiv \beta) \rightarrow (\exists v\ \alpha \equiv \exists v\ \beta)$.
- Let $AL = A1 \cup A2 \cup A3$ be the set of all *logical axioms* of J.
- For any $\alpha \in S(J)$ and $X \subseteq S(J)$ let $X \subseteq S(J)$: $\alpha \in Cn(X)$ iff α can be derived from $AL \cup X$ in a finite number of steps, using only the modus ponens rule.

- A *Cn*-theory T is called *invariant* w.r.t. generalization rule iff $Gn(T) \subseteq T$.
- *Cn* has all the properties of consequence C defined for SCI. Besides:
  1. $Cn(\emptyset)$ is invariant w.r.t. generalization rule.
  2. If $\alpha(v) \in Cn(\{\alpha_1, \ldots, \alpha_n\})$ and v does not occur in $\alpha_1, \ldots, \alpha_n$, then $\forall v\, \alpha(v) \in Cn(\{\alpha_1, \ldots, \alpha_n\}$.
- W-languages contain *sentences* (sentential formulas without free variables) and *names* (nominal formulas without free variables).

- Consequence Cn_0 in open W-language J_0 is defined by the axioms given below and the rule modus ponens $\frac{\alpha, \alpha \to \beta}{\beta}$ as the only rule of inference.
  1. axioms from TFA
  2. $\varphi \equiv \varphi$ for any (sentential or nominal) formula of J_0
  3. $(\alpha \equiv \beta) \to (\alpha \to \beta)$
  4. For any functor F from J_0:
     $$(\varphi_1 \equiv \psi_1) \wedge (\varphi_2 \equiv \psi_2) \wedge \ldots \wedge (\varphi_m \equiv \psi_m) \to F(\varphi_1, \varphi_2, \ldots, \varphi_m) \equiv F(\psi_1, \psi_2, \ldots, \psi_m)$$
- For any $X \subseteq S_0$ we have: $Cn_0(X) = Cn(X) \cap S_0$, and therefore Cn_0 is a non-creative extension of Cn.

Open W-languages.

- Let $\mu = (\mathbf{F}, \sigma)$ be the syntax of J_0.
- Let A_0 and A_1 be any disjoint sets such that $|A_0| \geqslant 2$ and $A_1 \neq \emptyset$. Sentential variables are interpreted in A_0, nominal variables in A_1.
- For any functor F such that $\sigma(F) = (k, m, n)$ let its interpretation be a function $o_F : A_0^m \times A_1^n \to A_k$, where $k = 0$ or $k = 1$.
- Any structure $(A_0, A_1, \{o_F\}_{F \in \mathbf{F}})$ is called *bialgebra* of type μ.
- Any language J_0 is a bialgebra absolutely free in the class $\mathcal{K}_\mu$ of all bialgebras of type μ.
- Let $\circ$ denote the interpretation of the identity connective and $\circledcirc$ the interpretation of the identity predicate.

- $\mathfrak{M}_0 = (\mathbf{M}, D)$ is a *W-model of type* μ iff $\mathbf{M} = (A_0, A_1, \{o_F\}_{F \in \mathbf{F}})$ is a bialgebra of type μ, $D \subseteq A_0$ (the set of distinguished elements) and for any $a, b \in A_0$ i $c, d \in A_1$:
  1. $\neg^{\mathbf{M}} a \in D$ iff $a \notin D$
  2. $a \wedge^{\mathbf{M}} b \in D$ iff $a \in D$ and $b \in D$
  3. $a \vee^{\mathbf{M}} b \in D$ iff $a \in D$ or $b \in D$
  4. $a \rightarrow^{\mathbf{M}} b \in D$ iff $a \notin D$ or $b \in D$
  5. $a \leftrightarrow^{\mathbf{M}} b \in D$ iff $a, b \in D$ or $a, b \notin D$
  6. $a \circ^{\mathbf{M}} b \in D$ iff $a = b$
  7. $c \circledcirc^{\mathbf{M}} d \in D$ iff $c = d$
- Any function $h : V_0 \cup V_1 \rightarrow A_0 \cup A_1$ such that $h(V_0) \subseteq A_0$ and $h(V_1) \subseteq A_1$ is called a *valuation* of variables of J_0 in $\mathfrak{M}_0$.
- Any valuation of variables of J_0 can be extended to a homomorphism of J_0 in the algebra of the model $\mathfrak{M}_0$.

