Notre Dame Journal of Formal Logic
Volume IX, Number 1, January 1968

ONTOLOGY IN THE *TRACTATUS* OF L. WITTGENSTEIN

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Abstract mathematics
may be a genuine philosophy.

The *Tractatus Logico-Philosophicus* of Ludwig Wittgenstein is a very unclear and ambiguous metaphysical work. Previously, like many formal logicians, I was not interested in the metaphysics of the *Tractatus*. However, I read in 1966 the text of a monograph by Dr. B. Wolniewicz of the University of Warsaw[2] and I changed my mind. I see now that the conceptual scheme of *Tractatus* and the metaphysical theory contained in it may be reconstructed by formal means. The aim of this paper* is to sketch a formal system or formalized theory which may be considered as a clear, although not complete, reconstruction of the ontology contained in Wittgenstein's *Tractatus*.

It is not easy to say how much I am indebted to Dr. Wolniewicz. I do not know whether he will agree with all theorems and definitions of the formal system presented here. Nevertheless, I must declare that I could not write the present paper without being acquainted with the work of Dr. Wolniewicz. I learned very much from his monograph and from conversations with him. However, when presenting in this paper the formal system of Wittgenstein's ontology I will not refer mostly either to the monograph of Dr. Wolniewicz or to the *Tractatus*. Also, I will not discuss here the problem of adequacy between my formal construction and *Tractatus*. I think that the Wittgenstein was somewhat confused and wrong in certain points. For example, he did not see the clear-cut distinction between language (theory) and metalanguage (metatheory): a confusion between use and mention of expressions.

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*Presented in Polish at the Conference on History of Logic, April 28-29, 1967, Cracow, Poland. cf. footnote 1.

Received November 16, 1967

Ludwig Wittgenstein attempted in the *Tractatus* to build a theory of the epistemological opposition:

Mind (language) - Reality (being)

One may distinguish in the *Tractatus* the three following components:

1. Ontology, i.e., a theory of being,
2. Syntax, i.e., a theory of the structure of language (mind),
3. Semantics, i.e., a theory of the epistemological relations between linguistic expressions and reality.

I present below the formalized version of Wittgenstein's ontology. The syntax and semantics contained in *Tractatus* will be not considered here. Wittgenstein's ontology is general and a formal theory of being. It may be called here shortly: ontology. It concerns (independently of time and space)[3], situations (facts, negative facts, atomic and compound situations) and objects. Thus, the ontology is composed of two parts:

1. *s*-ontology, i.e., the ontology of situations (*Sachlagen*),
2. *o*-ontology, i.e., the ontology of objects (*Gegenstände*).

The link between the two parts of ontology consists in the somewhat mysterious concept of a state of affairs (*Sachverhalt*) and that of a configuration of objects. The *s*-ontology is an original theory of Ludwig Wittgenstein. It is related in a sense to certain conceptions of G. Frege and to the formalized system of protothetics of St. Leśniewski. The theories of Frege and Leśniewski make use of sentential variables and of operators (e.g. quantifiers) binding them. The *s*-ontology is also to be formalized by means of sentential variables and corresponding operators binding them. This is the cause of a certain strangeness of *s*-ontology and, consequently, of the whole of Wittgenstein's philosophy. Firstly, most formalized languages of contemporary mathematical logic do not use bound sentential variables. On the other hand, the *Tractatus* essentially uses natural language and the notions and statements of *s*-ontology formulated in this language may seem to be produced by hypostatising thinking. Certainly, thinking in natural language is much more appropriate to the *o*-ontology than to *s*-ontology. Consequently, mathematical thinking in its historical development up to to-day is concerned with (abstract) objects and not situations. Consider for example the following sentence of *s*-ontology:

(1) Some situations are not facts.

It has the same grammatical structure as the following sentence:

(2) Some philosophers are not logicians.

Both sentences (1) and (2) are existential sentences. But there is a very deep difference between them. The terms "philosopher" and "logician" in (2) are unary predicates. The terms "situation" and "fact" are not predicates. They are unary sentential connectives like the word "not" which converts any sentence φ into its negation: not-φ. To see this, let us make the first step in formalizing (1) and (2). We write:

(3) for some p, Sp and not-Fp
(4) for some x, Px and not-Lx.

The letter p is a sentential variable and the letter x is a nominal variable. They are bound above in (3) and (4), respectively, by the existential quantifier: "for some". The symbols P and L are unary predicates and the letters S and F are unary sentential connectives like the connective "not". The difference between sentences (sentential variables, sentential formulae) and names (nominal variables, nominal formulae)[4] is very deep and fundamental in every language. It must be observed in any rigorous thinking. However, natural language leads sometimes to confusion on this point. Having in mind the categorial difference mentioned above we consider the sentence (1) quite as legimate and meaningful as the sentence (2). Moreover, both sentences (1) and (2) are true because:

1. It is a situation that London is a small city but it is not a fact.
2. Dr. B. W. is a good philosopher but he is not a logician.

1. *The language of ontology*. The language $\mathfrak{L}_0$ of ontology contains variables of two kinds:

1. sentential variables: $p, q, r, s, t, u, w, p_1, p_2, \ldots, p_k, \ldots$
2. nominal variables: $x, y, z, A, B, C, R, S, x_1, x_2, \ldots, x_k, \ldots$

There are in $\mathfrak{L}_0$ some sentential constants, i.e., simple sentences. There may be in $\mathfrak{L}_0$ some nominal constants, i.e., simple names. The meaningful expressions of $\mathfrak{L}_0$, or well formed formulae of $\mathfrak{L}_0$, are divided into sentential formulae and nominal formulae. The variables may occur in formulae as free or bound (by suitable operators). Any sentential or nominal formula which contains no free variable is called a sentence or name, as the case may be. The formulae of $\mathfrak{L}_0$ are built of variables and constants by means of certain syncategorematic expressions (and parentheses) like predicates, functors[5], sentential connectives, quantifiers, etc.

At first we mention the predicate of logical identity and the connective of logical identity. Both are denoted by the same symbol: =. Thus, if η, δ are nominal formulae and φ, ψ are sentential formulae then the expressions:

(1.1) $$\eta = \delta \qquad\qquad \varphi = \psi$$

are sentential formulae. The language $\mathfrak{L}_0$ contains the customary compound sentential formulae: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \equiv \psi$, $N\varphi$ built by means of classical sentential connectives (conjunction, alternation, material implication, material equivalence, negation, respectively). The quantifiers: $\forall, \exists$ (general and existential) are allowed to bind sentential variables and nominal variables as well. Thus we have in $\mathfrak{L}_0$ sentential formulae of the following forms:

(1.2) $\forall x\ \varphi(x)$ $\exists x\ \varphi(x)$

(1.3) $\forall p\ \psi(p)$ $\exists p\ \psi(p)$

Strictly speaking, we have in $\mathfrak{L}_0$ two kinds of quantifiers: (1.2) and (1.3). But we do not need to use different symbols for them. The language $\mathfrak{L}_0$ contains also the description operator (unifier in the sense of P. Bernays, 1958) which is allowed to bind nominal and sentential variables. It will be not used explicitly here however.

All the symbols mentioned above ($=$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$, N, $\forall$, $\exists$ and the unifier) are used in $\mathfrak{L}_0$ with their customary meaning. This means that the relation of logical derivability in $\mathfrak{L}_0$ is determined by the classical rules of interference. We assume that a sentential formula φ follows logically (is logically derivable) from sentential formulae $\varphi_1, \ldots, \varphi_k$ if and only if φ may be obtained in finitely many steps from $\varphi_1, \ldots, \varphi_k$ according to the rules of classical logical calculus. This calculus includes: two-valued sentential tautologies, modus ponens (for material implication), the rules for introducing and omitting quantifiers, two rules of substitution for free variables, the axioms for logical identity and the axioms for the unifier. A formula φ is called a logical theorem if and only if φ follows logically from the empty set of formulae. We decide to allow in the system of ontology the definitions of equational type only, i.e., identities (1.1) of some special form.

