Preliminary draft

INEXPRESSIBLE LONGING FOR THE INTENDED MODEL

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ABSTRACT. This note is a summary of the talk given on June 10, 2010 at the University of Opole during a meeting of The Group of Logic, Language and Information. We limit ourselves to some major points of the talk, skipping all the minor details. The main hero of this note is the concept of the *intended model* of a theory. Some emphasis is put on the role of *extremal axioms* in the characterization of such models.

1 Monomathematics and polymathemathics

There are two types of mathematical theories. Theories of the first type start with a specific mathematical structure, given in advance. This structure is well recognized in mathematical investigations. It is said in this case that we have deep *intuitions* about it and we want to develop a theory which is supposed to formalize these intuitions. Examples of such structures are: *natural numbers*, *real numbers*, *geometric Euclidean space* and *the universe of all sets*. You may call this *monomathematics* (Tennant 2000). The other type, *polymathematics*, deals with *classes* of structures, with no single specific structure fixed in advance. This is the case of, e.g.: *abstract algebra*, *general topology*, *graph theory*, and numerous other subareas of mathematics.

We may talk about *intended models* in monomathematics. The intended model of a theory T is the mathematical structure for the description of which the theory T was formulated. This characterization has a *pragmatic* character. It may happen that T has only one model, up to isomorphism (i.e. all its models are *isomorphic*, in which case we say that T is *categorical*). It may happen that all models of T are semantically indistinguishable, i.e. *elementarily equivalent*; in this case we say that T is *complete*. If T is complete, then for any sentence ψ from its language we have either $T \vdash \psi$ or $T \vdash \neg\psi$. If a theory is categorical, then it is complete, but not vice versa. A theory T is *κ-categorical*, where κ is an infinite cardinal, if all models of T of power κ are isomorphic.

Thus, in the case of monomathematics we are looking for categorical (categorical in power) or complete theories which can then characterize the intended model in a unique way, either algebraically (via isomorphism) or semantically (via elementary equivalence). As it happens, this goal is hard to achieve, due to some metamathematical facts. Anyway, we may look for some *metamathematical* characterization of intended models, pointing at some properties distinguishing them from all other possible models of the investigated theory.

We do not talk about intended models in polymathematics. Rather, we are looking for some *representation theorems* characterizing all models of a theory (as e.g. the *Cayley Representation Theorem* stating that every group is isomorphic to some group of permutations, the *Stone Representation Theorem* saying that every Boolean algebra is isomorphic to a field of sets, etc.).

2 Expressive power of a logical system

Possibilities of unique representation of intended models depend on the language and logic in question. The first order logic (FOL) has very poor *expressive power*: one cannot define in it several uppermost important mathematical concepts, as e.g. these of *infinity* or *continuity*. Due to the Löwenheim-Skolem-Tarski Theorem no consistent theory (without finite models) in the language of FOL is categorical. Some theories in FOL are categorical in particular powers, but in general the notion of categoricity seems to be rather a mathematical concept and not a purely logical one. Ryll-Nardzewski Theorem provides necessary and sufficient conditions for $\aleph_0$-categoricity and Morley Theorem provides an answer for the uncountable case (if a theory is categorical in *some* uncountable power, then it is categorical in *all* uncountable powers).

But FOL has many properties desirable from a logical point of view, e.g. it is complete and compact. One may say that FOL has great *deductive power*. On the other hand, logical systems stronger than FOL (second order logic, infinitary logics, logics with generalized quantifiers) usually have considerably great expressive power, but not all fundamental deductive properties of FOL have immediate counterparts in these strong systems. Rather, we must coin new concepts of completeness or compactness applicable in these cases. In this sense, strong logical systems have poor deductive power (compared with that of FOL).

Let us consider a few examples. We limit ourselves to the languages L_{Q_α} and $L_{\alpha\beta}$. An expression $Q_\alpha x \ \psi(x)$ has the meaning: "there are at least $\aleph_\alpha$ objects having the property ψ". Here Q_α is a *new logical constant*; one provides semantics for it in a known way. $L_{\alpha\beta}$ is an infinitary language with conjunctions and disjunctions of length less than α and quantifier prefixes of length less than β. In particular, $L_{\omega_1\omega}$ is a language in which countable conjunctions and disjunctions are allowed, but only finite quantifier prefixes are acceptable. By saying that a language L has some properties we mean that a corresponding logic formulated in L has these properties.

- The standard model of Peano Arithmetic PA can be characterized in L_{Q_0} up to isomorphism. It suffices to add to the axioms of discrete linear order $<$ a new axiom: $\forall x \neg Q_0 y\ y < x$.
- In L_{Q_0} the *Löwenheim-Skolem-Tarski Theorem* does not hold.
- L_{Q_0} is not recursively axiomatizable.
- In L_{Q_0} the *Compactness Theorem* does not hold.
- L_{Q_0} satisfies the *Downward Löwenheim-Skolem Theorem.*
- We obtain a complete system of (infinitary) proofs in L_{Q_0} by adding the infinitary rule:

$$\frac{\exists^{\geqslant 1} x\ \psi(x), \exists^{\geqslant 2} x\ \psi(x), \ldots}{Q_0 x\ \psi(x)}.$$

- L_{Q_0} is a fragment of $L_{\omega_1\omega}$, which is obviously seen from:

$$Q_0 x\ \psi(x) \equiv \bigwedge_{n<\omega} \exists^{\geqslant n} x\ \psi(x).$$

  The sentence $Q_0 x\ \psi(x)$ from L_{Q_0} has exactly the same models as the following sentence from $L_{\omega_1\omega}$:
  $\neg \bigvee_{n\in\omega} \exists x_1 \ldots \exists x_n \forall x\ (\psi(x) \rightarrow (x = x_1 \vee \ldots \vee x = x_n))$.
- L_{Q_1} is (!) axiomatizable.
- The notion of *well ordering* is not definable in $L_{\omega_1\omega}$, but it is definable in $L_{\omega_1\omega_1}$ by the conjunction of axioms of a linear ordering and the sentence:
  $(\forall x_n)_{n\in\omega} \exists x\ (\bigvee_{n\in\omega} (x = x_n) \wedge \bigwedge_{n\in\omega} (x \leqslant x_n))$.