- A formula α of J_0 is called:
  1. *satisfied* in the model $\mathfrak{M}_0 = (\mathbf{M}, D)$ by the valuation h, if $h(\alpha) \in D$; $Sat_h(\mathfrak{M}_0) = \{\alpha \in S_0 : h(\alpha) \in D\}$
  2. *true* in the model $\mathfrak{M}_0 = (\mathbf{M}, D)$, if α is satisfied by every valuation in $\mathfrak{M}_0$;
     $TR(\mathfrak{M}_0) = \bigcap\limits_h Sat_h(\mathfrak{M}_0)$.
- Sentential formula α of a language of type μ is a *tautology* of J_0, if it is true in every model of type μ.
- For any theory T in J_0 the relation $\sim_T$ on $S_0 \cup N_0$ defined by $\varphi \sim_T \psi$ iff $\varphi \equiv \psi \in T$ is a congruence of J_0 such that:
  1. if $\varphi \sim_T \psi$, then $\varphi, \psi \in N_0$ or $\varphi, \psi \in S_0$
  2. if $\alpha \sim_T \beta$ and $\alpha \in T$, then $\beta \in T$.
- Let $\mathcal{M}(J_0, T)$ denote the quotient structure $(J_0/\sim_T, T/\sim_T)$.

- **Theorem**. For any W-language J_0 of type μ and any $\alpha \in S_0$ and $X \subseteq S_0$: $\alpha \in Cn_0(X)$ iff for every W-model $\mathfrak{M}_0$ of type μ: if $X \subseteq Sat_h(\mathfrak{M}_0)$, then $\alpha \in Sat_h(\mathfrak{M}_0)$. □
- **Theorem**. T is a quasi-complete theory in J_0 iff there exists a W-model $\mathfrak{M}$ of J_0 such that $T = TR(\mathfrak{M})$. □
- **Theorem**. If $\mathfrak{M}_0 = (\mathbf{M}, D)$ is a W-model of type μ, then there exists an open language J_0 with the syntax μ and a theory T in J_0 such that the Lindenbaum-Tarski model $\mathcal{M}(J_0, T)$ and the model $\mathfrak{M}_0$ are isomorphic. □
- **Theorem**. T is a complete theory in J_0 iff there exists a model $\mathfrak{M}_0$ such that for some valuation h of variables of J_0 in $\mathfrak{M}_0$: $Sat_h(\mathfrak{M}_0) = T$. □
- **Theorem**. T is a quasi-complete theory in J_0 iff there exists a complete theory T_0 in J_0 such that $T \subseteq T_0$ and T is the largest theory closed under substitutions included in T_0. □

W-languages with quantifiers.

- Let $\mathfrak{M}_0 = (A, B, \{o_F\}_{F \in \mathbf{F}}, D)$ be any model of the open language J_0.
- We are going to extend this structure in order to get interpretations of quantifiers (of both types).
- Let h be a valuation of variables in $\mathfrak{M}_0$. The value of (sentential or nominal) formula φ under h in $\mathfrak{M}_0$ is denoted by $||\varphi, h||_{\mathfrak{M}_0}$. We omit the index if the model is clear from the context.
- Let h^v_t denote the valuation such that v is interpreted as t and $h^v_t(u) = h(u)$ for all variables $u \neq v$. It is understood that $t \in A$ if v is a sentential variable and $t \in B$, if v is a nominal variable.