Our task consists in determining two sets of sentential formulae: the set of ontological axioms and the set of ontological definitions. Consequently, the ontology or the set of ontological theorems may be identified with the set of all sentential formulae which follow logically from ontological axioms and definitions. Clearly, every logical theorem is an ontological theorem but not conversely. Many theorems will be formulated below with free variables. The reader may convert them into sentences by preceding them by general quantifiers. The ontological axioms and definitions proposed below cannot be considered as the ultimate solution of our problem. To avoid unnecessary derivations, many ontological theorems are included here in the set of axioms. Therefore, our axiom system is not independent. It may be replaced by another equivalent one which is shorter and more elegant. On the other hand, perhaps there may be some reasons (other than simplification) either to strengthen the proposed axiom system (e.g., to replace the equivalencies (4.7), (4.8) and (5.6) by corresponding equalities) or to change in some way the adopted definitions. I am not here to systematically study the logical properties (independent axiom system, the derivations of theorems, the proof of consistency) of the formal system of ontology. I intend only to present a formal theory which is identical in content with Wittgenstein's ontology.[6]

2. *Logical identity* For logical identity we have the two following axioms:

(2.1) $\forall s\ (s = s)$

(2.2) $\forall z\ (z = z)$

and the two following schemes of axioms:

(2.3) $$x = y \rightarrow (\varphi(x) \equiv \varphi(y))$$

(2.4) $$p = q \rightarrow (\psi(p) \equiv \psi(q))$$

The axioms for identity belong to the logical calculus. They are logical theorems. It is easy to see that the following schemes of formulae:

(2.5) $$x = y \rightarrow (\varphi(x) = \varphi(y))$$

(2.6) $$p = q \rightarrow (\psi(p) = \psi(q))$$

follow logically from the axioms for identity. Clearly, we have:

(2.7) $$p = q \rightarrow p \equiv q$$

Observe that the formula:

(*) $$p \equiv q \rightarrow p = q$$

is equivalent to the following scheme of formulae:

(**) $$p \equiv q \rightarrow (\varphi(p) \equiv \psi(q))$$

In the sequel we will use the auxiliary definitions:

(2.8)	$(p \neq q) = \mathrm{N}(p = q)$	
(2.9)	$(x \neq y) = \mathrm{N}(x = y)$	
(2.10)	$pDq = \mathrm{N}(p \equiv q)$	(p or q, but not both)
(2.11)	$Sp = (p = p)$	(it is a situation that p)
(2.12)	$Ox = (x = x)$	(x is an object)

The term "situation" is a universal sentential connective and the term "object" is a universal predicate. We have the theorems:

(2.13) $$\forall p\ Sp \qquad \forall x\ Ox$$

The logical axioms of identity have interesting consequences in o-ontology. We use here the connective L of necessity which will be introduced later. We obtain immediately from (2.5) that: $x = y \rightarrow (L\varphi(x) = L\varphi(y))$. It is a logical theorem. By (4.9) we have: $L(x = x)$. It follows that:

$$x = y \rightarrow L(x = y)$$

On the other hand, let us suppose that our language contains an operator B binding no variable and such that the expression $\delta B\varphi$ is a sentential formula for any nominal formula δ and any sentential formula φ. Thus, the symbol B is a mixed operator (predicate-connective) studied first by J. Łoś in 1948. It follows from (2.5) and (2.6) that:

$$x = y \rightarrow (zB\varphi(x) = zB\varphi(y))$$

In view of this logical theorem one may say that the operator B above cannot be used to formalize the notion of believing which occurs, e.g., in the true sentence:

I believe that Wittgenstein was a great philosopher.

3. *S-ontology* We assume the axiom:

(3.1) $Np = Nq \to p = q$

The connective "fact" or "positive fact" is introduced by the definition:

(3.2) $Fp = p$

The formula Fp means: it is a (positive) fact that p . The formula Np may be read: it is a negative fact that p . It follows from (3.2) that: $Fp = FFp =, \ldots, = FF \ldots Fp$ and:

(3.3) $FNp = NFp = Np$ (double equality)
(3.4) $F(p \wedge q) = Fp \wedge Fq$
(3.5) $F(p \vee q) = Fp \vee Fq$

(3.6) $\forall r(Sr \to FrDNr)$

Let σ be an arbitrary sentence. We assume the definitions:

(3.7) $1 = F\sigma \vee N\sigma$ $0 = F\sigma \wedge N\sigma$

Clearly, $0D1$. It follows from (2.7) that:

(3.8) $1 \neq 0$

Of course, we have:

(3.9) $F1$ and, consequently, $\exists p(Sp \wedge Fp)$
(3.10) $N0$ and, consequently, $\exists p(Sp \wedge Np)$

Thus, there exist (positive) facts and negative facts.

The formula (*), i.e., $\forall p \forall q(p \equiv q \to p = q)$ may be called the condition of ontological two-valuedness. To explain this terminology let us remark that theorems (3.9) and (3.10) state the existence of (positive) facts and negative facts. On the other hand, the formula (*) above is equivalent to each of the following formulae:

(1) $N(\exists p \exists q(p \neq q \wedge Fp \wedge Fq))$
(2) $N(\exists p \exists q(p \neq q \wedge Np \wedge Nq))$
(3) $Sp \to p = 1 \vee p = 0$
(4) $Fp \equiv (p = 1)$
(5) $Np \equiv (q = 0)$
(6) $\forall r \varphi(r) \equiv \varphi(1) \wedge \varphi(0)$
(7) $\exists r \varphi(r) \equiv \varphi(1) \vee \varphi(0)$

Thus, the formula (*) "means" that (1) there are exactly two situations, (2) there exists exactly one (positive) fact, 1, and exactly one negative fact, 0, and (3) the role of quantifiers binding sentential variables is not essential.[7] The formula (*) is a theorem of the system of Frege and of Leśniewski's protothetics.

Postulate: *the formula* (*) *is not an ontological theorem.*

4. *Modality* The *s*-ontology contains the sentential connectives of necessity L and of possibility M. We assume the following equalities as ontological axioms for modal connectives:

(4.1)	$NNp = p$	
(4.2)	$NLr = MNr$	$NMr = LNr$
(4.3.1)	$LLr = Lr$	$MMr = Mr$
(4.3.2)	$LMLMr = LMr$	$MLMLr = MLr$
(4.4)	$LMLr = MLr$	$MLMr = LMr$
(4.5)	$MLr = Lr$	$LMr = Mr$

The equalities above reduce so called modalities.[8] We have here 6 modalities as in Lewis' system S5: Fp, Np, Lp, Mp, NLp, NMp. The following laws of distributivity:

(4.6) $$L(p \wedge q) = Lp \wedge Lq \qquad M(p \vee q) = Mp \vee Mq$$

also serve to simplify the formulae containing modal operators. The formulae (4.6) are assumed here as axioms. We assume as ontological axioms the formulae:

(4.7) $$Lr \equiv (r = 1) \qquad Mr \equiv (r \neq 0)$$

(4.8) $$(p = q) \equiv L(p \equiv q)$$

and all formulae of the form:

(4.9) $$L\varphi$$

where the formula φ is an arbitrary logical theorem. The comprehensive axiom scheme (4.9) may be considerably simplified if we take into account that the logical theorems are suitable "axiomatized" in a logical calculus. Clearly, we have the theorems:

(4.10) $$Lp \rightarrow Fp \qquad Fp \rightarrow Mp$$

(4.11) $$N(Fp \wedge Np) = (Np \vee Fp) = (p \rightarrow q)$$ (double equality)

(4.12) $$L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq)$$

We assume as axioms the following important axiom schemes (Barcan formulae)[9]:

(4.13) $$L\forall p\varphi(p) \equiv \forall pL\varphi(p)$$

(4.14) $$M\exists p\varphi(p) \equiv \exists pM\varphi(p)$$

5. *Boolean algebra of situations* A situation p may include another situation q. Then we say also that the situation p entails the situation q or that the situation q occurs in the situation p and we write simply: pEq. Note that: $pEq = FpEFq$. The connective E corresponds to Lewis' strict implication. However, we do not consider it as an interpretation of the conditional sentences in natural language. This would be a confusion. Inclusion or entailment is a special relation between situations in the same sense as necessity and possibility are properties of situations. They are connected together as stated in the following axioms:

(5.1) $L(p \to q) = pEq = NM(Fp \wedge Nq)$ (double equality)

(5.2) $Lp = 1Ep \qquad NMp = pE0$

It is clear that only one of the connectives L, M, E need be taken as a primitive undefined symbol. We assume as ontological axioms all formulae of the form:

(5.3) $\varphi E \psi$

where φ is a sentence and the sentential formula ψ follows logically from φ. It may be shown that the axioms (4.9) and (5.3) are equivalent. Clearly, we have the theorem:

(5.4) $pEq \to (p \to q)$

The following axioms state that the universe of all situations is a Boolean algebra with the relation of inclusion E. The situations $p \vee q$, $p \wedge q$ and Np may be called: the sum and product of situations p, q and the complement of the situation p.