- *The predicate of truth* of formulas of a language with countably many symbols is definable in $L_{\omega_1\omega}$.
- In $L_{\omega_1\omega}$ any countable structure with a countable number of relations can be characterized up to isomorphism by a single sentence (*Scott Theorem*).

Several mathematical structures can be characterized up to isomorphism in the full second order logic. However, this logic does not admit any effective deductive system, so the ordinary completeness fails in this case. It is a Logician, who cares a lot about completeness. The Mathematicians prefer to have categorical descriptions of mathematical structures (in particular, their intended models) and do not worry that much about completeness of the underlying system of logic.

3 Lessons from metatheory

The Löwenheim-Skolem-Tarski Theorem says that FOL does not distinguish infinite powers. There are several other *limitative theorems* which describe what is and what is not possible in a given system of logic. For example, the Lindström Theorem says that no logic satisfying both: the Completeness Theorem (or the Compactness Theorem) and the Downward Löwenheim-Skolem Theorem is stronger than FOL. Further examples of limitative theorems may concern particular theories formulated in one or another system of logic. The well known examples are results about incompleteness and undecidability of theories, most notably the first order Peano arithmetic PA and the first order axiomatic set theory ZF. We are not going to report here on these results; they can be found in any advanced logic textbook.

For our purposes it is sufficient to note that as a consequence of these limitative theorems we obtain theorems on existence of *non standard* models of theories belonging to monomathematics. Hence our hope to win categorical or at least complete characterization of intended models appear in these cases unjustified. For example, Peano arithmetic PA has $2^{\aleph_0}$ pairwise elementarily non-equivalent (and hence also pairwise non-isomorphic) countable models. Only one of them (the one defined in the known way in set theory) is *standard* and coincides with the *intended model*. Actually, PA has in any infinite cardinality κ the maximum possible number of pairwise non-isomorphic models, i.e. 2^κ. Thus, PA is a *wild* theory.

The situation is even more complicated in the case of set theory ZF. This theory inherits the incompleteness phenomena from arithmetic PA (which is interpretable in ZF). The very existence of a lot of sentences independent from the axioms of ZF shows that the notions of *set* and the relation $\in$ are characterized by the axioms rather weakly. If ZF is consistent, then it has countable models which obviously differ from the *intended* interpretation of set theory. The power set operation may be interpreted in several different ways, giving raise to several distinct *universes* of sets. We do not have a *single intended model* of set theory ZF. Rather, we still are looking for *new axioms* which could characterize the notion of *set* in a more complete way. Some examples are provided below.

There is still another possibility for the future of set theory. At the present moment it is customary to base all mathematical investigations on set theory. But we may not exclude that a time will come when we will treat different models of set theory in the same way as today we treat, say, different topological spaces. Notice that Thoralf Skolem and John von Neumann, two of the Fathers of Modern Set Theory were both very sceptical about set theory as the basis of the whole of mathematics. Zermelo's opinion was different, of course.

4 Extremal axioms: examples

Extremal axioms have been formulated in order to determine intended models in a unique way. They are either the axioms of *maximality*, or the axioms of *minimality*. Moreover, it is not only the mere volume of the universe of a model that counts: the "richness" of the structure of a model is taken into account as well.

4.1 Induction

Axiom of induction may be considered either as a single sentence in a second order language or as an axiom scheme in a first order language:

- Second order axiom:

$$\forall X \ (0 \in X \wedge \forall x \ (x \in X \rightarrow s(x) \in X) \rightarrow \forall x \ (x \in X)),$$

  where s is the symbol for successor.

- First order scheme:

$$(\psi(0) \wedge \forall x \ (\psi(x) \rightarrow \psi(s(x)))) \rightarrow \forall x \ \psi(x),$$

  where $\psi(x)$ is any formula with one free variable of the language of Peano arithmetic.

In each of these cases *induction* (axiom or schema) is a certain *minimality* condition imposed on the universe of all natural numbers. The very existence of non standard models of (first order) PA shows that the existence of *alien intruders* (i.e.

non standard numbers) cannot be prohibited by PA and FOL. There are several methods of proving the existence of non standard models of PA: you may use an argument from compactness, or the ultraproduct construction or the tree of expansions of PA, etc.

The schema of induction cannot be replaced by any finite number of axioms equivalent to it: PA is not finitely axiomatizable. Neither can we restrict the complexity of formulas in it and simultaneously keep the full force of PA.

Let $\mathfrak{N}_0$ denote the standard model of PA. We know that it cannot be uniquely characterized in FOL either in terms of isomorphism or in terms of elementary equivalence. We can only distinguish it from all other countable models of PA on the metalevel:

- $\mathfrak{N}_0$ is the only *well-founded* model of PA.
- $\mathfrak{N}_0$ is a *prime* model of PA.
- *Tennenbaum Theorem.* $\mathfrak{N}_0$ is the only *recursive* model of PA.

All countable non standard models of PA have the same *ordinal type*: $\omega + (\omega^* + \omega) \cdot \eta$ (a copy of natural numbers followed by that many copies of integers as there are rational numbers). They differ in properties of addition and multiplication: as a consequence of the Tennenbaum Theorem, *both* these operations cannot be given recursive definitions in non standard models.

The investigations of models of arithmetic are already highly advanced, with many very sophisticated mathematical tools used in them. We are not going to even roughly summarize these results. An interested reader is kindly invited to compare in this respect e.g. Kaye 1991 or Hájek, Pudlák 1993.

The first axiomatizations of arithmetic, i.e. these given by Giuseppe Peano and Richard Dedekind were essentially second order. In such an approach you can of course determine the standard model of arithmetic up to isomorphism, as Dedekind did in his *Kettentheorie*. Dedekind in *Was sind ud was sollen die Zahlen?* (1888) was not really interested in the *logical* aspects of his system. He defined natural numbers as the least infinite set being the universe of a structure $(N, f, 1)$ with a function $f : N \to N$ and a distinguished element 1 outside the range of f.

The schema of induction was also explicitly present in Grassmann's *Lehrbuch der Arithmetik* (1861). It is claimed that already Pascal used induction. Notably, inductive arguments took place in Ancient Greece, in the reasoning based on *infinite regress*. In order to prove that no (natural) number has the property ψ it suffices to show that for any number n with the property ψ there exists a number $m < n$ which also has ψ. If there would exist a number with the property ψ, then we could get smaller and smaller numbers with that property, which was conceived as absurd. This method was later rediscovered by Fermat.