- For any given formula α of J_0 and a valuation h let $\lambda_t||\alpha, h_t^v||$ be the function which associates with any $t \in A$ the value $||\alpha, h_t^v||$. If v does not occur in α, then the function $||\alpha, h_t^v||$ associates with any $t \in A$ the value $||\alpha, h||$ (which is independent from v): in this case that function is constant.
- Interpretation of quantified formulas should satisfy the following conditions, for any formula α, sentential variable p and valuation h:
  1. $||\forall p\, \alpha, h|| \in D$ iff for every $t \in A$: $||\alpha, h_t^p|| \in D$
  2. $||\exists p\, \alpha, h|| \in D$ iff for some $t \in A$: $||\alpha, h_t^p|| \in D$.
- These conditions mean that:
  1. $||\forall p\, \alpha, h|| \in D$ iff $\{t \in A : ||\alpha, h_t^p|| \in D\} = A$
  2. $||\exists p\, \alpha, h|| \in D$ iff $\{t \in A : ||\alpha, h_t^p|| \in D\} \neq \emptyset$.

- Interpretation of quantified formulas should also satisfy the following conditions, for any formula α, nominal variable x and valuation h:
  1. $||\forall x\ \alpha, h|| \in D$ iff for every $t \in B$: $||\alpha, h^x_t|| \in D$
  2. $||\exists x\ \alpha, h|| \in D$ iff for some $t \in B$: $||\alpha, h^x_t|| \in D$.
- These conditions mean that:
  1. $||\forall x\ \alpha, h|| \in D$ iff $\{t \in B : ||\alpha, h^x_t|| \in D\} = B$
  2. $||\exists x\ \alpha, h|| \in D$ iff $\{t \in B : ||\alpha, h^x_t|| \in D\} \neq \emptyset$.

- $\mathfrak{M} = (A, B, \{o_F\}_{F\in\mathbf{F}}, \bigwedge^A, \bigvee^A, \bigwedge^B, \bigvee^B)$ is a *partial pseudo-model* for a W-language J in the alphabet $(V_0, V_1, \mathbf{F}, Q, \sigma)$ iff:
  1. $(A, B, \{o_F\}_{F\in\mathbf{F}})$ is a bialgebra similar to the open fragment of J;
  2. $\bigwedge^A, \bigvee^A$ are functions from an arbitrary but fixed subset $\triangle_A$ of the set A^A of all functions from A to A, which means that if $f \in \triangle_A$, then $\bigwedge^A f, \bigvee^A f \in A$;
  3. $\bigwedge^B, \bigvee^B$ are functions from an arbitrary but fixed subset $\triangle_B$ of the set A^B of all functions from B to A, which means that if $f \in \triangle_B$, then $\bigwedge^B f, \bigvee^B f \in A$;
- $h : V_0 \cup V_1 \to A \cup B$ is a *valuation* of variables of J in a partial pseudo-model $\mathfrak{M}$ iff $h(V_0) \subseteq A$ and $h(V_1) \subseteq B$.

- Let a valuation h be fixed. The *value* of a formula of J under h is defined inductively:
  1. If $\varphi \in V_0 \cup V_1$, then $||\varphi, h|| = h(\varphi)$;
  2. If $F \in \mathbf{F}$ i $\sigma(F) = (k, m, n)$, then $||F(\alpha_1, \ldots, \alpha_m, \eta_1, \ldots, \eta_n), h|| = o_F(||\alpha, h||, \ldots, ||\alpha_m, h||, ||\eta_1, h||, \ldots, ||\eta_n, h||)$;
  3. For any formula α, if $\lambda_t ||\alpha, h_t^p|| \in \triangle_A$, then
     $||\forall p\ \alpha, h|| = \bigwedge^A ||\alpha, h_t^p||$
     $||\exists p\ \alpha, h|| = \bigvee^A ||\alpha, h_t^p||$
  4. For any formula α, if $\lambda_t ||\alpha, h_t^x|| \in \triangle_B$, then
     $||\forall x\ \alpha, h|| = \bigwedge^B ||\alpha, h_t^x||$
     $||\exists x\ \alpha, h|| = \bigvee^B ||\alpha, h_t^x||$.
- If the value $||\varphi, h||$ is defined for any (sentential or nominal) formula φ and any valuation h, then $\mathfrak{M}$ is called a *pseudo-model* of J.