(5.5) pEp

(5.6) $pEq \wedge qEp \equiv (p = q)$

(5.7) $pEq \wedge qEr \to pEr$

(5.8) $(p \wedge q = p) \equiv pEq \equiv (p \vee q = q)$ (double equivalence)

(5.9) $pEq \equiv NqENp \qquad pEq \equiv (p \wedge Nq = 0)$

(5.10) $pE1 \qquad 1 = N0 \qquad 0Ep$

(5.11) $p \wedge q = q \wedge p \qquad p \vee q = q \vee p$

(5.12) $p \wedge (q \wedge r) = (p \wedge q) \wedge r \qquad p \vee (q \vee r) = (p \vee q) \vee r$

(5.13) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(5.14) $p \wedge 1 = p \qquad p \vee 0 = p$

(5.15) $p \wedge Np = 0 \qquad p \vee Np = 1$

Remark. According to the terminological convention assumed above the formula pEq is to be read: "p includes q". In customary Boolean terminology it is read: p is included in q. If we had replaced the equivalencies (4.7), (4.8) and (5.6) by corresponding equalities then the connectives L, M, E would be definable as follows: $Lp = (p = 1)$, $Mp = (p \neq 0)$, $pEq = ((p \to q) = 1)$. Let us introduce now the definition of the independence of situations. The formula pIq is to be read: the situations p, q are independent.

(5.16) $pIq = M(Fp \wedge Fp) \wedge M(Fp \wedge Nq) \wedge M(Np \wedge Fq) \wedge M(Np \wedge Nq)$

From this we infer that:

(5.17) $N(pIq) = (pENq) \vee (pEq) \vee (qEp) \vee (NpEq)$

(5.18) $N(pIp) \wedge N(pINp) \wedge N(qI1) \wedge N(qI0)$

According to Dr. Wolniewicz the Wittgenstein's principle of logical atomism consists of two statements. One of them is concerned with the independence of states of affairs and will be discussed in section 8. The second one asserts the existence of at least two independent situations. It is formulated as the following axiom:

(5.19) $$\exists q \exists q(Sp \wedge Sq \wedge pIq)$$

It follows from this that $\exists q(1 \neq p \neq 0)$. Consequently,

$$\exists q \exists q(p \equiv q \wedge p \neq q)$$

Thus, we have the negation of the formula (*) discussed in section 3.

We assume that the Boolean algebra of all situations is, in the usual terminology, atomic (5.24) and complete (6.1), (6.2). Let us introduce the following definition:

(5.20) $$Pw = Mw \wedge \forall r(Sr \rightarrow wEFr \vee wENr)$$

The formula Pw may be read: "the situation w is a point".[10] Following E. Stenius (1960) the points may be called the possible worlds. They constitute, in the terminology of *Tractatus*, logical space (*logischer Raum*). If Pw then the formulae $wEFr$, $wENr$ may be read as follows: "the situation that r occurs (does not occur) in the possible world w". It is easy to prove the following theorems:

(5.21) $$Pw \rightarrow (wENq \equiv N(wEq))$$
(5.22) $$Pw \rightarrow (wE(p \wedge q) \equiv wEp \wedge wEq)$$
(5.23) $$pEq \equiv \forall w(Pw \rightarrow (wEp \rightarrow wEq))$$

The first axiom announced above (Boolean atomicity) is represented by the formula:

(5.24) $$Mp \rightarrow \exists t(Pt \wedge tEp)$$

It may be read: "every possible situation occurs in some possible world."

If pIq then the four situations p, q, Np, Nq are different and they differ from 1 and 0. Therefore, it follows from (5.19) that there exist at least four situations r such that Mr and MNr. There exist possible situations. Therefore, the possible worlds exist, too. Clearly, if $0 \neq r \neq 1$ then $0 \neq Nr \neq 1$. Consider a situation r such that Mr and MNr. The situations r, Nr cannot occur in the same possible world. Consequently, there exist at least two possible worlds:

(5.25) $$\exists w \exists u(Pw \wedge Pu \wedge u \neq w)$$

There is, of course, the question how many possible worlds exist. The number of them depends on the number of states of affairs and determines the number of situations. This will discussed later.

6. *Boolean completeness* Let us introduce the sum operator $\bigvee$ binding sentential variables like the quantifiers. Using it we build formulae of the form $\bigvee r\, \varphi(r)$. The situation that $\bigvee r\, \varphi(r)$ is the sum (ontological alternation) of all situations r such that $\varphi(r)$. This is expressed in the following scheme of axioms (Boolean completeness):

(6.1) $$\forall p(\varphi(p) \rightarrow pE \bigvee r\,\varphi(r))$$
(6.2) $$\forall p(\varphi(p) \rightarrow pEq) \rightarrow \bigvee r\varphi(r)\,E\,q$$

The sum-operator $\bigvee$ may be defined by means of the unifier (description operator) if we assume instead of (6.1) and (6.2) the corresponding existential axioms. It is easy to prove that:

(6.3) $$\forall s(\varphi(s) \equiv s = p_1 \vee s = p_2) \rightarrow \bigvee r\varphi(r) = p_1 \vee p_2$$

One may define the product-operator $\bigwedge$ binding sentential variables which corresponds by duality to the operator $\bigvee$. We assume the definition:

(6.4) $$\bigwedge r\,\varphi(r) = N \bigvee r\,\varphi(Nr)$$

The situation that $\bigwedge r\,\varphi(r)$ is the product (ontological conjunction) of all situations r such that $\varphi(r)$. This is illustrated by the theorem:

(6.5) $$\forall s(\varphi(s) \equiv s = p_1 \vee s = p_2) \rightarrow \bigwedge r\varphi(r) = p_1 \wedge p_2$$

Of course, two schemes of axioms, being dual to (6.1) and (6.2), hold for the operator $\bigwedge$. It is easy to prove the following theorems:

(6.6) $$tE \bigwedge r\,\varphi(r) \equiv \forall r(\varphi(r) \rightarrow tEr)$$
(6.7) $$\bigvee r\,\varphi(r)\,E\,t \equiv \forall r(\varphi(r) \rightarrow rEt)$$

It follows from these theorems that if $\forall r(\varphi(r) \equiv \psi(r))$ then:

$$\bigwedge r\,\varphi(r) = \bigwedge r\,\psi(r) \quad \text{and} \quad \bigvee r\,\varphi(r) = \bigvee r\,\psi(r) \; .$$

It may be interesting to compare the operators $\bigwedge, \bigvee$ with the quantifiers $\forall$, $\exists$. Let us consider only $\bigvee$ and $\exists$. It is clear that: $\psi(q)\,E\,\exists t\,\psi(t)$. Consequently, we have the implication:

(6.8) $$\exists q(p = \psi(q)) \rightarrow p\;E\exists t\,\psi(t)$$

Suppose now that: $\forall p(\exists t(p = \psi(t)) \rightarrow pEq)$. It follows that: $\forall t(\psi(t)\,Eq)$, i.e. $L\forall t(\psi(t) \rightarrow q)$. The formula: $\forall t(\psi(t) \rightarrow q) \rightarrow (\exists t\psi(t) \rightarrow q)$ is a logical theorem. Therefore, we have: $L(\exists t\psi(t) \rightarrow q)$, i.e., $\exists t\psi(t)\,E\,q$. Thus, we have proved the formula:

(6.9) $$\forall p(\exists t(p = \psi(t)) \rightarrow pEq) \rightarrow \exists t(\psi(t)\,E\,q) \; .$$

If we now compare (6.8) and (6.9) with (6.1) and (6.2) then we see that:

(6.10) $$\exists t\psi(t) = \bigvee r(\exists t(r = \psi(t))) \; .$$

Both operators $\bigvee$ and $\exists$ are generalizations in a sense of the connective of alternation $\vee$. However, there is a sharp difference between them. We explain it in the following intuitive but not quite exact way. The formula $\exists t\psi(t)$ is the alternation of all sentences of the form $\psi(\sigma)$ where σ is any sentence. The formula $\bigvee r\psi(r)$ is the alternation of all sentences σ such that the sentence $\psi(\sigma)$ is true. The reader may compare the operators $\bigwedge$ and $\forall$ and reconstruct a theorem dual to (6.10). It follows directly from (6.1) that:

(6.11) $$\exists p(\varphi(p) \wedge wEp) \rightarrow wE \bigvee r\varphi(r)$$

We prove that the implication (6.11) may be reversed if the situation that w is a possible world. Suppose that:

1. Pw
2. $wE \bigvee r\varphi(r)$
3. $\forall p(\varphi(p) \rightarrow N(wEp))$

According to (5.20) we infer that: $\forall p(\varphi(p) \rightarrow wENp)$. Consequently, $\forall p(\varphi(p) \rightarrow pENw)$. Using (6.2) we obtain: $\bigvee r\varphi(r) E Nw$. Therefore, by (3): $wENw$, i.e. $w = 0$, contrary to (1). Thus, we have proved the theorem:

(6.12) $$Pw \rightarrow (wE \bigvee r\varphi(r) \equiv \exists p(\varphi(p) \wedge wEp)$$

One may read this theorem as follows: if w is any possible world then the situation that $\bigvee r\varphi(r)$ occurs in w if and only if some situation p such that $\varphi(p)$ occurs in w. Using (6.12), (5.23) and (5.6) one obtains easily the theorem:

(6.13) $$\bigvee r\psi(r) \vee \bigvee r\varphi(r) = \bigvee r(\psi(r) \vee \varphi(r))$$

The reader may formulate three theorems which are dual to (6.11), (6.12) and (6.13). Take notice of the last case!