4.2 Continuity

Investigations of *continuity* origin in difficulties with mathematical description of the *geometric continuum*. These difficulties were connected, among others, with: the problem of infinite divisibility (of the continuum itself), the question how is it possible to obtain the continuum out of points which do not have extension, etc. The discovery of *irrational* numbers showed that the dense ordering of rational numbers contains *gaps*. By the way, it was also a revolutionary change in the world-perspective according to which the ultimate structure of reality should be based on numbers. The mysterious structure of the geometric continuum was hidden in mathematical applications in physics, e.g. the description of *movement*, *velocity* or *change*. Today, we see the essence of these difficulties in the very notion of *continuity*.

Two early opposite positions concerning the structure of the geometric continuum were the following ones:

- the continuum does not consist of atoms but of infinitely divisible parts (Aristotle, Awerroes, Bradwardine, Kepler, Cavaglieri);
- the continuum consists of (ultimately non-divisible) parts: *atoms* (Democritus) or non-divisible *points* (Plato, Pythagoras); there was also a controversy concerning the number of these parts – should it be finite or infinite?

As Bradwardine wrote: *Nullum continuum ex athomis integrari*. The continuum should rather be *integrated* from other smaller continua of the same kind.

The view that *velocity* may be related to a single point (moment) was alien to Aristotle. This view changed to the opposite one in the Middle Ages and we find the latter in an elaborated form in the work of Galileo.

Only after the rapid development of Analysis in the works of Newton, Leibniz, Euler and others there appeared a necessity of establishing it on solid logical background. First, Lagrange formulated some restrictions. Then the notions of an completely arbitrary function has been coined. As it is known, to that time by *a function* one understood a kind of a rule, or recipe, or algorithm according to which the value of a function was associated with each of its arguments.

The program of *arithmetization* of Analysis belongs to XIX century. According to it, one obtains an arithmetical representation of the continuum and the arguments, as well as the values of the investigated functions vary in arithmetic domains. Already in the works of Gauss and Cauchy we find representations of the most important concepts of Analysis: these of *limit* and *continuity*. Bolzano shows that a continuous function takes all the intermediate values between any given two of its values. He also explicitly expresses the view that the continuum consists of *points*. The highest degree of precision in Analysis is reached in the works of Weierstrass. Here we find the well known $\varepsilon - \delta$ convention and the explicit use of quantifiers. In the second part of XIX century several *theories of real numbers* are proposed, e.g.: Méray (1869), Cantor (1872), Heine (1872), Dedekind (1858, published in 1872), Weber (1898). In 1890 Schwartz proves in a precise way that if the derivative of f is everywhere equal to 0, then f is a constant function. This fact was taken as a dogma, without proof, in Newton's system.

Dedekind's theory of real numbers is based on the notion of *cut* of the set $\mathbb{Q}$ of all rational numbers with their usual ordering $<$. Recall that a pair (A, B) of subsets of $\mathbb{Q}$ is a *cut*, if $A \cup B = \mathbb{Q}$ and for all $a \in A$ and $b \in B$ we have $a < b$. The ordering of $\mathbb{Q}$, though dense, does contain *gaps*, i.e. there are cuts (A, B) with no greatest element in A and no smallest element in B. Dedekind's idea was to *associate* a new number with each cut of $\mathbb{Q}$. This *completion* of $\mathbb{Q}$ gives the set of all *real numbers*. Dedekind's theory required two auxiliary components: *theory of rational numbers* and *theory of sets*. The first was provided by Weber in 1898, who based the theory of rational numbers on the arithmetic of natural numbers. The second originated in the Cantor's set theory (1972, 1983). The first axiomatization of set theory was given by Zermelo in 1908. Peano gave his axiomatization of the arithmetic of natural numbers in 1889.

Another well known construction of real numbers is that proposed by Cantor. Here real numbers are understood as abstraction classes of sequences of rational numbers satisfying the Cauchy condition.

Do we now finally have a complete representation of the geometric continuum? Should it be simply represented by the continuous ordering of the real numbers? Well, at the present moment Analysis is soundly based on the structures of real and complex numbers. The notion of an *infinitely small magnitude* can be represented in a precise way in the *non standard analysis*. However, some problems still remain open, just to mention the *Continuum Hypothesis*, which cannot be either proved or refuted in the Zermelo-Fraenkel set theory. Cf. also e.g. the remarks concluding Mioduszewski's monograph on continuity (Mioduszewski 1996), as well as the last sentence from Bukovský monograph on the structure of the real line (Bukovský 1979, 206):

> Z toho, čo sme uviedli, môžeme však dôjst' k rovnakému poučeniu, aké predpovedal N. Luzin v [59], str. 322: že prišiel čas, ked' *je potrebné vykonat' reformu našich predstaví o kontinuu*. [Here [59] refers to: Luzin, N. 1930. *Leçons sur les ensembles analytiques*. Gauthier-Villars, Paris.]

4.2.1 Algebra

There are two methods of construction of number systems: *genetic* and *axiomatic*. In the first case we begin with natural numbers (together with addition and multiplication of them) and then we *construct* the other number systems: the integers, rational, real and complex numbers, in each case with the corresponding arithmetical operations on them. These step-by-step extensions are connected with the fact that some arithmetical operations are not in general applicable in the "smaller" number system (as e.g. subtraction of natural numbers, division of integers) and we construct a "larger" system in which these operations are applicable. The second, axiomatic method begins with a list of axioms which should be satisfied by all objects belonging to the number system in question. Thus, we have separate axiom systems for, e.g., natural numbers and real numbers.

The axiom schema of induction is an example of an extremal axiom of *minimality*, as we have seen. On the other hand, it is the *Axiom of Continuity* which is an extremal axiom specific for the *real numbers*: it is an axiom of *maximality*, in turn. *The Axiom of Continuity* is added to the usual axioms of an ordered field $(R, +, \cdot, 0, 1, <)$ and may have e.g. one of the following forms (cf. Błaszczyk 2007, 306):

1. For any cut (A, B) in $(R, <)$ either in A there exists the greatest element, or in B there exists the smallest element.
2. Any non-empty bounded from above subset $A \subseteq R$ has the lowest upper bound in R.
3. Any infinite and bounded subset $A \subseteq R$ has a limit point in R (in order topology).
4. $(R, +, \cdot, 0, 1, <)$ is an Archimedean field and for any sequence $(a_n) \subseteq R$ there exists $a \in R$ such that $\lim\limits_{n\to\infty} a_n = a$.
5. $(R, +, \cdot, 0, 1, <)$ is an Archimedean field and for any descending chain of closed intervals (A_n) we have $\bigcap\limits_n A_n \neq \emptyset$.