- Let a pseudo-model $\mathfrak{M}$ be fixed. A function $f \in A^A$ is *determined by a formula* of J iff there exists a sentential formula α in J and a sentential variable p such that for every $t \in A$: $f(t) = ||\alpha, h_t^p||$. In a similar manner we define functions from A^B and B^B determined by a formula of J.
- $\mathfrak{M} = (A, B, \{o_F\}_{F \in \mathbf{F}}, \bigwedge^A, \bigvee^A, \bigwedge^B, \bigvee^B, D)$ is called a *model* of a W-language J iff:
  1. $(A, B, \{o_F\}_{F \in \mathbf{F}}, \bigwedge^A, \bigvee^A, \bigwedge^B, \bigvee^B)$ is a pseudo-model of J;
  2. $(A, B, \{o_F\}_{F \in \mathbf{F}}, D)$ is a model for the open fragment of J;
  3. For any function $f \in A^A$ determined by a formula of J:
     $\bigwedge^A f \in D$ iff $f(t) \in D$ for every $t \in A$
     $\bigvee^A f \in D$ iff $f(t) \in D$ for some $t \in A$
  4. For any function $f \in A^B$ determined by a formula of J:
     $\bigwedge^B f \in D$ iff $f(t) \in D$ for every $t \in B$
     $\bigvee^B f \in D$ iff $f(t) \in D$ for some $t \in B$.

- A formula α of J is called:
  1. *satisfied* in a model $\mathfrak{M}$ by a valuation h ($\alpha \in Sat_h(\mathfrak{M})$) iff $||\alpha, h|| \in D$;
  2. *true* in a model $\mathfrak{M}$ ($\alpha \in TR(\mathfrak{M})$) iff α is satisfied by every valuation in $\mathfrak{M}$;
  3. *a tautology* of J ($\alpha \in Taut(J)$) iff it is true in every model of J.
- It follows from these definitions that:
  1. $Sat_h(\mathfrak{M}) = \{\alpha : ||\alpha, h|| \in D\}$
  2. $TR(\mathfrak{M}) = \bigcap\limits_{h} Sat_h(\mathfrak{M})$
  3. $Taut(J) = \bigcap\limits_{\mathfrak{M}} TR(\mathfrak{M})$.
- A model $\mathfrak{M}$ of J is called a *model of a set of formulas* X iff $X \subseteq TR(\mathfrak{M})$.

- **Theorem** (Bloom 1971). Let J be a W-language. For any X and α: $\alpha \in Cn(X)$ iff for every model $\mathfrak{M}$ of J, if $X \subseteq TR(\mathfrak{M})$, then $\alpha \in \mathfrak{M}$.
- **Proof outline**. It is convenient to divide the proof into three lemmas:
  1. **Lemma 1**. $Cn(\emptyset) \subseteq Taut(J)$.
  2. **Lemma 2**. If X is a consistent set of sentences of a W-language J, then there exists a model $\mathfrak{M}$ of J such that $X \subseteq TR(\mathfrak{M})$.
  3. **Lemma 3**. $Taut(J) \subseteq Cn(\emptyset)$.