We conclude this section with the theorem:

(6.14) $$p = \bigvee w(Pw \wedge wEp)$$

which may be easily obtained from (5.23), (6.1) and (6.2). It states that every situation is the sum (ontological alternation) of all possible worlds in which it occurs. The situation $\bigvee w(Pw \wedge wEp)$ is composed of points in the logical space. One may say that every situation is a place (*Ort*) in the logical space.

7. *The real world* Let us consider now three situations s_0, s_1, and $\dot{w}$ which are defined as follows:

(7.1) $$s_0 = \forall rFr \qquad s_1 = \exists rFr$$

(7.2) $$\dot{w} = \bigwedge rFr$$

Clearly, $Fs_1 \wedge Ns_0$. This is a logical theorem. It follows that:

(7.3) $$s_1 = 1 \wedge s_0 = 0$$

Using the theorems dual to (6.1) and (6.2) we obtain the equality:

(7.4) $$\bigwedge rFr = \bigwedge r(Fr \wedge r \neq 1)$$

Starting now with the theorem dual to (6.1) we obtain two formulae:

(7.5) $$Fp \rightarrow \dot{w}EFp \qquad Np \rightarrow \dot{w}ENp$$

It follows from this that: $\forall p(Sp \rightarrow \dot{w}EFp \vee \dot{w}ENp)$. Thus, we have proved the remarkable theorem:

(7.6) $$\dot{w} = 0 \vee P\dot{w}$$

It means that either the situation $\dot{w}$ is impossible or it is a point (possible world). The formula:

$$M\dot{w} \tag{7.7}$$

is assumed as an axiom. It states that the product (ontological conjunction) of all facts is a possible situation. Consequently, the situation $\dot{w}$ is a point in the logical space and will be called the real world.

Now, using (5.21), we infer from (7.5) the following theorem:

$$Fp \equiv \dot{w}Ep \tag{7.8}$$

Any situation is a (positive) fact if and only if it occurs in the real world. In another words, the real world includes a situation if and only if it is a (positive) fact.

From (7.8), (5.21), (5.22) and (6.12) we obtain the theorems:

$$F\dot{w} \tag{7.9}$$
$$Np \equiv \dot{w}ENp \tag{7.10}$$
$$F(p \wedge q) \equiv \dot{w}EFp \wedge \dot{w}EFq \tag{7.11}$$
$$F(\bigvee r\varphi(r)) \equiv \exists p(\varphi(p) \wedge Fp) \tag{7.12}$$

(Remark, added November 1967.) As observed by B. Wolniewicz and P. T. Geach, Wittgenstein would not say that $F\sigma$ where σ is a logical theorem. This means, in my opinion, that Wittgenstein's notion of a fact may be expressed more adequately by the connective $\dot{F}$ defined as follows: $\dot{F}p = Fp \wedge p \neq 1$. However, observe that the real world may be defined using F and $\dot{F}$ as well. See (7.4). In many theorems we may replace F by $\dot{F}$. Also, we have: $\dot{F}\dot{F}p = \dot{F}p$. The difference between F and $\dot{F}$ is not very deep. Figuratively speaking, it is like that between nonnegative integers and positive integers. The theorem (7.4) corresponds to the equality: $0 + 1 + 2 = 1 + 2$.

8. *States of affairs* The notion of state of affairs or atomic situation is represented in the system of ontology by the sentential connective SA. The formula SAp is to be read: the situation that p is a state of affairs. Wittgenstein uses the notion of (positive) atomic fact (the connective FA) but he does not use the notion of a negative atomic fact (the connective NA). We introduce both in the following definitions:

$$FAp = SAp \wedge Fp \tag{8.1}$$
$$NAp = SAp \wedge Np \tag{8.2}$$

States of affairs are neither necessary nor impossible. We formulate this as an axiom:

$$SAp \rightarrow Mp \wedge MNp \tag{8.3}$$

It follows from this (a formula of Wolniewicz):

$$SAp \rightarrow (Fp \equiv (p \wedge MNp)) \tag{8.4}$$

There exists only one necessary fact ($LF1$) and only one impossible fact ($NM0$). The atomic facts (positive and negative) are contingent situations. We assume now the axiom:

(8.5) $$\bigvee r\, SAr = 1$$

This axiom is equivalent to each of two following formulae:

(8.6) $$\bigwedge r\, SA(Nr) = 0 \qquad L(\bigvee r\; SAr)$$

It follows from (6.12) and (8.5) that in every possible world (including the real world) there occur at least one state of affairs, i.e.:

(8.7) $$Pw \rightarrow \exists p\,(SAp \wedge wEp)$$

Points exist (5.25). Therefore, states of affairs exist too:

(8.8) $$\exists p\; SAp$$

The existence of situations is a logical theorem: $\exists p\, Sp$.

The difference between different possible worlds consists in the occurrence of different states of affairs in them. We see from (8.6) that the product of all complements of states of affairs equals the impossible situation 0. One may consider also an arbitrary product of certain states of affairs and of complements of all other ones. Every such product will be called a combination of states of affairs. In particular, the product of all states of affairs is a combination of them. We assume that every positive combination of states of affairs, i.e. every combination except that mentioned in (8.6), is not equal zero. Moreover, we assume that every positive combination is a possible world (8.10). To make it precise, let us suppose that we have a sentential formula $\varphi(r)$ such that $\exists r(SAr \wedge \varphi(r))$. The situation that $\bigwedge r(SAr \wedge \varphi(r))$ is the product of all states of affairs r such that $\varphi(r)$. On the other hand, the situation that $\bigwedge r(SA(Nr) \wedge N\varphi(Nr))$ is the product of all complements of states of affairs q such that $N\varphi(q)$. To see this it is enough to check the equality: $\bigwedge r(SA(Nr) \wedge N\varphi(Nr)) = \bigwedge r(\exists q(r = Nq \wedge SAq \wedge N\varphi(q)))$.

One may easily verify the following equality (compare the dual equality to (6.13)): $\bigwedge r(SAr \wedge \varphi(r)) \wedge \bigwedge r(SA(Nr) \wedge N\varphi(Nr)) = \bigwedge r((SAr \wedge \varphi(r)) \vee (SA(Nr) \wedge N\varphi(Nr)))$. Thus, the product $\bigwedge r((SAr \wedge \varphi(r)) \vee (SA(Nr) \wedge N\varphi(Nr)))$ is a combination of states of affairs. We assume the definition:

(8.9) $$\supset r(\psi(r), \varphi(r)) = \bigwedge r((\psi(r) \wedge \varphi(r)) \vee (\psi(Nr) \wedge N\varphi(Nr)))$$

Here, a new operator $\supset$ binding sentential variables is introduced. It follows from above that for any formula $\varphi(r)$ the situation that $\supset r(SAr, \varphi(r))$ is a combination of states of affairs. It is a positive combination if $\exists r(SAr \wedge \varphi(r))$. As announced above, we assume the following scheme of axioms (for any arbitrary formula $\varphi(r)$):

(8.10) $$\exists r(SAr \wedge \varphi(r)) \rightarrow P(\supset r(SAr, \varphi(r)))$$

This scheme of formulae may be read: to every positive combination of states of affairs corresponds (in the sense of identity) a uniquely determined possible world. We assume that this correspondence is biunique. In other words, two combinations of states of affairs which differ in some of their factors are different. It is expressed by the scheme of axioms:[11]

(8.11) $\supset r(SAr,\ \varphi(r)) = \supset r(SAr,\ \psi(r)) \rightarrow \forall r(SAr \rightarrow (\varphi(r) \equiv \psi(r)))$

Suppose now that there exists a situation p such that SAp and $SA(Np)$. Then, it follows from the theorem dual to (6.1) that the product of all states of affairs, i.e., $\supset r(SAr, Sr)$ is equal to the impossible situation 0, contrary to (8.10). Therefore:

(8.12) $SAp \rightarrow N(SA(Np)) \qquad SA(Np) \rightarrow NSAp$

It follows from (6.6) that

(8.13) $tE \supset r(SAr,\ \varphi(r)) \equiv \forall p(SAp \wedge \varphi(p) \rightarrow tEp) \wedge \forall q(SA(Nq) \wedge N\varphi(Nq) \rightarrow tEq)$

Let us introduce now the definition:

(8.14) $sa(t) = \supset r(SAr,\ tEr)$

Clearly, the combination $sa(t)$ is the product of all states of affairs occurring in t and of all complements of those states of affairs which do not occur in t. If t is a possible world then the situation $sa(t)$ may be called the combination of states of affairs in the possible world t. In view of (8.7) this combination is a positive one. Therefore, if t is a point then $sa(t)$ is also a point. It may be inferred from (8.13) that they are identical. Thus, we have the theorem:

(8.15) $Pt \rightarrow sa(t) = t$

Every possible world is a positive combination of states of affairs. Therefore, the biunique correspondence (in the sense of identity) between positive combinations of states of affairs and possible worlds is concerned not only with all positive combinations but also with all possible worlds.[12] We have seen (6.14) that every situation is the sum (ontological alternation) of certain possible worlds. This includes also the impossible situation 0 although it occurs in no possible world. On the other hand, every possible world is a product of certain states of affairs and of complements of other states of affairs. In view of this, we say that the states of affairs are generators of the Boolean algebra of all situations, i.e., the Boolean algebra of all situations is generated by the states of affairs. Every situation may be obtained from the states of affairs by means of the generalized sum-operation $\bigvee$, generalized product-operation $\bigwedge$ and complement operation N.

