# Reverse Mathematics of Model Theory

## Or: What I Would Tell My Graduate Student Self About Reverse Mathematics

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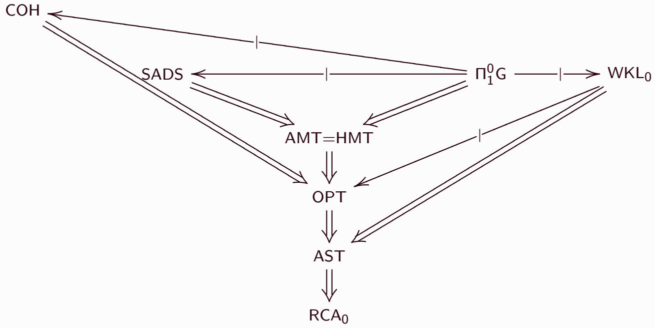


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We'll examine some of these in the context of model-theoretic principles.

The Completeness Theorem is provable in $\mathsf{RCA}_0$.

But what if we want to produce models with particular properties?

# Conventions and Basic Definitions I

All our theories $T$ are countable, complete, and consistent.

All our models $\mathcal{M}$ are countable.

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$T$ is decidable if it is computable.

$\mathcal{M}$ is decidable if its elementary diagram is computable.

In reverse mathematics, we identify $\mathcal{M}$ with its elementary diagram.

# Conventions and Basic Definitions II

A partial type $\Gamma$ of $T$ is a set of formulas $\{\psi_n(\vec{x})\}_{n\in\omega}$ consistent with $T$.

$\Gamma$ is a (complete) type if it is maximal.

$\Gamma$ is principal if there is a consistent $\varphi$ s.t. $\forall\psi \in \Gamma\ (T + \varphi \vdash \psi)$.

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We write $\vec{a} \equiv \vec{b}$ if $\vec{a}$ and $\vec{b}$ have the same complete type.

$\mathcal{M}$ realizes $\Gamma$ if some $\vec{a} \in \mathcal{M}$ has type $\Gamma$. Otherwise $\mathcal{M}$ omits $\Gamma$.

The type spectrum of $\mathcal{M}$ is the set of types it realizes.

# Homogeneous models

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Equivalently, $\mathcal{M}$ is homogeneous if for all $\vec{a} \equiv \vec{b} \in \mathcal{M}$ and all $c \in \mathcal{M}$, there is a $d \in \mathcal{M}$ s.t. $\vec{a}c \equiv \vec{b}d$.

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**HOM**: Every theory has a homogeneous model.

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# Building Homogeneous Models

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A Turing degree is PA if it is the degree of a nonstandard model of PA.

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$\mathbf{d}$ is PA iff every infinite binary tree has an infinite $\mathbf{d}$-computable path.

# A Reversal

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**Thm (Csima, Harizanov, Hirschfeldt, and Soare).** There is a decidable $T$ s.t. any homogeneous model of $T$ has PA degree.

**Thm (Lange).** $\mathsf{RCA}_0 \vdash \mathsf{HOM} \rightarrow \mathsf{WKL}_0$.

# Atomic and homogeneous models

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If $\mathcal{M}$ is atomic then it is homogeneous.

Any two atomic models of $T$ are isomorphic.

$T$ is atomic if every formula consistent with $T$ can be extended to a principal type of $T$.

$T$ has an atomic model iff $T$ is atomic.

# The Atomic Model Theorem I

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**Thm (Hirschfeldt, Shore, and Slaman).** AMT and $\mathsf{WKL}_0$ are incomparable over $\mathsf{RCA}_0$.

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SADS is strictly weaker than ADS, which is strictly weaker than $RT^2_2$.

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**Thm (Hirschfeldt, Shore, and Slaman).** $RCA_0 + SADS \vdash AMT$.
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- Closure under unions of types on disjoint sets of variables.
- Closure under type / type amalgamation.
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If $S$ satisfies these conditions, we say it is closed.

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**HMT:** If $S$ is closed then there is a homogeneous model of $T$ with type spectrum $S$.

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Computability theoretic results suggest that HMT behaves like AMT:

**d** is low if $\Delta_2^{0,\mathbf{d}} = \Delta_2^0$.

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**Thm (Csima).** Every decidable atomic $T$ has a low atomic model.

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**Thm (Lange).** For every computable closed $S$, there is a low homogeneous model with type spectrum $S$.

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# The Homogeneous Model Theorem and AMT II

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**Thm (Csima, Hirschfeldt, Knight, and Soare).** TFAE if $\mathbf{d} \leqslant \mathbf{0}'$:

- Every decidable atomic $T$ has a $\mathbf{d}$-decidable atomic model.
- $\mathbf{d}$ is nonlow$_2$.