- **Outline of proof of Lemma 1**. Firstly, one has to check that each axiom is a tautology.
- Let us prove, for example, that $\forall p\ (\alpha \rightarrow \beta) \rightarrow (\forall p\ \alpha \rightarrow \forall p\ \beta)$ is a tautology.
- Let $\mathfrak{M} = (A, B, \{o_F\}_{F \in \mathbf{F}}, \bigwedge^A, \bigvee^A, \bigwedge^B, \bigvee^B, D))$ be any W-model.
- Then for some $F \in \mathbf{F}$ the operation o_F is the denotation of $\rightarrow$, that is $o_F = \rightarrow^{\mathfrak{M}}$, and we have: $a \rightarrow^{\mathfrak{M}} b \in D$ iff $a \notin D$ or $b \in D$, for any $a, b \in A$.
- In order to prove that $\forall p\ (\alpha \rightarrow \beta) \rightarrow (\forall p\ \alpha \rightarrow \forall p\ \beta)$ is a tautology it suffices to show that there does not exist a valuation h such that: $||\forall p\ \alpha, h|| \in D$, $||\forall p\ (\alpha \rightarrow \beta)|| \in D$ i $||\forall p\ \beta, h|| \notin D$.

- Suppose the contrary holds, that is for some valuation h: $||\forall p\, \alpha, h|| \in D$, $||\forall p\, (\alpha \to \beta), h|| \in D$ ale $||\forall p\, \beta, h|| \notin D$.
- Since $||\forall p\, \alpha, h|| \in D$ and $||\forall p\, \beta, h|| \notin D$, then according to the definition of $\bigwedge_A$:
  1. $||\forall p\, \alpha, h|| \in D$ iff $||\alpha, h^p_t|| \in D$ for every $t \in A$
  2. $||\forall p\, \beta, h|| \notin D$ iff $||\beta, h^p_{t_0}|| \notin D$ for some $t_0 \in A$.
- Therefore for some $t_0 \in A$: $||\alpha, h^p_{t_0}|| \in D$ and $||\beta, h^p_{t_0}|| \notin D$.
- This means that for some $t_0 \in A$: $(||\alpha, h^p_{t_0}|| \to^{\mathfrak{M}} ||\beta, h^p_{t_0}||) \notin D$, contrary to the assumption that $||\forall p\, (\alpha \to \beta), h|| \in D$.
- It follows from the definition of $\to^{\mathfrak{M}}$ that the modus ponens rule preserves tautologies, and this finishes the proof. □

- **Outline of proof of Lemma 2**. This proof uses the well-known Henkin's technique of model construction for a consistent theory.
- We extend the alphabet of J by adding to it a countable set of sentential constants $(r_i)_{i \in \mathbb{N}}$ and a countable set of individual constants $(c_i)_{i \in \mathbb{N}}$. Let us denote:
  1. J^*: the language J with the alphabet expanded as described above
  2. S: the set of all sentences of J^*
  3. N: the set of all names of J^*
  4. S_{J^*}: the set of all sentential formulas of J^*
  5. N_{J^*}: the set of all nominal formulas of J^*.
- The countable set S can be arranged as a sequence: $\gamma_1, \gamma_2, \gamma_3, \ldots$
- Let c_{i_1} be the first element of the sequence $(c_i)_{i \in \mathbb{N}}$, which does not occur in the sentence γ_1. If the sequence $(c_{i_1}, \ldots, c_{i_n})$ is already defined, let $c_{i_{n+1}}$ be the first element of the sequence $(c_i)_{i \in \mathbb{N}}$, which does not occur in the sentences $\gamma_1, \ldots, \gamma_n, \gamma_{n+1}$ and $i_n < i_{n+1}$.

- Let r_{i_1} be the first element of the sequence $(r_i)_{i \in \mathbb{N}}$, which does not occur in the sentence γ_1. If the sequence $(r_{i_1}, \ldots, r_{i_n})$ is already defined, let $r_{i_{n+1}}$ be the first element of the sequence $(r_i)_{i \in \mathbb{N}}$, which does not occur in the sentences $\gamma_1, \ldots, \gamma_n, \gamma_{n+1}$ and $i_n < i_{n+1}$.
- We define a sequence $(A_i)_{i \geqslant 0}$ of sets as follows. $A_0 = X$; if A_n is already defined, let A_{n+1} be defined as follows:
  1. If the sentence γ_n is of the form $\forall x\, \alpha$, then $A_{n+1} = A_n \cup \{\alpha[x/c_{i_n}] \to \forall x\, \alpha\}$
  2. If the sentence γ_n is of the form $\forall p\, \alpha$, then $A_{n+1} = A_n \cup \{\alpha[p/r_{i_n}] \to \forall p\, \alpha\}$
  3. $A_{n+1} = A_n$ in the remaining cases.
- Let $A = \bigcup\limits_{i=0}^{\infty} A_i$. The proof that A is consistent is a routine. It is also well-known that A can be extended to a complete consistent set, say T. Then:
  1. $X \subseteq A \subseteq T \subseteq S$.
  2. If $\alpha \in S$ i $\alpha \notin T$, then $Cn(T \cup \{\neg\alpha\}) = S_{J^*}$.