The roots of the theory of real numbers possessing such continuity properties can be found, among others, in:

- David Hilbert. *Über den Zahlbegriff* (1900).
- Georg Cantor. *Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen* (1872). This was later developed in § 9 of *Über unendliche lineare Punktmannigfaltigkeiten* (1883).

- Richard Dedekind. *Stetigkeit und irrationale Zahlen* (1872).
- Eduard Heine. *Die Elemente der Functionenlehre* (1872).

Despite some differences in formulation or the basic constructions, all these approaches have some properties in common, e.g.:

- A clear distinction is made between the *geometric continuum* and the set of all *real numbers*.
- The *creative* character of the introduction of real numbers is stressed.
- The authors explicitly state that there are no rigorous proofs that the structure of reality, space, time, degrees of intensiveness of features, etc. has indeed a *continuous* nature. It may well be the case that the reality has ultimately a discrete nature and the concept of continuity is a free projection of the mind, present in mathematical constructions only.

Now, what about the *intended model* of a continuous number system, like the real numbers? One can prove several *isomorphism theorems* which characterize the corresponding systems up to isomorphism, by taking into account: arithmetical operations, ordering and topological properties of the systems in question. Here are a few most known examples ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ correspond, respectively, to: real numbers, complex numbers, quaternions and octonions; in each of the theorems below a suitable structure on these sets is presupposed):

- *Frobenius Theorem*. Each associative algebra with division over $\mathbb{R}$ is isomorphic either with $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$.
- *Hurwitz Theorem*. Any normed algebra with division is isomorphic either with $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$ or $\mathbb{O}$.
- *Ostrowski Theorem*. Any field complete with respect to an Archimedean norm is isomorphic with either $\mathbb{R}$ or $\mathbb{C}$ and the norm is equivalent with the usual norm determined by the absolute value.
- *Pontriagin Theorem*. Any connected locally compact topological field is isomorphic with either $\mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$.
- *1,2,4,8–Theorem (Bott, Milnor, Kervaire)*. Each algebra with division over $\mathbb{R}$ has dimension 1, 2, 4 or 8.
- *Hopf Theorem*. Each commutative algebra with division over $\mathbb{R}$ has dimension $\leqslant 2$.

Let us recall that another Ostrowski Theorem says that any non-trivial absolute value on the rational numbers $\mathbb{Q}$ is equivalent to either the usual real absolute value or the p-adic absolute value.

Some further properties allow for distinctions between the above number systems, just take two examples:

- $\mathbb{R}$ is linearly and densely ordered. This ordering is also continuous (on behalf of the axiom of continuity). The rational numbers $\mathbb{Q}$ form a dense countable subset of $\mathbb{R}$. On the other hand, the complex numbers $\mathbb{C}$ cannot be linearly ordered in such a way that the ordering will be compatible with the arithmetic operations.

- $\mathbb{R}$ is commutative (with respect to multiplication) and $\mathbb{H}$ is not.

According to the theorems mentioned above there are only a few algebraic structures (up to isomorphism) which may serve as a basis for all usual arithmetical operations. In this sense, we may say that the intended models for number systems are characterized in a unique way. Let us only mention that non standard analysis offers still another mathematical representation of a number system (the *hyperreal numbers*) which is obtained from the "usual" real numbers by the construction of an appropriate ultrapower. The field of hyperreal numbers is not Archimedean, and the field of real numbers is Archimedean. Among hyperreal numbers there are *infinitely small* numbers, which can be used for explication of some statements of classical Analysis previously formulated in a very vague manner.

Still another (second order) axiomatization of the real numbers was given in 1936 by Tarski. It describes the structure $(\mathbf{R}, <, +, 1)$, where $<$ is a linear dense and Dedekind-complete ordering of $\mathbf{R}$, $+$ is the operation of addition (compatible with $<$) and 1 is a distinguished element satisfying $1 < 1 + 1$. The axioms imply that this structure is a linearly ordered Abelian group under addition with distinguished element 1. It is also Dedekind-complete and divisible. It can be shown that the axioms imply the existence of a binary operation having all the expected properties of multiplication.

This note is not supposed to be a complete survey of *closure properties* investigated in algebra which may show close affinity to the extremal axioms. An interested reader should consult any advanced book in abstract algebra in this respect. Let us close this section with the following sketchy remarks:

- The (first order) axiomatic theory of real closed fields is complete. It admits elimination of quantifiers. Hence it is decidable.

- Real closed fields have exactly the same first order properties as the real numbers.

- *Artin-Schreier Theorem*. Let F be an ordered field (i.e. with a definite ordering $<$ on it). Then F has an algebraic extension, say E, called *the real closure* of F such that E is a real closed field and its ordering, say $\prec$, is an extension of $<$. Such E is unique, up to isomorphism.

On the other hand, the real numbers remain still a little bit mysterious. There are indeed many statements concerning the set of real numbers, as well as its subsets which are independent from the axioms of Zermelo-Fraenkel set theory. The famous examples are e.g.: The Continuum Hypothesis, The Suslin Hypothesis, the sentence PM (all projective sets are Lebesgue measurable). The continuum (i.e. the cardinal number $2^{\aleph_0}$) may take almost every value on the scale of alephs (with exception of cardinal numbers with countable cofinality, as e.g. $\aleph_\omega$). One formulation of The Suslin Hypothesis states that each set with a linear order without first and last element satisfying the ccc (countable chains condition, stating that every *antichain* is at most countable) and such that the corresponding order topology is connected is isomorphic to the set of all real numbers with their usual ordering.

4.2.2 Geometry

Hilbert's *Axiom of Completeness* from his *Grudlagen der Geometrie* has the following form in editions 2–6:

> The elements (points, lines, planes) of geometry constitute a system of things which cannot be extended while maintaining simultaneously the cited axioms, i.e., it is not possible to add to this system of points, lines, and planes another system of things such that the system arising from this addition satisfies axioms AI–V1.