Let us now discuss the problem of the number of states of affairs, of possible worlds and of situations. It is clear that

(1) if there exist exactly n states of affairs then there exist exactly $2^n - 1$ positive combinations of them, i.e., there exist exactly $2^n - 1$ possible worlds. On the other hand,

(2) if the (atomic and complete) Boolean algebra of all situations contains exactly m points (possible worlds) then there exist exactly 2^m situations.

In the infinite case, n and m are infinite cardinal numbers and we have only to replace $2^n - 1$ by 2^n in (1) above. The number of states of affairs is determined in some degree by the postulate of their independence. This is the second component of the principle of logical atomism in *Tractatus*; compare (5.19). Notice that the formula (5.19) follows from our assumption (8.16) below. It doesn't suffice to assume that any two different states of affairs are independent: $\forall p \forall q(SAp \wedge SAq \wedge p \neq q \rightarrow pIq)$. To formulate the independence of states of affairs we must introduce the notions of independence of three, four, . . . , states of affairs. Let us write $I_2(p,q)$ instead of pIq. Subsequently, in analogy to (5.16), we may define for each $k = 3, 4, \ldots$, the k-ary connective I_k such that the formula $I_k(p_1, p_2, \ldots, p_k)$ means: the situations $p_1, p_2, \ldots, p_k$ are independent. For example, we define the formula $I_3(p_1, p_2, p_3)$ using a formula which is the product of 8 factors:

$$M(Fp_1 \wedge Fp_2 \wedge Fp_3) \wedge M(Fp_1 \wedge Fp_2 \wedge Np_3) \wedge, \ldots, \wedge M(Np_1 \wedge Np_2 \wedge Fp_3) \wedge M(Np_1 \wedge Np_2 \wedge Np_3)$$

Now, we assume as axioms all formulae of the following kind:

$$(8.16) \qquad (SAp_1 \wedge, \ldots, \wedge SAp_k \wedge p_1 \neq p_2 \wedge, \ldots, \wedge p_i \neq p_j \wedge, \ldots, \wedge p_{k-1} \neq p_k) \rightarrow I_k(p_1, \ldots, p_k)$$

They mean that each k different states of affairs are independent; $i < j < k = 2, 3, 4, \ldots$. Suppose that there exist finitely many, e.g. k, states of affairs: $p_1, p_2, \ldots, p_k$. We infer from (8.16) that the situation $Np_1 \wedge Np_2 \wedge, \ldots \wedge Np_k$ is possible. This contradicts (8.6). Thus:

(8.17) *There exist infinitely many states of affairs.*

On the other hand, all the axioms (8.16) may be inferred from (8.17) and (8.10). Therefore, one may consider the independence of states of affairs as equivalent to the existence of infinitely many of them. It follows from (1) and (2) above that the number of possible worlds is much greater than the number of states of affairs and the number of all situations is much greater than the number of possible worlds. This holds also in our infinite case (8.17). The number of all situations is enormously large. This applies also to the number of all (positive) facts because in our infinite case the cardinal number of all situations occurring in a given possible world is equal to the number of all situations.[13]

9. *Configurations of objects* The link between the *s*-ontology and *o*-ontology is given by the following thesis of *Tractatus*:

(1) *The states of affairs are configurations of objects*.

It is an important but very difficult point in the philosophy of Wittgenstein. I cannot give here an ultimate and complete explication of thesis (1). The considerations in this section must be seen as preliminary remarks only. Firstly, the main difficulty consists in the precise presentation of the principles of the o-ontology. The o-ontology has its own serious problems and will be not presented here in a definite form. On the other hand, the notion of configuration of objects and thesis (1) give rise to certain special problems which cannot be solved here. I will propose a definition of the notion of configuration of objects (9.0). The main idea of this definition is quite right, in my opinion. It will be applied to formulate thesis (1). However, it will be shown that the formula (9.1) which corresponds to (1) is not adequate.

Let us reflect for a moment on the notion of configuration. Here, I consider the configurations of finite number of objects. When speaking about configurations of certain objects $x_1, \ldots, x_n$ we take usually into account many possible configurations of these objects. In other words, there may be many configurations of the given objects $x_1, \ldots, x_n$. One may infer from this that the notion of configuration must contain a hidden parameter which changes from one configuration of $x_1, \ldots, x_n$ to another one. To reveal this parameter we may use the phrase:

(2) the R-configuration of objects $x_1, \ldots, x_n$

where the letter R is a nominal variable. To explain the role of the parameter R we suppose that every configuration of objects $x_1, \ldots, x_n$ consists in a certain n-ary relation holding between the objects $x_1, \ldots, x_n$. In particular, it is quite natural to use phrase (2) for the situation that the n-ary relation R holds between $x_1, \ldots, x_n$. Therefore, we assume that the phrase (2) means:

(3) the situation that n-ary relation R holds for $x_1, \ldots, x_n$

Instead of (3) one may use the sentential expression:

(4) the n-ary relation R holds for $x_1, \ldots, x_n$

or the symbolic sentential formula:

(5) $R * x_1, \ldots, x_n$

Thus we have the definition:

(9.0) the R-configuration of $x_1, \ldots, x_n = R * x_1, \ldots, x_n$

It allows us to dispense with phrases of the form (2) and to formulate thesis (1) as follows:

(9.1) $$SAp \equiv \exists R \exists x_1 \ldots \exists x_n (p = R * x_1, \ldots, x_n)$$

The formula (9.1) should be assumed as an ontological axiom. It might read: a situation p is a state of affairs if and only if p consists in a certain (given) relation holding between some (given) objects. Note that formula (9.1) is not quite precise because first of all the number of existential quantifiers occurring in it is indefinite. There are several subtle points connected with the formula (9.1). Some of them will be mentioned below.

It follows immediately from (9.1) that:

(9.2) $$SA(R * x_1, \ldots, x_n)$$

for every n-ary relation R and objects $x_1, \ldots, x_n$.

(A) The identity on the right hand side of (9.1) must not be replaced by equivalence. For, suppose that:

(6) $$(p \equiv R * x_1, \ldots, x_n) \rightarrow SAp$$

If $F(R * x_1, \ldots, x_n)$ then $R * x_1, \ldots, x_n \equiv 1$. Consequently, $SA1$, contrary to (8.3). On the other hand, one may see that the formula:

(7) $$p \equiv q \rightarrow SAp \equiv SAq$$

does not hold; compare the scheme (**) in the section 2.

(B) Let R be an n-ary relation and S be an m-ary relation. Consider arbitrary objects $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}$. If $n \neq m$ then the states of affairs:

(8) $$R * x_1, \ldots, x_n \qquad S * x_{n+1}, \ldots, x_{n+m}$$

are not only different, their structures are also different. Suppose now that $n = m$. One may say that the states of affairs:

(9) $$R * x_1, \ldots, x_n \qquad S * x_{n+1}, \ldots, x_{n+m}$$

are of the same structure in a weak sense. A stronger version of this notion may be introduced in the case of isomorphic relations. Let C be a one-to-one correspondence between the fields of the relations R and S. It assigns to the elements x, y of the field of R certain objects $C * x$ and $C * y$ in the field of S.[14] If we also have:

(10) $$x \neq y \rightarrow C * x \neq C * y$$

(11) $$R * x_1, \ldots, x_n \equiv S * (C * x_1), \ldots, (C * x_n)$$

then the correspondence C is called an isomorphism between R and S. We say that the states of affairs (9) are of the same structure in the strong sense and with respect to the correspondence C if and only if C is an isomorphism between R and S and $x_{n+k} = C * x_k$ for $k = 1, \ldots, n$.