The set T of sentences of J^* has the following properties:

- There exists a complete theory T^* in J^* such that $T = T^* \cap S$.
- For each sentence of the form $\forall x\, \alpha$ from J^*: $\forall x\, \alpha \in T$ iff $\alpha[x/a] \in T$ for every name $a \in N$.
  If $\forall x\, \alpha \in T$, then it follows from $\forall x\, \alpha \to \alpha[x/a]$ that $\alpha[x/a] \in T$ for every name $a \in N$. If, in turn, $\alpha[x/a] \in T$ for every name $a \in N$, then $\alpha[x/c_{i_n}] \in T$ for some i_n, and hence also $\forall x\, \alpha \in T$, on the basis of the definition of the sets A_n. We prove similarly that:
- For each sentence of the form $\forall p\, \alpha$ from J^*: $\forall p\, \alpha \in T$ iff $\alpha[p/\gamma] \in T$ for every sentence $\gamma \in S$.
- For each sentence of the form $\exists x\, \alpha$ from J^*: $\exists x\, \alpha \in T$ iff $\alpha[x/a] \in T$ for some name $a \in N$.
- For each sentence of the form $\exists p\, \alpha$ from J^*: $\exists p\, \alpha \in T$ iff $\alpha[p/\gamma] \in T$ for some sentence $\gamma \in S$.

- We construct the structure $\mathfrak{M}^* = (S, N, \{o_F\}_{F \in \mathbf{F}}, \bigwedge_S, \bigwedge_N, \bigvee_S, \bigvee_N, T)$:
  1. S is the universe for sentential variables of J.
  2. N is the universe for nominal variables of J.
  3. T is the set of distinguished elements of $\mathfrak{M}^*$.
  4. If $F \in \mathbf{F}$, $\sigma(F) = (k, m, n)$, $\gamma_1, \ldots, \gamma_m \in S$ and $\eta_1, \ldots, \eta_n \in N$, then: $o_F(\gamma_1, \ldots, \gamma_m, \eta_1, \ldots, \eta_n) = F(\gamma_1, \ldots, \gamma_m, \eta_1, \ldots, \eta_n)$.
  5. The domain of functions $\bigwedge_S$ and $\bigvee_S$ is the set of all functions $\lambda_t \gamma[p/t]$, where $\gamma(p)$ is a sentential formula of J^* with one free variable p and $t \in S$. The values of $\bigwedge_S$ and $\bigvee_S$ for the argument $\lambda_t \gamma[p/t]$ are defined as follows: $\bigwedge_S \lambda_t \gamma[p/t] = \forall p\, \gamma(p)$, $\bigvee_S \lambda_t \gamma[p/t] = \exists p\, \gamma(p)$.
  6. Similarly, if $\gamma[x/a]$ is a sentential formula of J^* with one nominal variable x and $a \in N$, then: $\bigwedge_N \lambda_a \gamma[x/a] = \forall x\, \gamma(x)$, $\bigvee_N \lambda_a \gamma[x/a] = \exists x\, \gamma(x)$.
- Let $\varphi, \psi \in S \cup N$. The relation $\sim_T$ defined by $\varphi \sim_T \psi$ iff $\varphi \equiv \psi \in T$ is a congruence of $\mathfrak{M}^*$