As it is well known, the above axiom was later reformulated to the following *Linear Completeness Axiom* (Hilbert 1999[14], 30):

> V 2 (Axiom der linearen Vollständigkeit). *Das System der Punkte einer Geraden mit seinen Anordnungs- und Kongruenz-beziehungen ist keiner solchen Erweiterun fähig, bei welcher die zwischen den vorigen Elementen bestehenden Beziehungen sowie auch die aus den Axiomen I–III folgenden Grundeigeschaften der linearen Anordnung und Kongruenz, und V 1 erhalten bleiben.*

This formulation, in turn, was replaced by the usual *Axiom of Completeness* for the real numbers system. In such a form the axiom in question must be formulated in a second order language. This system of geometry is categorical, i.e. it has exactly one model, up to isomorphism.

The system of geometry presented in Borsuk, Szmielew 1975 has the following primitive terms:

- the space (understood as a set of all *points*),
- the families of *lines* and *planes*,
- the three-argument relation $\mathbf{B}xyz$ ($\mathbf{B}xyz$ is to be read: the point y *lies between* the points x and z),
- the four-argument relation $\mathbf{D}xyuv$ ($\mathbf{D}xyuv$ is to be read: *the distance between* x *and* y *is the same as the distance between* u *and* v).

It can be shown that the Axiom of Completeness is independent from the other axioms of the system of *absolute geometry* (i.e. the above system without the Euclid paralel postulate). Thus, absolute geometry admits models which are not continuous. Moreover, absolute geometry is not categorical: it has a Cartesian model as well as the Klein's model and these models are not isomorphic. Its extensions, obtained by either taking the Euclid paralel axiom (i.e. the Euclidean geometry) or its negation (the hyperbolic geometry) are categorical.

The Axiom of Completeness is necessary for distinguishing the *intended model* of Euclidean geometry (i.e. the Cartesian model, known from the school). By the way, notice that the Euclidean geometry is privileged on historical grounds and most likely on grounds connected with our (mostly visual) perception of the physical world on the medium scale. Should we be, say, clouds living in the cloudy environment (without any rigid solid bodies around us), then we could possibly begin with another geometric representation of reality.

The Axiom of Continuity has the following form in the case of the system of geometry from Borsuk, Szmielew 1975 (this is the only second order axiom of the system):

> If X, Y are non empty sets of points and there exists a point a such that $p \in X$ and $q \in Y$ imply $\mathbf{B}apq$, then there exists a point b such that $p \in X - \{b\}$ and $q \in Y - \{b\}$ imply $\mathbf{B}pbq$.

The axiomatic system of geometry proposed by Tarski has as primitive terms the predicates $\mathbf{B}xyz$ and $\mathbf{D}xyuv$ (read as above), but it is an *elementary* (i.e. first order) system. Here the Axiom of Continuity is not a single sentence, but a scheme of the following form:

$$\exists z \forall x \forall y \ ((\varphi(x) \wedge \psi(y)) \rightarrow \mathbf{B}xyz) \rightarrow \exists u \forall x \forall y \ ((\varphi(x) \wedge \psi(y)) \rightarrow \mathbf{B}xuz),$$

where $\varphi(x)$ is any formula in which y, z, u are not free and $\psi(y)$ is any formula in which x, z, u are not free.

This system is complete and decidable. Its axioms contain only the primitive terms. It may be added that the system has many "nice" metamathematical properties. However, it also has some disadvantages from the point of view of practical applications.

Again, this note is not supposed to present any summary of applications of extremal axioms in geometry: we have limited ourselves to a few examples only.

Let us also marginally add that the notion of completeness is applicable in the case of general topological spaces. However, in this case we of course do not speak about *intended models* – it seems that there was no one, fixed in advance complete topological space when the theory of such spaces was developed. Recall that by *a complete* topological space we understand any metric space in which every Cauchy sequence has the limit belonging to this space.

4.3 Axioms of restriction in set theory

Historically, first axiom of *minimality* in set theory was the *axiom of restriction* formulated by Fraenkel in 1922 and then repeated in his *Einleitung in die Mengenlehre*. It says, roughly speaking: "there are no other sets besides these, whose existence can be proven from the axioms of set theory." It is thus a proposal to understand the notion of *set* as narrowly as possible. Fraenkel intended to obtain some version of *completeness* in set theory with this axiom. It should be remembered that these considerations took place before Gödel's results about incompleteness of arithmetic (and, consequently, set theory, in which one can interpret arithmetic). Fraenkel aimed also at exclusion of infinite descending $\in$-chains of sets; this goal has been later achieved by accepting the axiom of regularity.

The idea of an axiom of restriction was criticized both by John von Neumann and Ernst Zermelo. On the other hand, Roman Suszko and independently John Myhill tried to attach a precise mathematical content to the restriction axiom in Fraenkel's style (cf. Suszko 1951, Myhill 1952). The idea that each set should be associated with its name is also present later in Paul Cohen's origin of the forcing method (cf. Cohen 1966).

Kurt Gödel's *Axiom of Constructibility* ($V = L$, to be read: all sets are constructible) was not conceived as a restriction axiom, though it has a form of an axiom of minimality in set theory. The inner model of all constructible sets was devised in order to prove that if set theory ZF is consistent, then also ZF plus the axiom of choice and the Generalized Continuum Hypothesis is consistent.

Let us recall that at successor stages in building the constructible universe one makes use of the *poorest* powerset operation possible: the powerset of x contains only *definable* subsets of x. At limit stages we take of course unions of all stages constructed so far. The class of all constructible sets is a minimal countable transitive model of set theory containing all ordinal numbers.

The method of inner models has its own limitations, as shown in Shepherdson 1951–1953. However, it is a very convenient point of departure for some more subtle constructions, including the celebrated method of forcing, due to Paul Cohen.

Kurt Gödel himself was against axioms of restriction in set theory and he overtly expressed his view in favor of axioms of maximality (Gödel 1964, quotation after *Collected Works* II, 262–263):

> On the other hand, from an axiom in some sense opposite to this one [i.e. to the Axiom of Constructibility — JP], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas axiom A [i.e. the Axiom of Constructibility — JP] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set [. . .]