(C) It is clear that the states of affairs (9) are identical if $R = S$ and $x_k = x_{n+k}$ for $k = 1, \ldots, n$. Does the inverse implication hold? In other words, is it true that:

(12) $$R * x_1, \ldots, x_n = S * x_{n+1}, \ldots, x_{2n} \rightarrow R = S \wedge x_1 = x_{n+1} \wedge, \ldots, \wedge x_n = x_{2n}$$

(D) Let *Id* be the identity relation. This means that for all elements x, y of some class of objects (which does not need to be specified): $(Id * x,y) = (x = y)$. In view of (2.2), (4.8), (4.9) and (4.12) we infer that $L(Id * x,x)$. Consequently, according to (8.3) the situation that $Id * x,x$ is not a state of affairs which contradicts (9.2). On the other hand, suppose that the relation S is the complement of the relation R in the sense for all elements x, y of some class of objects (which does not need to be specified): $(S * x,y) = N(R * x,y)$. We infer from (9.2) that $SA(S * x,y)$, i.e., $SA(N(R * x,y))$. It follows from (8.12) that $N(SA(R * x,y))$, contrary to (9.2). Thus we see that the formula (9.2) cannot hold in full generality. Therefore, the formula (9.1) must also be suitably modified. The problem is: how to restrict the range of nominal variables (R, $x_1, \ldots, x_n$) bound by the existential quantifiers in (9.1).

One may conjecture that the formula (9.2) holds if the objects $x_1, \ldots, x_n$ are individuals, i.e., non-abstract objects (*Urelemente*), and the relation R is a Wittgensteinean one. I think that Wittgensten believed in the existence of a certain absolute class of relations between individuals which generates the states of affairs according to (9.2). Of course, only *Der liebe Gott* could define the class of Wittgensteinean relations. However, I think, it is possible to formulate certain necessary conditions which must be satisfied by Wittgensteinean relations. Our considerations above (point D) suggest two following conditions:

(1) No Wittgensteinean relation is an invariant under all permutations of individuals. In particular, the relation of identity is not a Wittgensteinean one.

(2) The Wittgensteinean relations are mutually independent. Thus, for example, the complement of a Wittgensteinean relation is not a Wittgensteinean relation.

Both conditions call for further explanation. Especially, the condition of independence is somewhat complicated because the Wittgensteinean relations may be unary, binary, . . . , n-ary, ... However, I do not enter into these details here.

Remark, suggested by Dr. Wolniewicz, December 1967. Anyway we define states of affairs there exist infinitely many of objects. If there were finitely many objects then there would exist only finitely many configurations of them, i.e., finitely many states of affairs, contrary to (8.17). Note, however, that the existence of infinitely many objects is not equivalent to the existence of an infinite class of objects.

10. *O-ontology*. Traditional opposition between realism and nominalism belongs to the field of ontology of objects and is concerned with the problem of existence of abstract objects. We assume here (!) the realistic point of view. Nominalism is followed by poor and mostly negative conclusions. In particular, nominalism (e.g., reism of T. Kotarbiński) seems to be unable to reconstruct classical mathematics. On the other hand, it is known that classical mathematics may be reconstructed within realistic ontology (e.g., set theory). Starting from the realistic point of view we introduce the notion of an abstract object as a primitive one. We assume also certain specific axioms concerning the existence of abstract objects. Realistic formal ontology does not formulate any specific assumptions concerning individuals.

There are many contemporary versions of realistic ontology (e.g., the theory of types, the set theory of Zermelo-Fraenkel or of von Neumann-Bernays-Gödel, the systems NF and ML of Quine). Moreover, most of them offer certain difficult special problems which seem to be solvable by decision only. Under such circumstances I do not feel inclined to present here any elaborated system of o-ontology. When speaking about realistic o-ontology I do not intend either to choose a particular system known in mathematical logic or to construct a new one. I will restrict my considerations to the so called principle of abstraction (Cantor, Frege) which is the kernel of every realistic ontology. The principle of abstraction formulated below is concerned with the abstract objects derived from objects and situations as well. I will discuss also a classification of abstract objects and some problems connected with the principle of extensionality.

There exists, of course, the question as to whether the ontology of objects contained in the *Tractatus* is realistic or nominalistic. Wittgenstein gives no direct answer to this question. However, I think that the *Tractatus* must be interpreted in a realistic way. This follows from the interpretation of configurations of objects given in the section 9. The revealed parameter R represents relations which, certainly, are abstract objects. There is also an indirect historical argument. Clearly, Wittgenstein was influenced by Russell's theory of types. Every strict formulation of the theory of types (Church's simple theory of types, the so-called ontology of St. Leśniewski, Tarski's general theory of classes) proves to be a restricted version of set theory and, thereby, a version of realistic ontology of objects. In former times the realism of the theory of types was not obvious because, mostly, of certain shortcomings in *Principia Mathematica*. The theory of types is a many sorted theory. It uses essentially many sorts of nominal, i.e., nonsentential variables. This peculiar syntatic feature of the theory of types (formed sometimes in the style of the so-called higher-order-system) gave rise to certain misinterpretations of this theory in the past. Now, only simpleminded people believe in the nominalistic character of the theory of types.

Certainly, the realistic ontology we are speaking about is not the theory of types. Our formalized language L_0 contains just one kind of nominal, i.e., non-sentential variable. Therefore, the principle of abstraction formulated below states explicitly the existence of abstract objects.

11. *Principle of abstraction* The formulae which are neither sentences nor names contain some free variables. For simplicity, we consider only formulae which contain as free exactly the nominal variables $x_1, \ldots, x_m$

and sentential variables $p_1, \ldots, p_n$. We assume that $m,n = 0, 1, 2, \ldots$ with the exception of the case when $m + n = 0$. There formulae will be called of rank (m,n). Sentential or nominal formulae of rank (m,n) will be called formulae of rank $(0;m,n)$ or $(1;m,n)$, respectively.

One might use the following suggestive notation:

$$\chi(x_1, \ldots, x_m, p_1, \ldots, p_n)$$

for arbitrary formulae of rank (m,n). But I will not use this notation.

When we use (!) certain formulae φ and ψ of rank $(0;m,n)$ and $(1;m,n)$, respectively, then we think about indeterminate object ψ and situation φ which depend on indeterminate objects $x_1, \ldots, x_m$ and situations $p_1, \ldots, p_n$. In other words, to any objects $x_1, \ldots, x_m$ and situations $p_1, \ldots, p_n$ there correspond an object ψ and situation φ. This correspondence is univocal. Thus, one may say that every meaningful formula of rank (m,n) determines an univocal correspondence, i.e., assignment. Indeed, the formula ψ of rank $(1;m,n)$ assigns to arbitrary objects $x_1, \ldots, x_m$ and situations $p_1, \ldots, p_n$ the corresponding object ψ. On the other hand, the formula φ of rank $(0;m,n)$ assigns to arbitrary objects $x_1, \ldots, x_m$ and situations $p_1, \ldots, p_n$ the corresponding situation φ.

Certainly, the assignments determined by the formulae of some rank are abstract objects. One may assume that every abstract object is an assignment. This does not mean that every abstract object is determined by some formula. The formulae only refer in an indirect way to some abstract objects. The existence of abstract objects determined by the formulae of any rank is stated in the principle of abstraction.

(Remark, added November 1967.) The existence of abstract objects is meant here in the sense of the existential quantifier binding nominal variables. This applies to the formulation of PA_0 and PA_1, below.

When speaking about assignments it may be useful to introduce a suitable notation. Let C be an assignment. The symbolic formula:

$$(*) \qquad C * x_1, \ldots, x_m, p_1, \ldots, p_n$$

will be used for the object or situation which is assigned by C to the objects $x_1, \ldots, x_m$ and situations $p_1, \ldots, p_n$. It may be read:

the value of C at $x_1, \ldots, x_m, p_1, \ldots, p_n$

The value of an assignment C is an object or situation. Strictly speaking there are exactly two cases:

Case 0; for every $x_1, \ldots, x_m, p_1, \ldots, p_n$ the value (∗) of C is a situation.

Case 1; for every $x_1, \ldots, x_m, p_1, \ldots, p_n$ the value (∗) of C is an object.

The assignment C is called of kind (0) or (1) correspondingly to the case 0 or 1, above. Consequently, formula (∗) is either a sentential formula (case 0) or nominal formula (case 1). Therefore, the asterisk ∗ is a syntactically ambiguous symbol. It is, in general, an operator (not binding any variable) which forms a formula (sentential or nominal)

$$\psi_0 * \psi_1, \ldots, \psi_m, \varphi_1, \ldots, \varphi_n$$

together with $m + 1$ nominal formulae $\psi_0, \psi_1, \ldots, \psi_m$ and n sentential formulae $\varphi_1, \ldots, \varphi_n$ for any natural numbers m, n such that $m + n = 0$. If $n = 0$ then the asterisk $*$ is either an $(m + 1)$-ary predicate (case 0) or an $(m + 1)$-ary functor (case 1). I hope, the high syntactic ambiguity of the asterisk $*$ will not give rise to any confusion.