- We build the quotient structure $\mathfrak{M} = \mathfrak{M}^* / \sim_T$. Then: $\mathfrak{M} = (S/\sim_T, N/\sim_T, \{o_F/\sim_T\}_{F \in \mathbf{F}}, \bigwedge_S/\sim_T, \bigwedge_N/\sim_T, \bigvee_S/\sim_T, \bigvee_N/\sim_T, T/\sim_T)$.
- One proves that $\mathfrak{M}$ is a W-model and that $X \subseteq TR(\mathfrak{M})$. Note that:
  1. Elements of X are sentences, and hence their values in any model does not depend on valuations in the model: $||\alpha, h|| = [\alpha]_{\sim_T}$ for every sentence $\alpha \in X$ and any valuation h.
  2. If $\alpha \in X$, then $[\alpha]_{\sim_T} \in T/_{\sim_T}$. □

- **Outline of proof of Lemma 3**. Assume that α is a tautologu of a W-language J. If $\overline{\alpha}$ is a generalization of formuły α, which does not contain free variables, then $\overline{\alpha}$ is a W-tautology of J as well.
- Suppose that $\overline{\alpha}$ is not a theorem.
- Then $\neg\overline{\alpha}$ is a consistent sentence, because if $\neg\overline{\alpha}$ were inconsistent, then $\neg\overline{\alpha} \in Cn(\{\neg\overline{\alpha}\})$, and from the deduction theorem we would have $(\neg\overline{\alpha} \rightarrow \overline{\alpha}) \in Cn(\emptyset)$.
- In this case, on the basis of the theorem $(\neg\overline{\alpha} \rightarrow \overline{\alpha}) \rightarrow \overline{\alpha}$ we would have that $\overline{\alpha}$, as well as α are theorems.
- We can thus maintain the assumption that $\neg\overline{\alpha}$ is consistent.
- It follows from Lemma 2 that there exists a model $\mathfrak{M}$ of J such that $\neg\overline{\alpha} \in TR(\mathfrak{M})$.
- But then both α and $\neg\alpha$ would be true in $\mathfrak{M}$, which is impossible.
- Therefore α is a theorem. □

- **Outline of proof of the completeness theorem**. Assume that $\alpha \notin Cn(X)$. Then the set $X \cup \{\neg\alpha\}$ is consistent.
- It follows from Lemma 2 that there exists a model $\mathfrak{M}$ such that $X \cup \{\neg\alpha\} \subseteq TR(\mathfrak{M})$, which means that there exists a model $\mathfrak{M}$ such that $X \subseteq TR(\mathfrak{M})$ and $\alpha \notin TR(\mathfrak{M})$.
- Assume, in turn, that for some model $\mathfrak{M}$: $X \subseteq TR(\mathfrak{M})$ and $\alpha \notin TR(\mathfrak{M})$.
- Then $\neg\alpha \in TR(\mathfrak{M})$, which implies that the set $X \cup \{\neg\alpha\}$ is consistent.
- Hence there exists a complete theory T such that $X \cup \{\neg\alpha\} \subseteq T$.
- It follows from the above that $\alpha \notin Cn(X)$, because every Cn-theory is the intersection of all complete theories including it.
- This finishes the proof of the completeness theorem. □

- Bloom, S. 1971. A completeness theorem for "Theories of kind W". *Studia Logica* 27, 43–55.
- Bloom, S., Suszko, R. 1972. Investigations into the sentential calculus with identity. *Notre Dame Journal of Formal Logic* 13 (3), 289–308.
- Omyła, M. 1986. *Zarys logiki niefregowskiej*. Państwowe Wydawnictwo Naukowe, Warszawa.
- Suszko, R. 1968. Ontologia w Traktacie L. Wittgensteina. *Studia Filofoziczne* 1 (52), 97–121.
- Suszko, R. 1975. Abolition of the Fregean axiom. *Lecture Notes in Mathematics* 453, 169–239.