It seems that nobody in the community of set theoreticians has ever seriously taken into account a possibility of adjoining the Axiom of Constructibility to the body of fundamental axioms of set theory. "Normal" mathematicians may have different opinion in this respect – cf. Friedman's judgment (Feferman, Maddy, Steel, Friedman 2000, 436–437):

> The set theorist is looking for deep theoretic phenomena, and so $V = L$ is anathema since it restricts the set theoretic universe so drastically that all sorts of phenomena are demonstrably not present. Furthermore, for set theorist, any advantage that $V = L$ has in terms of power can be obtained with more powerful axioms of the same rough type that accommodate measurable cardinals and the like – e.g., $V = L(\mu)$, or the universe is a canonical inner model of a large cardinal.
>
> However, for the normal mathematician, since set theory is merely a vehicle for interpreting mathematics as to establish rigor, and not mathematically interesting in its own right, the less set theoretic difficulties and phenomena the better.
>
> I.e., less is more and more is less. So if mathematicians were concerned with the set theoretic independence results – and they generally are not – then $V = L$ is by far the most attractive solution for them.

> This is because it appears to solve all set theoretic problems (except for those asserting the existence of sets of unrestricted cardinality), and is also demonstrably relatively consistent.
>
> Set theorist also say that $V = L$ has implausible consequences – e.g., there is a PCA well ordering of the reals, or there are nonmeasurable PCA sets.
>
> The set theorists claim to have a direct intuition which allows them to view these as so implausible that this provides "evidence" against $V = L$.
>
> However, mathematicians disclaim such direct intuition about complicated sets of reals. Some say they have no direct intuition about all multivariate functions from $\mathbb{N}$ into $\mathbb{N}$!

Nevertheless, the Axiom of Constructibility, taken as a working assumption, has many consequences of considerable interest, in combinatorics, algebra, model theory, theory of recursive functions, etc. However, the Axiom of Constructibility implies e.g. the nonexistence of measurable cardinals as well as the negation of Suslin hypothesis. The prize to be paid, if one accepts this axiom seems to be too high, compared with its alleged naturalness and evident economy. We prefer to stay in the *Cantor's Paradise*.

The most destructive critique of minimal axioms is presented in Fraenkel, Bar-Hillel, Levy 1973. The authors formulate two axioms of restriction. The main idea captured by the first of them is the following.

- THE FIRST AXIOM OF RESTRICTION. *If Q is a property such that each set whose existence follows from the axioms has this property, then every set has the property Q.*

Now, the property Q should be closed with respect to the set-forming operations described in the axioms. Thus, e.g., if x and y have Q, then $\{x, y\}$ has Q, if x has Q, then $\bigcup x$ has Q, etc. Also the axiom schemas of comprehension and replacement can be translated into the suitable closure conditions with respect to Q. One can then show the following facts, among others:

- The First Axiom of Restriction is equivalent to the conjunction of axiom of regularity and the sentence saying that there are no strongly inaccessible cardinals. Obviously, all the consequences of nonexistence of strongly inaccessible cardinals are also provable.

- If we consider set theory in a second order language with a suitable version of The First Axiom of Restriction, then we can prove categoricity of such a theory.
- No consequences concerning the Continuum Hypothesis can be drawn from The First Axiom of Restriction.

THE SECOND AXIOM OF RESTRICTION is the conjunction of the following sentences:

1. *All sets are constructible (in Gödel's sense).*
2. *There are no transitive sets which are models of ZF.*

It follows from 1) that all sets are well founded. As it is known from Gödel's work, 1) implies the GCH. The sentence 2), in turn, implies that there are no strongly inaccessible cardinals. Thus, The Second Axiom of Restriction implies the first one.

Both Axioms of Restriction share some common features:

- Each of them states that some "big" cardinals or sets with high rank do not exist:
  1. First Axiom of Restriction — inaccessible cardinal numbers;
  2. Second Axiom of Restriction — transitive sets which are models of ZF.
- Some complicated sets do not exist:
  1. First Axiom of Restriction — non well founded sets.
  2. Second Axiom of Restriction — non-constructible sets.

The author's arguments against axioms of restriction may be summarized as follows.

- *Analogy*. "In the case of the axiom of induction in arithmetic and the axiom of completeness in geometry, we adopt these axioms not because they make the axiom systems categorical or because of some metamathematical properties of these axioms, but because, once these axioms are added, we obtain axiomatic systems which perfectly fit our intuitive ideas about arithmetic and geometry. In analogy, we shall have to judge the axioms of restriction in set theory on the basis of how the set theory obtained after adding these axioms fits our intuitive ideas about sets" (Fraenkel, Bar Hillel, Levy 1973, 117). Observe that this argument has mostly a *pragmatic* character.

- *Faith.* One could restrict the notion of a set to the narrowest possible only if one could have absolute faith in the axioms of ZF, which does not seem to be the case. Even if one had such a faith, it is more likely that one would look for *maximality* axioms (as in geometry), rather than for restriction axioms.

Two more arguments against axioms of restriction are correlated with author's attitude to the axiom of constructibility (cf. Fraenkel, Bar Hillel, Levy 1973, 108–109):

- *Mathematical elegance*. Axioms of restriction do not improve mathematical elegance of set theory, in the sense that one can prove more powerful theorems based on them. Rather, they may be involved only in proofs that some sets do not exist.

- *Platonistic point of view*. Axioms of restriction are unnatural also when we consider the universe of all sets as an entity capable of growing, in the sense that we can always produce new and new sets. If an axiom of restriction forces us to accept that the universe of all sets is a fixed entity, then why couldn't we consider it as a new set in a still bigger universe? "In other words, there is no property expressible in the language of set theory which distinguishes the universe from some "temporary universes". These ideas are embodied in the *principles of reflection*, which are, mostly, strong axioms of strong infinity." (Fraenkel, Bar Hillel, Levy 1973, 118).

4.4 Axioms of maximality in set theory

The idea of an axiom of maximality in set theory has been investigated even before the formulation of set theory ZF in its present shape (cf. Baer 1928). In a sense, Zermelo's demand concerning the existence of a transfinite sequence of inaccessible cardinals can also be viewed as an axiom of maximality (cf. Zermelo 1930). More recently, maximal axioms, in form of the *axioms of existence of very large cardinal numbers* are just one of the central topics in the contemporary set theory.