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**Thm (Lange).** TFAE if $\mathbf{d} \leqslant \mathbf{0}'$:

- For every computable closed $S$ there is a $\mathbf{d}$-decidable homogeneous model of $T$ with type spectrum $S$.
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# The Homogeneous Model Theorem and AMT III

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**Thm (Hirschfeldt, Lange, and Shore).** $\mathsf{RCA}_0 \vdash \mathsf{AMT} \leftrightarrow \mathsf{HMT}$.

# Atomic models and type omitting

# Omitting Partial Types

**Thm (Millar).** Let $T$ be decidable.

Let $A$ be a computable set of complete types of $T$.
There is a decidable model of $T$ omitting all nonprincipal types in $A$.

Let $B$ be a computable set of nonprincipal partial types of $T$.
There is a decidable model of $T$ omitting all partial types in $B$.

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**Thm (Millar).** There is a decidable $T$ and a computable set of partial types $C$ of $T$ s.t. no decidable model of $T$ omits all nonprincipal partial types in $C$.

# Omitting Partial Types II

**Thm (Millar).** There is a decidable $T$ and a computable set of partial types $C$ of $T$ s.t. no decidable model of $T$ omits all nonprincipal partial types in $C$.

# Omitting Partial Types II

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**Thm (Csima).** Let $T$ be decidable and let $C$ be a computable set of partial types of $T$. If $\mathbf{0} < \mathbf{d} \leqslant \mathbf{0}'$ then there is a $\mathbf{d}$-decidable model of $T$ omitting all nonprincipal partial types in $C$.

# AMT With Computable Types

**Thm (Goncharov and Nurtazin; Harrington).** Let $T$ be a decidable atomic theory s.t. the types of $T$ are uniformly computable.
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**Thm (Goncharov and Nurtazin; Millar).** There is a decidable atomic $T$ s.t. each type of $T$ is computable, but $T$ has no decidable atomic model.

**Thm (Csima).** Let $T$ be a decidable atomic theory s.t. each type of $T$ is computable, and let $\mathbf{0} < \mathbf{d} \leqslant \mathbf{0}'$. Then $T$ has a $\mathbf{d}$-decidable atomic model.

# Omitting Types and Atomic Models

**Thm (Csima).** Let $T$ be decidable and let $C$ be a computable set of partial types of $T$. If $\mathbf{0} < \mathbf{d} \leqslant \mathbf{0}'$ then there is a $\mathbf{d}$-decidable model of $T$ omitting all nonprincipal partial types in $C$.

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Let $T$ be a decidable atomic theory s.t. each type of $T$ is computable.

There is a computable set of partial types $C$ containing every complete type of $T$.

Omitting $C$ yields an atomic model of $T$.

# Omitting Types and Atomic Models

**Thm (Hirschfeldt).** Let $T$ be a decidable atomic theory s.t. each type of $T$ is computable, and let $\mathbf{d} > \mathbf{0}$. Then $T$ has a $\mathbf{d}$-decidable atomic model.

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$\mathbf{d}$ is *hyperimmune* if there is a $\mathbf{d}$-computable $g$ not majorized by any computable $f$.

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**Thm (Hirschfeldt, Shore, and Slaman).** Let $T$ be decidable and let $C$ be a computable set of partial types of $T$. If $\mathbf{d}$ is hyperimmune then there is a $\mathbf{d}$-decidable model of $T$ omitting all nonprincipal partial types in $C$.

There is a decidable $T$ and a computable set $C$ of partial types of $T$ s.t. every model of $T$ that omits $C$ has hyperimmune degree.

# Reverse Mathematical Versions

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**HYP:** For every $X$ there is a $g$ not majorized by any $X$-computable $f$.

**Thm (Hirschfeldt, Shore, and Slaman).** $\mathrm{RCA}_0 \vdash \mathrm{OPT} \leftrightarrow \mathrm{HYP}$.

# Reverse Mathematical Versions II

Partial types $\Gamma$ and $\Delta$ of $T$ are equivalent if they imply the same formulas over $T$.

$(\Delta_n)_{n\in\omega}$ is a subenumeration of the partial types of $T$ if for every partial type $\Gamma$ of $T$ there is an $n$ s.t. $\Gamma$ and $\Delta_n$ are equivalent.

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**AST:** If $T$ is atomic and its partial types have a subenumeration, then $T$ has an atomic model.

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**Thm (Hirschfeldt, Shore, and Slaman).** $\mathsf{RCA}_0 \vdash \mathsf{AST} \leftrightarrow \forall X \exists Y (Y \not\leq_{\mathbf{T}} X)$.

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