The principle of abstraction is comprised of two components. Both are schemes of statements corresponding to nominal and sentential formulae of any rank.

Principle of abstraction PA_0. Given a formula φ of rank $(0;m,n)$ there exists an abstract object A being an assignment of kind (0) such that for all $x_1, \ldots, x_m, p_1, \ldots, p_n$:

(11.0) $$(A * x_1, \ldots, x_m, p_1, \ldots, p_n) = \varphi$$

Principle of abstraction PA_1. Given a formula ψ of rank $(1;m,n)$ there exists an abstract object B being an assignment of kind (1) such that for all $x_1, \ldots, x_m, p_1, \ldots, p_n$:

(11.1) $$(B * x_1, \ldots, x_m, p_1, \ldots, p_n) = \psi$$

Observe that the symbol of identity is either a binary connective in (11.1) or a binary predicate in (11.1).

(Remark, added November 1967.) The principles PA_0 and PA_1 may be formulated in a more general way. One has only to allow the formulae φ and/or ψ to contain some additional free variables.

It is well known that the general principle of abstraction is a contradictory scheme. It gives rise to the antinomies. Therefore, one cannot assume the principle of abstraction in full generality, i.e., for all formulae. The general principle of abstraction must be limited in some way. This is the principal problem in the foundations of o-ontology. It may be resolved in many ways. But I will not discuss this point here. Instead, I will consider shortly the vicious element in the principle of abstraction. The abstract objects, i.e. assignments, are objects. Therefore, we may construct the formula:

$$x * x$$

which will be called the diagonal formula. It contains only one free variable. Observe that the diagonal formula is either a sentential (case 0) or a nominal (case 1) formula.

Let us consider the case of a nominal diagonal formula. Suppose that there exists a nominal formula $\chi(z)$ which contains exactly one free variable z and that for all objects z:

$$\chi(z) \neq z$$

According to PA_1 there exists an assignment B of kind (1) such that for all objects x: $(B * x) = \chi(x * x)$. It follows that $B * B = \chi(B * B)$ which is impossible. This reasoning shows that the principle PA_1 is incompatible with the existence of the nominal formula metioned above.

In the case of sentential diagonal formula the reasoning is quite analogous. Observe, however, that for every situation p:

$$Np \neq p$$

It follows according to the principle of abstraction PA_0 that there exists an assignment A of kind (0) such that for all objects x: $(A * x) = N(x * x)$. Consequently, $A * A = N(A * A)$ which is a contradiction (Russell's antinomy).

12. *Classification of abstract objects* We have divided abstract objects into two kinds. We may introduce a more precise division into the realm of abstract objects. It corresponds exactly to the classification of formulae according to their ranks. We say that the assignment determined by some formula of rank $(i;m,n)$ according to the principle of abstraction is of kind $(i;m,n)$ where $i = 0, 1$ and $m, n = 0, 1, 2 \ldots$ (the case $m = n = 0$ being excluded). Consequently, the formulation of the principle of abstraction (PA_0, PA_1) given in the preceding section must be suitable precise. We have to put the expression "kind $(i;m,n)$" instead of the expression "kind (i)" where $i = 0,1$. Strictly speaking, we have infinitely many notions of abstract objects of some kind. They are unary predicates $\mathrm{Abs}_i^{m,n}$. The notation $\mathrm{Abs}_i^{m,n}(x)$ is to be read:

x is an abstract object of kind $(i;m,n)$

On the other hand, we have the general notion of abstract object given by the unary predicate Abs. If we wish to state that our classification of abstract objects is exhaustive then we meet a serious difficulty. Indeed, we have to assert the general statement of the form:

(1) $$\forall x(\mathrm{Abs}(x) \rightarrow \ldots)$$

where the three point blank represents an infinite alternation of all formulae of the form: $\mathrm{Abs}_i^{m,n}(x)$. Certainly, this alternation is somewhat too long for our purpose. This is the point where the problem of reduction of abstract objects arises. It will be discussed later. The abstract objects may be grouped into three broad kinds. A terminology stemming from arithmetic may be applied to them. An abstract object is called (a) *real*, (b) *imaginary* and (c) *complex* if and only if it is of the following kind, respectively:

(a) $(i;m,0)$ where $m \neq 0$
(b) $(i;0,n)$ where $n \neq 0$
(c) $(i;m,n)$ where $m \neq 0 \neq n$

The real abstract objects are intensively studied in set theory. The mathematical set theory may be considered as a realistic ontology of objects but limited to real abstract objects. Indeed, the abstract objects of kind $(0;m,0)$ are usually called m-ary relations (between objects); compare

(4) and (5) in section 9. Especially, the abstract objects of kind (0;1,0) are identical with classes or sets (of objects). In this case the formula $A * x$ may be read: object x is an element of the class A. On the other hand, abstract objects of kind $(1;m,0)$ are usually called functions or operations of m arguments (on objects); for the case $m = 1$ compare (10) and (11) in section 9.

The imaginary abstract objects of kind $(0;0,n)$ may be called n-ary relations between situations or classes of situations, in the special case when $n = 1$. These abstract objects are taken into account by Leśniewski in his protothetics. However, this theory is an oversimplification because it states that there exist exactly two situations; compare section 3. Consequently, protothetics distinguishes exactly 2^{2^n} abstract objects of the kind $(0;0,n)$. There are also complex abstract objects. For example, formulae like the one (xBp) mentioned at the end of section 2, determine abstract objects of kind (0;1,1).

Now, I intend to sketch how to solve the problem of reduction of abstract objects. First, let us consider real abstract objects. Thus, we are within set theory. It is known that von Neumann (1927) has shown that all real abstract objects may be reduced to those of kind (1;1,0), i.e., functions of one argument. In other words, von Neumann introduced the predicate $\mathrm{Abs}_1^{1,0}$ as a primitive notion and then, defined all remaining predicates $\mathrm{Abs}_i^{m,0}$. The von Neumann system is not popular with mathematicians and philosophers, however. The customary systems of set theory reduce real abstract objects to those of kind (0;1,0), i.e. classes or sets. This reduction is carried over as follows. According to Peano's definition (1911) functions of m arguments may be identified with $(m + 1)$-ary relations which satisfy the condition of univocality:

(2) $$R * x_1, \ldots, x_m, y \wedge R * x_1, \ldots, x_m, z \rightarrow y = z$$

Subsequently, a device of Wiener-Kuratowski is to be used. We define ordered pairs and, in general, ordered k-tuples $(k \geqslant 2)$ of objects. They are defined as certain classes of objects, i.e., objects of kind (0;1,0). Finally, the k-ary relations are identified with classes of ordered k-tuples. Thus, we may introduce the predicate $\mathrm{Abs}_0^{1,0}$ as a primitive notion and, then, define all remaining predicates $\mathrm{Abs}_i^{m,0}$.

It is easy to realize that the reduction procedure sketched above may be applied to imaginary abstract objects, i.e., abstract objects of kind $(i;0,n)$ where $n \neq 0$. The only difference consists in the definition of ordered pairs and, in general, k-tuples $(k \geqslant 2)$ of situations. There are, of course, certain abstract objects. It results that the abstract objects of kind $(0;0,n)$ where $n \geqslant 2$, i.e., relations between situations reduce to certain classes of objects, i.e., abstract objects of kind (0;1,0). On the other hand, abstract objects of kind $(1;0,n)$ where $n \geqslant 2$ reduce to those of kind (1;1,0) and, finally, to classes of abstract objects, i.e., abstract objects of kind (0;1,0). Our reduction procedure does not affect abstract objects of kind (0;0,1) and (1;0,1). All remaining imaginary abstract objects reduce to certain real abstract objects.

The reduction procedure which uses the notion of ordered k-tuples of objects or of situations may be applied also to complex abstract objects. Abstract objects of kind $(0,m,n)$ where $m \geqslant 1$ and $n \geqslant 2$ reduce to binary relations between objects and, finally, to classes of objects. On the other hand, abstract objects of kind $(1;m,n)$ where $m \geqslant 1$ and $n \geqslant 2$ reduce to functions of two arguments (binary operations on objects) and, finally, to classes of objects. The complex abstract objects of kind (0;1,1) or (1;1,1) remain unreduced. All other complex abstract objects reduce to them or to certain real abstract objects.