There are several criteria to be met when formulating new axioms (of existence of large cardinal numbers), among others: *necessity* (or *non-arbitrariness*) and *fruitfulness in their consequences*. Adding the axiom of infinity to (ZF minus this axiom) enables us to prove theorems about infinite sets. In a similar way, adding an axiom stating the existence of inaccessible cardinals makes it possible to extend operations of set formation beyond what is provable in ZFC. Large cardinals axioms have decisive importance for Descriptive Set Theory and in this sense they appeared fruitful. The same concerns their applications in, say, infinitary combinatorics.

Joan Bagaria recalls fundamental principles by which (according to Hao Wang quotations of Gödel ideas in Wang 1974, 1996) new axioms of set theory should be introduced (Bagaria 2005, 5–6):

> According to Gödel there are five such principles: *Intuitive Range*, the *Closure Principle*, the *Reflection Principle*, *Extensionalization*, and *Uniformity*. The first, *Intuitive Range*, is the principle of intuitive set formation, which is embodied into the ZFC axioms. The *Closure Principle* can be subsumed into the principle of *Reflection*, which may be summarized as follows: The universe V of all sets cannot be uniquely characterized, i.e., distinguished from all its initial segments, by any property expressible in any reasonable logic involving the membership relation. A weak form of this principle is the ZFC-provable reflection theorem of Montague and Levy (see Kanamori 1994):
>
> *Any sentence in the first-order language of Set Theory that holds in V holds also in some V_α.*
>
> Gödel's *Reflection* principle consists precisely of the extension of this theorem to higher-order logics, infinitary logics, etc.
>
> The principle of *Extensionalization* asserts that V satisfies an extensional form of the Axiom of Replacement and it is introduced in order to justify the existence of inaccessible cardinals. [...]
>
> The principle of *Uniformity* asserts that the universe V is uniform, in the sense that its structure is similar everywhere. In Gödel's words (Wang 1996, 8.7.5): *The same or analogous states of affairs reappear again and again (perhaps in more complicated versions).* He also says that this principle may also be called *the principle of proportionality of the universe*, according to which, analogues of the properties of small cardinals lead to large cardinals. Gödel claims that this principle makes plausible the introduction of measurable or strongly compact cardinals, insofar as those large-cardinal notions are obtained by generalizing to uncountable cardinals some properties of ω.

Bagaria discusses then "some heuristic principles, which may be regarded as *Meta-Axioms of Set Theory*, that provide a criterion for assessing the naturalness of the set-theoretic axioms." Axioms in question are either axioms of existence of large cardinal numbers or some *forcing axioms*.

It is of course not possible to give even a rough summary of all the problems concerning axioms of existence of large cardinals in a short paper like this one.

The interested reader is kindly invited to consult e.g., Kanamori 1994 in this respect. Below, we limit ourselves to few remarks pointing at some interconnections between large cardinal axioms and the consistency strength. We follow the presentation contained in Koellner 2010.

According to Mostowski (Mostowski 1967) there are two principles of introducing new axioms of infinity:

1. *The principle of passing from potential to actual infinity*. We build new sets using the axioms of infinity and replacement of ZF. The universe of all sets is potentially infinite and closed with respect to some operations. We postulate the existence of a *set* which itself is closed with respect to these operations. In this way we obtain for instance *inaccessible* cardinals.

2. *The principle of existence of peculiar sets*. Suppose that while constructing sets according to the known operations on them we always meet sets with a certain property P. If there are no evident reasons which should force us to assume that *all* sets have P, then we propose a new axiom saying that there exist sets *without* the property P. In this way we obtain for example *measurable* cardinals.

In the last few decades several kinds of large cardinals have been investigated. Postulating the existence of a large cardinal (whose existence can not be proved from the axioms of ZF) is, of course, a kind of a maximality condition. But it is not only a mere demand on the volume of the universe of set theory: large cardinal axioms are also closely related to the deductive strength of the theories obtained by adjoining such axioms. Let us look at some very elementary examples.

Let Z_0 denote ZFC without the axioms of infinity and replacement. The standard model for this theory is V_ω. The existence of this set follows from the axiom of infinity. Let Z_1 denote Z_0 with the axiom of infinity. Then we can prove in Z_1:

- Z_0 is consistent.
- There exists a standard model for Z_0.

The standard model for Z_1 is $V_{\omega+\omega}$. The existence of this set follows from the axiom of replacement. Let Z_2 denote Z_1 with the axiom of replacement. Then we can prove in Z_2:

- Z_1 is consistent.
- There exists a standard model for Z_1.

The standard model for Z_2 is V_κ, where κ is an inaccessible cardinal. Thus, the next axiom of infinity in this hierarchy will be the sentence "There exists an inaccessible cardinal." The next theory, i.e. ZFC together with this sentence proves the existence of a level of the cumulative hierarchy which is a model for ZFC. And so on: in this way we obtain stronger and stronger set theories.

Let $Con(PA_n)$ denote the sentence expressing consistency of the n-th order arithmetic PA_n. Then $Con(PA_n)$ can not be decided in PA_n, but it can be decided in PA_{n+1}. These sentences are connected with the levels of the cumulative hierarchy of sets. Recall that PA_1, i.e. the first order system of Peano arithmetic PA is mutually interpretable with ZF minus the axiom of infinity, because V_ω, the first infinite level of the cumulative hierarchy consists of hereditarily finite sets which can be coded by natural numbers.

However, the sentences $Con(PA_n)$ are not the only undecidable statements (cf. Koellner 2010, 3–4):

> The trouble is that when one climbs the hierarchy of sets in this fashion the greater expressive resources that become available lead to more intractable instances of undecidable sentences and this is true already of the second and third infinite levels. For example, at the second infinite level one can formulate the statement PM (that all projective sets are Lebesgue measurable) and at the third infinite level one can formulate CH (Cantor's continuum hypothesis).
>
> [...][1]
>
> These instances of independence are more intractable in that no simple iteration of the hierarchy of types leads to their resolution. They led to a more profound search for new axioms.