The reduction procedure sketched above allows one to draw the following conclusion. It is possible to introduce the predicates:

$$\mathrm{Abs}_0^{1,0},\ \mathrm{Abs}_0^{0,1},\ \mathrm{Abs}_1^{0,1},\ \mathrm{Abs}_0^{1,1},\ \mathrm{Abs}_1^{1,1}$$

as primitive notions and, subsequently, to define all other predicates $\mathrm{Abs}_i^{m,n}$. It follows that we can now formulate and, also, assume as an axiom, the statement that our classification of abstract objects is exhaustive. It is enough to put the alternation:

(3) $$\mathrm{Abs}_0^{1,0}(x) \vee \mathrm{Abs}_0^{0,1}(x) \vee \mathrm{Abs}_1^{0,1}(x) \vee \mathrm{Abs}_0^{1,1}(x) \vee \mathrm{Abs}_1^{1,1}(x)$$

in the three point blank in (1).

13. *The principle of extensionality* The reduction of abstract objects may seem to be guided by the philosophical idea known as Occam's razor. This idea seems also to be the source of the principle of extensionality.

The general problem of extensionality arises already in connection with the principle of abstraction. The last one states, roughly speaking, the existence of abstract objects determined by the formulae. It is not excluded thereby that there may exist a formula which determines two different assignments. Occam's razor suggests excluding this possibility and to formulate a stronger version of the principle of abstraction according to which every formula determines exactly one abstract object.

It is not necessary to formulate here the stronger version of the principle of abstraction. The reader will easily verify that it follows from the usual principle (PA_0, PA_1) and the principle of extensionaltiy PE formulated below. It will be enough to mention here that the stronger version of the principle of abstraction serves as a principle of defining abstract objects. It is used in the reduction procedure to define at least the k-tuples of objects or situations. In general, the following symbol:

$$\{x_1, \ldots, x_m, p_1, \ldots, p_n | \chi\}$$

may be used to denote the only assignment determined in the stronger version of the principle of abstraction by the formula χ of rank (m,n).

(Remark, added November 1967.) Given a situation q, we define: $\{q\} = \{p_1 | p_1 = q\}$. Clearly, $\{q\}$ is the unit class such that the situation q is the only element of it. Thus, to every situation q there corresponds an abstract object $\{q\}$. This correspondence is biunique (one-to-one). Therefore, instead of studying situations and their properties we may investigate the corresponding abstract objects. Also, note that the states of affairs (compare (5) and (9.2) in section 9) may be put into one-to-one correspondence with k-tuples $<R, x_1, \ldots x_n>$ of objects, $k = n + 1$.

The principle of extensionality PE. Let A and B be any abstract objects of the same kind, say $(i;m,n)$. If the equality:

(1) $$A * x_1, \ldots x_m, p_1, \ldots p_n = B * x_1, \ldots, x_m, p_1, \ldots, p_n$$

holds for all $x_1, \ldots, x_m, p_1, \ldots, p_n$ then $A = B$.

The principle PE allows one to strengthen the principle of abstraction. Therefore, one may say that the principle PE follows in a sense from Occam's principle. On the other hand, PE serves as a necessary and sufficient condition (criterion) for identifying and distinguishing abstract objects of the same kind. To see this it suffices to observe that the implication contained in PE may be reversed. It is clear that the principle of extensionality PE may be decomposed into two principles PE_0 and PE_1 corresponding to two cases: $i = 0$ or $i = 1$. The reader may formulate partial principles PE_0 and PE_1, and observe that the symbol of identity in (1) is either a connective (PE_0) or a predicate (PE_1).

The case when $i = 0$ gives rise to certain problems concerning extensionality. The symbol of identity serves to identify either objects or situations. However, besides the connective of logical identity of situations there is in our language $\mathfrak{L}_0$ the connective of material equivalence. It corresponds to the fundamental dyadic division of all situations into positive and negative facts (ultrafilter and dual ultraideal in the Boolean algebra of situations). This correspondence is expressed in the following theorem (compare definition (2.10)):

(13.1) $$(p \equiv q) = (Fp \wedge Fq)\, D\, (Np \wedge Nq)$$

Let us replace in PA_0 and PE_0 the connective of logical identity of situations by the symbol of material equivalence. The principles PA_0 and PE_0 are transformed in this way into the weak principle of abstraction WPA_0 and strong principle of extensionality SPE_0. Principles of abstraction and of extensionality occur in the usual set theory in the form of WPA_0 and SPE_0, respectively. Note that there is in the domain of objects nothing which would correspond to the distinction between positive and negative facts and to material equivalence. Therefore, there is no analogous possibility to strengthen PE_1 and weaken PA_1.

We may state, because of (2.7) that:

(1) WPA_0 follows from PA_0 and
(2) PE_0 follows from SPE_0.

Moreover, PA_0 and SPE_0 reduce to WPA_0 and PE_0, respectively, in the case of ontological two-valuedness (section 2 and 3).

In my opinion, the principle PA_0 does not offer any serious difficulty besides the antinomies. Therefore, the principles PA_0 and PA_1 as well, limited in certain ways, may be assumed as axiom schemes. As to the principles of extensionality I would like to assume the principles SPE_0 and PE_1 as axioms of ontology. If one assumes PE_0 instead of SPE_0 then one cannot exclude the possibility that there exist two different coextensional objects A, B of some kind $(0;m,n)$, i.e., $A \neq B$ and for all $x_1, \ldots, x_m, p_1, \ldots, p_n$: $A * x_1, \ldots, x_m, p_1, \ldots, p_n \equiv B * x_1, \ldots, x_m, p_1, \ldots, p_n$. The abstract objects A, B would be intensional entities.

Our preference for SPE_0 against PE_0 rests in Occam's principle and accords with usual mathematical thinking. The principle SPE_0 excludes the existence of intensional abstract objects. There may be only one argument against SPE_0. We will be hindered in assuming SPE_0 only if somebody will prove that it contradicts the postulate formulated at the end of section 3. Indeed, this postulate is the heart of Wittgenstein's ontology. The formula (*) is sometimes called the Principle of Extensionality of Situations, PES. Clearly, SPE_0 follows from PES. I hope, PES does not follow from SPE_0.

NOTES

1. The original version in Polish "Ontologia w Traktacie L. Wittgensteina," has been published in *Studia Filozoficzne* 1(52), 1968, 97-120, Warsaw.

2. *Rzeczy i Fakty* (Things and Facts). Państwowe Wydawnictwo Naukowe (Polish State Publishers in Science), 1968, Warsaw, Poland.

3. There is an opinion that mereology, a formal theory built by St. Leśniewski, is a suitable basis for the theory of spatiotemporal relations.

4. The predicates "philosopher", "logician" are called sometimes nominal predicates or, simply names. However, they are not names in our sense.

5. If Q is a k-ary predicate, f is a k-ary functor and $\Theta_1, \ldots, \Theta_k$ are nominal formulae then the expression:

$$Q(\Theta_1, \ldots, \Theta_k)$$

is sentential formula and the expression:

$$f(\Theta_1, \ldots, \Theta_k)$$

is nominal formula.

6. For a concise axiom system see my "Non-Fregean logic and theories," submitted to *The Journal of Symbolic Logic*.

7. It is clear that the quantifiers play an essential role only when there exist infinitely many situations.

8. If we assume (4.1), (4.2), (4.3.1), (4.3.2) then we have 14 modalities like Lewis' system S4. The additional assumption of (4.4) reduces the number of modalities to 10; compare the system S4.2 of M. A. E. Dummett and E. J. Lemmon (and of P. T. Geach).

9. We assume also (in o-ontology) Barcan formulae with nominal variables: $L \forall x \phi(x) \equiv \forall x L \phi(x)$, $M \exists x \phi(x) \equiv \exists x M \phi(x)$.

10. Usually, in the theory of Boolean algebras the points are called atoms.

11. The converse of (8.11) is a logical theorem formulated with defined terms. It follows from definitions (8.9) and (6.4).

12. One might consider the reflexive, antisymmetric and transitive relation of "entailment" between points represented by the connective *W* and defined as follows: *pWq* means that *p, q* are points and every state of affairs occurring in *p* occurs also in *q*. Every point entails in this sense the point:

$$\dot{\supset} r(SAr, Sr)$$

Remember, however, that every two different possible worlds *p, q* exclude themselves, i.e. the product of them is impossible ($p \wedge q = 0$). It may also be shown that there exists a situation (state of affairs) which occurs in one of the points *p, q* and does not occur in the other.

13. R. Wojcicki (University of Wrocław) observed in a letter to the writer that the axiom system for the states of affairs reduces to the formulas (8.5), (8.10), (8.16). All the remaining theorems (8.3, 8.4, 8.6, 8.7, 8.8, 8.11, 8.12, 8.13, 8.14, 8.15, 8.17) are provable.

14. There is a systematic ambiguity in our use of the asterisk *. This point will be explained later.

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