Due to Gödel's and Cohen's results concerning the independence of CH from ZFC one can see that ZFC is mutually *interpretable* with ZFC+CH, as well as with ZFC+¬CH. The situation with PM is, however, different. The method of inner models shows that ¬PM holds in the constructible universe L. Hence ZFC and ZFC+¬PM are mutually interpretable. But Shelah has shown that ZFC+PM implies the consistency of ZFC and therefore, due to Gödel's second incompleteness theorem, ZFC+PM *is not* interpretable in ZFC. It follows that in order to establish the independence of PM from ZFC we need to assume the consistency of some *stronger* theory – namely that of ZFC plus the sentence "There exists a strongly inaccessible cardinal".

[1] Here Koellner briefly summarizes Gödel's and Cohen's results showing together the independence of CH from ZFC.

This was only a very elementary example. One considers a plentitude of axioms of existence of large cardinals which have relevant impact on the independence proofs. Let us only add that there exists a pattern of formulating large cardinal axioms in terms of elementary embeddings. Generally speaking, one considers non trivial (i.e. different from identity) elementary embeddings $j : V \to M$ of the cumulative hierarchy V into a transitive class M. The least ordinal moved by such an embedding is called the *critical point* of j and denoted by crit(j). For example, a cardinal is *measurable* if and only if it is the critical point of some such embedding. Further conditions imposed on j and M enable us to create several sorts of large cardinal axioms. As Kunen has shown, there is no elementary embedding $j : V \to V$, so there exists an upper bound for this procedure.

The structure of degrees of interpretability of theories is very complicated. However, *natural* theories having practical mathematical applications happen to be orderly comparable, which of course is only an *empirical* fact. Theories can be compared through large cardinal axioms corresponding to them (cf. Koellner 2010, 10–11):

> Given ZFC+φ and ZFC+ψ one finds large cardinal axioms Φ and Ψ such that (using the methods of inner and outer models) ZFC+φ and ZFC+Φ are mutually interpretable and ZFC+ψ and ZFC+Ψ are mutually interpretable. One then compares ZFC+φ and ZFC+ψ (in terms of interpretability) by mediating through the natural interpretability relationship between ZFC+Φ and ZFC+Ψ. So large cardinal axioms (in conjunction with the dual method of inner and outer models) lie at the heart of the remarkable empirical fact that natural theories from completely distinct domains can be compared in terms of interpretability.

Sometimes the procedure sketched above is the only known way to compare theories, which provides a *pragmatic* justification for the investigations of large cardinal axioms.

5 Intended model: a purely pragmatic concept?

We have seen that FOL does not provide sufficient tools for unique characterization of intended models. We may characterize such models either in some stronger logical systems or at the level of metatheory.

This shortcoming may bother a logician, but it is not very important for the working mathematicians. The latter cares first of all about characterization of models up to isomorphism, paying less attention to logical matters. As Jon Barwise has put it (Barwise 1985, 7):

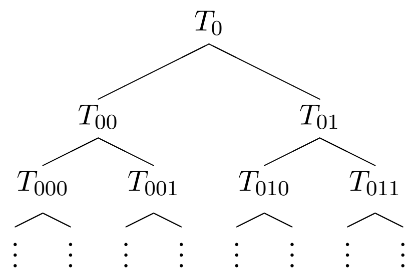
But if you think of logic as the mathematicians in the street, then the logic in a given concept is what it is, and if there is no set of rules which generate all the valid sentences, well, that is just a fact about the complexity of the concept that has to be lived with.

We have pointed at some possibilities of a *purely mathematical* characterization of intended models: Tennenbaum theorem with respect to Peano arithmetic, isomorphism theorems with respect to number systems, theorems concerning categoricity of some chosen systems of geometry. These characterizations are all given at the level of metalanguage. The same concerns the role of maximal axioms in set theory in the context of comparing theories with respect to interpretability. Finally, let us give one more example connected again with Peano arithmetic.

Let $T_0 = PA$ and let ψ_0 be any sentence undecidable in T_0. Further, let $T_{00} = PA \cup \{\psi_0\}$ and $T_{01} = PA \cup \{\neg\psi_0\}$. For any finite sequence σ of elements being equal either to 0 or 1 let:

- $T_{\sigma 0} = T_\sigma \cup \{\psi_\sigma\}$
- $T_{\sigma 1} = T_\sigma \cup \{\neg\psi_\sigma\}$.

(where ψ_σ is any sentence undecidable in T_σ). We get an infinite binary tree:



This tree has $2^{\aleph_0}$ branches. Due to the compactness theorem the union of theories on each branch is consistent (under the assumption that PA itself is consistent). On behalf of the downward Löwenheim-Skolem theorem each such union has a countable model. No two of these models are elementarily equivalent, due to the construction of the tree.

If we let ψ_0 to be $Con(PA)$ and ψ_α to be $Con(T_\alpha)$ then $\mathfrak{N}_0$, i.e. the standard model of PA is a model of the leftmost branch of the tree. All the other branches have non standard (countable) models. Each sentence of the form $\neg Con(T_\alpha)$ has a Gödel number which is a non standard number in the corresponding model.

We see that we can pick up the standard model of PA from all these models. But again, the rule underlying this choice belongs to metatheory.

The debate about intended models became vivid in the general methodology of the sciences after publication of Hilary Putnam's famous essay *Models and reality* (cf. Putnam 1980). It should be stressed that it is not only the Putnam's *model-theoretic argument* (based essentially on the Löwenheim-Skolem Theorem) which is relevant at this issue. We have seen that much more is involved in the problem of distinguishing the intended model of some theory and the situation depends on the area of mathematics we are dealing with.

We did not report in this short note about several positions taken in the general methodology of the sciences concerning intended models. An interested reader may consult, e.g.: Gaifman 2004, Grobler 2006, Nowaczyk 1998, Przełęcki 1988, Woleński 1993, 2005, Wójcicki 1974.

The above considerations contain no essentially new original reflections. All the problems discussed here have been widely known for several years. Anyway, we think that it is an interesting enterprize to look *collectively* at the *extremal axioms* formulated in different areas in mathematics. At the present moment, the works devoted to the extremal axioms *in general* are not that numerous yet (cf. e.g. Carnap, Bachmann 1981, Bernays 1955, Hintikka 1986, 1991, remarks on extremal axioms in several monographs on arithmetic, geometry, algebra and set theory). The present author works on a monograph *Extremal Axioms*. Hopefully, he will finish it before his Ultimate End.

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